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DIFFERENTIAL INEQUALITIES FOR A NONLINEAR PARTIAL DIFFERENTIAL EQUATION OF THE FOURTH ORDER

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In this paper, some results for the solution and the upper and lower functions of the nonlinear partial differential equation

$$u_{xxtt} = f(t, x, u, u_{xx}, u_{tt}) \tag{1}$$

are derived by means of the methods of differential inequalities [1], [2].

If we define:

$$Pu = u_{xxtt} - f(t, x, u, u_{xx}, u_{tt})$$

and

$$D = \{(t, x) : 0 \leqq x \leqq L,\ 0 \leqq t \leqq T\},$$

then we can prove the following

Theorem 1. *Assume that*

1) $v, w \in C[D, R]$; *the partial derivatives* $v_{xxtt}, v_{xx}, v_{tt}, w_{xxtt}, w_{xx}, w_{tt}$ *exist and are continuous in* D.
2) $f \in C[D \times R^3, R]$; *for* $(t, x) \in D$ *the function* $f(t, x, u, p, q)$ *is nondecreasing in* u, p, q *and fulfils the inequality* $Pv < Pw$.
3) *For* $0 \leqq x \leqq L$, $0 \leqq t \leqq T$ *the functions* v, w *satisfy the inequalities*

$$\begin{aligned} &v(t, 0) < w(t, 0), && v_x(t, 0) < w_x(t, 0), \\ &v_{tt}(t, 0) < w_{tt}(t, 0), && v_{ttx}(t, 0) < w_{ttx}(t, 0), \\ &v(0, x) < w(0, x), && v_t(0, x) < w_t(0, x), \\ &v_{xx}(0, x) < w_{xx}(0, x), && v_{xxt}(0, x) < w_{xxt}(0, x). \end{aligned} \tag{2}$$

Then we have in D:

$$v(t, x) < w(t, x), \quad v_{xx}(t, x) < w_{xx}(t, x), \quad v_{tt}(t, x) < w_{tt}(t, x). \tag{3}$$

Proof. Assume that the contrary is true. Let y_0 be the greatest lower bound of the numbers $y = t + x$ such that for the points (t, x), $t + x < y_0$ all the three inequalities (3) are satisfied.

For (t_0, x_0), $t_0 + x_0 = y_0$ the inequality $v_{xx} < w_{xx}$, for instance, is not true, so that

$$v_{xx}(t_0, x_0) = w_{xx}(t_0, x_0)\,, \tag{4}$$

$$v_{tt}(t_0, x_0) \leqq w_{tt}(t_0, x_0)\,, \quad v(t_0, x_0) \leqq w(t_0, x_0)\,. \tag{5}$$

Since $t_0 > 0$, we have for $h > 0$

$$v_{xx}(t_0 - h, x_0) < w_{xx}(t_0 - h, x_0)\,. \tag{6}$$

From (4) and (6) we obtain

$$v_{xxt}(t_0, x_0) \geqq w_{xxt}(t_0, x_0)\,. \tag{7}$$

From (4), (5) and from the assumption 2) it follows that

$$v_{xxtt}(t_0, x_0) < w_{xxtt}(t_0, x_0)\,. \tag{8}$$

Since for $0 \leqq t < t_0$ we have $v_{xx}(t, x_0) < w_{xx}(t, x_0)$, $v_{tt}(t, x_0) < w_{tt}(t, x_0)$, $v(t, x_0) <$ $< w(t, x_0)$ and

$$v_{xxtt}(t, x_0) - f[t, x_0, v(t, x_0), v_{xx}(t, x_0), v_{tt}(t, x_0)] <$$
$$< w_{xxtt}(t, x_0) - f[t, x_0, w(t, x_0), w_{xx}(t, x_0), w_{tt}(t, x_0)]\,,$$

it follows that

$$v_{xxtt}(t, x_0) < w_{xxtt}(t, x_0) \tag{9}$$

for $0 \leqq t < t_0$.

Because v_{xxtt} and w_{xxtt} are continuous in D, we obtain from (8) and (9)

$$\int_0^{t_0} v_{xxtt}(t, x_0)\, dt < \int_0^{t_0} w_{xxtt}(t, x_0)\, dt\,,$$

so that, with regard to the assumption 3), we have

$$v_{xxt}(t_0, x_0) < w_{xxt}(t_0, x_0)\,,$$

which is a contradiction with (7). Hence $v_{xx}(t_0, x_0) = w_{xx}(t_0, x_0)$ cannot hold.

It is clear that if we suppose that $v_{tt}(t_0, x_0) = w_{tt}(t_0, x_0)$ holds instead of (4), an argument similar to the foregoing one leads to a contradiction.

Since v_{xx} and w_{xx} are continuous in D and the inequality $v_{xx} < w_{xx}$ holds, we have with regard to the assumption 3),

$$v(t, x) < w(t, x) \quad \text{if} \quad (t, x) \in D\,.$$

Hence, the inequalities (3) are true.

Definition 1. Consider the differential equation (1) and functions $\varphi_0(x), \varphi_1(x) \in C^2[I_1, R]$, $\psi_0(t), \psi_1(t) \in C^2[I_2, R]$, where $I_1 = \{x : 0 \leqq x \leqq L\}$, $I_2 = \{t : 0 \leqq t \leqq T\}$. Let $u(t, x) \in C[D, R]$ and let its derivatives u_{xxtt}, u_{xx}, u_{tt} be continuous in D.

Let

$$u_{xxtt} = f(t, x, u, u_{xx}, u_{tt}), \tag{10}$$

$$u(0, x) = \varphi_0(x), \quad u_t(0, x) = \varphi_1(x) \quad \text{if} \quad x \in I_1, \tag{11}$$

$$u(t, 0) = \psi_0(t), \quad u_x(t, 0) = \psi_1(t) \quad \text{if} \quad t \in I_2, \tag{12}$$

where

$$\varphi_0(0) = \psi_0(0), \quad \varphi_1(0) = \psi_0'(0), \quad \varphi_0'(0) = \psi_1(0), \quad \varphi_1'(0) = \psi_1'(0).$$

Then the function $u(t, x)$ is called a solution of the problem (10), (11), (12).

If the function $u(t, x)$ fulfils in D the inequalities

$$u_{xxtt} > f(t, x, u, u_{xx}, u_{tt}), \tag{13}$$

$$u(0, x) > \varphi_0(x), \quad u_t(0, x) > \varphi_1(x), \quad u(t, 0) > \psi_0(t), \quad u_x(t, 0) > \psi_1(t),$$

then it is said to be an *upper function* of the problem (10), (11), (12).

If the function $u(t, x)$ fulfils in D the inequalities

$$u_{xxtt} < f(t, x, u, u_{xx}, u_{tt}), \tag{14}$$

$$u(0, x) < \varphi_0(x), \quad u_t(0, x) < \varphi_1(x), \quad u(t, 0) < \psi_0(t), \quad u_x(t, 0) < \psi_1(t),$$

it is said to be a *lower function* of the problem (10), (11), (12).

Applying the foregoing Theorem 1, we obtain for the upper and lower functions the following

Theorem 2. *Let $u(t, x)$ be a solution, $v(t, x)$ a lower function and $w(t, x)$ an upper function of the problem* (10), (11), (12). *Let the function f of the differential equation* (10) *fulfil the assumptions of Theorem* 1 *and let*

$$v_{xx}(0, x) < u_{xx}(0, x) < w_{xx}(0, x), \quad v_{xxt}(0, x) < u_{xxt}(0, x) < w_{xxt}(0, x),$$

$$v_{tt}(t, 0) < u_{tt}(t, 0) < w_{tt}(t, 0), \quad v_{ttx}(t, 0) < u_{ttx}(t, 0) < w_{ttx}(t, 0).$$

Then in the domain D we have the inequalities

$$v(t, x) < u(t, x) < w(t, x), \tag{15}$$

$$v_{xx}(t, x) < u_{xx}(t, x) < w_{xx}(t, x), \quad v_{tt}(t, x) < u_{tt}(t, x) < w_{tt}(t, x).$$

We can derive some estimates for the solution of the problem (10), (11), (12) in terms of the upper functions.

Theorem 3. *Assume that*

1) $f \in C[D \times R^3, R]$ *and for* $(t, x) \in D$ *it is*

(16) $$|f(t, x, u, p, q)| \leqq g(t, x, |u|, |p|, |q|),$$

where $g \in C[D \times R_+^3, R_+]$ *is nondecreasing in* u, p, q *if* $(t, x) \in D$.

2) $h(t, x) \in C[D, R_+]$ *is an upper function of the problem*

(17) $$z_{xxtt} = g(t, x, z, z_{xx}, z_{tt}),$$

(18) $$|\varphi_0(x)| = z(0, x), \quad |\varphi_1(x)| = z_t(0, x) \quad \text{if} \quad x \in I_1,$$
$$|\psi_0(t)| = z(t, 0), \quad |\psi_1(t)| = z_x(t, 0) \quad \text{if} \quad t \in I_2,$$

where $h_{xx}(t, x) > 0$, $h_{tt}(t, x) > 0$ *if* $(t, x) \in D$.

Let

(19) $$|\varphi_0''(x)| < h_{xx}(0, x), \quad |\varphi_1''(x)| < h_{xxt}(0, x),$$
$$|\psi_0''(t)| < h_{tt}(t, 0), \quad |\psi_1''(t)| < h_{ttx}(t, 0)$$

if $t \in I_2$, $x \in I_1$.

Then the solution $u(t, x)$ *of the problem* (10), (11), (12) *satisfies in* D:

(20) $$|u(t, x)| < h(t, x), \quad |u_{xx}(t, x)| < h_{xx}(t, x), \quad |u_{tt}(t, x)| < h_{tt}(t, x).$$

Proof. The proof is analogous to that of Theorem 1.

Suppose that the inequalities (20) hold for the points (t, x), $t + x < y_0$, but for (t_0, x_0), $t_0 + x_0 = y_0$ the inequality $|u_{xx}| < h_{xx}$, for instance, is not true, so that

(21) $$|u_{xx}(t_0, x_0)| = h_{xx}(t_0, x_0).$$

It holds

$$|u_{xx}(t_0, x_0)| \leqq \left|\int_0^{t_0}\int_0^{t_1} f[t, x_0, u(t, x_0), u_{xx}(t, x_0), u_{tt}(t, x_0)]\,dt\,dt_1\right| + $$
$$+ |u_{xx}(0, x_0)| + t_0|u_{xxt}(0, x_0)|.$$

Then the assumptions 1 and 2 imply that

$$|u_{xx}(t_0, x_0)| < h_{xx}(0, x) + t_0\, h_{xxt}(0, x) +$$
$$+ \int_0^{t_0}\int_0^{t_1} g[t, x_0, |u(t, x_0)|, |u_{xx}(t, x_0)|, |u_{tt}(t, x_0)|]\,dt\,dt_1.$$

The ineqalities $|u_{xx}(t, x_0)| \leqq h_{xx}(t, x_0)$, $|u_{tt}(t, x_0)| \leqq h_{tt}(t, x_0)$, $|u(t, x_0)| \leqq h(t, x_0)$, which hold for $0 \leqq t \leqq t_0$, together with the monotonicity of g in u, p, q imply

$$|u_{xx}(t_0, x_0)| < h_{xx}(0, x) + t_0\, h_{xxt}(0, x) +$$
$$+ \int_0^{t_0}\int_0^{t_1} g[t, x_0, h(t, x_0), h_{xx}(t, x_0), h_{tt}(t, x_0)]\,dt\,dt_1$$

and because the function $h(t, x)$ is an upper function of (17), (18), it is

$$|u_{xx}(t_0, x_0)| < h_{xx}(t_0, x_0),$$

which is a contradiction with (21). Similarly we obtain that also $|u_{tt}(t, x)| < h_{tt}(t, x)$ holds for all $(t, x) \in D$.

For arbitrary $(t_0, x_0) \in D$ it is

$$|u(t_0, x_0)| \leqq \int_0^{x_0} \int_0^{x_1} |u_{xx}(t_0, x)| \, dx \, dx_1 + |u(t_0, 0)| + x_0 |u_x(t_0, 0)| .$$

From $|u_{xx}| < h_{xx}$ and from the assumption 2 it follows that

$$|u(t_0, x_0)| < \int_0^{x_0} \int_0^{x_1} h_{xx}(t_0, x) \, dx \, dx_1 + h(t_0, 0) + x_0 \, h_x(t_0, 0)$$

which implies the inequality

$$|u(t_0, x_0)| < h(t_0, x_0),$$

so that the assertion of Theorem 3 is true.

Consider now the inequality

$$(22) \qquad |u_{xxtt} - f(t, x, u, u_{xx}, u_{tt})| \leqq \delta(t, x),$$

where $f \in C[D \times R^3, R]$, $\delta(t, x) \in C[D, R_+]$.

Definition 2. A function $u(t, x) \in C[D, R]$ possessing continuous partial derivatives u_{xxtt}, u_{xx}, u_{tt} if $(t, x) \in D$, and satisfying in D the differential inequality (22) is called a *δ-approximate solution* of the differential equation (10).

By means of the following theorem we can estimate the difference between a solution of the problem (10), (11), (12) and a δ-approximate solution.

Theorem 4. *Assume that*

1) *$u(t, x)$ is a solution of the problem (10), (11), (12) and $v(t, x)$ is a δ-approximate solution.*

2) *Let $f \in C[D \times R^3, R]$ and let for $(t, x) \in D$*

$$(23) \qquad |f(t, x, u, p, q) - f(t, x, \bar{u}, \bar{p}, \bar{q})| \leqq g(t, x, |u - \bar{u}|, |p - \bar{p}|, |q - \bar{q}|),$$

where $g \in C[D \times R_+^3, R]$ is in D d nondecreasing function in u, p, q.

3) *Let $h(t, x) \in C[D, R_+]$ be an upper function of the problem*

$$(24) \qquad z_{xxtt} = g(t, x, z, z_{xx}, z_{tt}) + \delta(t, x),$$

$$z(0, x) = |\varphi_0(x)|, \quad z_t(0, x) = |\varphi_1(x)| \quad \text{if} \quad x \in I_1,$$

$$z(t, 0) = |\psi_0(t)|, \quad z_x(t, 0) = |\psi_1(t)| \quad \text{if} \quad t \in I_2,$$

where $h_{xx} > 0$, $h_{tt} > 0$ *if* $(t, x) \in D$, $h_{xxt}(0, x) > 0$ *if* $x \in I_1$, $h_{ttx}(t, 0) > 0$ *if* $t \in I_2$, $\delta(t, x) \in C[D, R_+]$.

4) *Let*

$$(25) \qquad |v_{xx}(0, x) - \varphi_0''(x)| < h_{xx}(0, x), \quad |v_{xxt}(0, x) - \varphi_1''(x)| < h_{xxt}(0, x),$$
$$|v_{tt}(t, 0) - \psi_0''(t)| < h_{tt}(t, 0), \quad |v_{ttx}(t, 0) - \psi_1''(t)| < h_{ttx}(t, 0)$$

if $0 \leqq x \leqq L$, $0 \leqq t \leqq T$.

Then we have in D

$$(26) \qquad |v(t, x) - u(t, x)| < h(t, x),$$
$$|v_{xx}(t, x) - u_{xx}(t, x)| < h_{xx}(t, x), \quad |v_{tt}(t, x) - u_{tt}(t, x)| < h_{tt}(t, x).$$

The proof of this theorem is analogous to that of Theorem 3.

Note 1. Let $v(t, x)$ be a δ-approximate solution of the differential equation (10) fulfilling the initial conditions (11), (12) and let $u(t, x)$ be a solution of the problem (10), (11), (12). Let the assumption 2 of Theorem 4 hold. Let $h(t, x) \in C[D, R_+]$ be an upper function of the problem

$$z_{xxtt} = g(t, x, z, z_{xx}, z_{tt}) + \delta(t, x),$$
$$z(0, x) = z_t(0, x) = z(t, 0) = z_x(t, 0) = 0,$$

where $\delta(t, x) \in C[D, R_+]$, and $h_{xx} > 0$, $h_{tt} > 0$ if $(t, x) \in D$, $h_{xxt}(0, x) > 0$, $h_{ttx}(t, 0) > 0$. Then in D the relations (26) hold.

This assertion is an easy consequence of the foregoing Theorem 4.

Note 2. Let the function $f \in C[D \times R^3, R]$ be Lipschitz-continuous in D, so that

$$|f(t, x, u, p, q) - f(t, x, \bar{u}, \bar{p}, \bar{q})| \leqq M(|u - \bar{u}| + |p - \bar{p}| + |q - \bar{q}|).$$

For $0 < u < \infty$, $0 < p < \infty$, $0 < q < \infty$ the function $g(t, x, u, p, q) = M(u + p + q)$ is nondecreasing in u, p, q if $(t, x) \in D$. Let $u(t, x)$ be a solution and $v(t, x)$ a δ-approximate solution of the equation (10) fulfilling (11), (12), so that $\delta(t, x) = \delta e^{M(t+x)}$, where $\delta > 0$ is a constant and $M > 0$ is the Lipschitz constant. Let $M^4 - 2M^3 - M > 0$, $A \geqq \delta/(M^4 - 2M^3 - M)$.

Then we have in D

$$(27) \qquad |u(t, x) - v(t, x)| < Ae^{M(t+x)}.$$

Since $h(t, x) = Ae^{M(t+x)}$ is an upper function of the problem

$$z_{xxtt} = M(z + z_{xx} + z_{tt}) + \delta e^{M(t+x)},$$
$$z(0, x) = z_t(0, x) = z(t, 0) = z_x(t, 0) = 0$$

if $x \in I_1$, $t \in I_2$, the relation (27) is a consequence of the foregoing Note 1 because all the assumptions of this note are fulfilled.

It is evident that Theorem 4 can be used for numerical estimation of errors when solwing the problem (10), (11), (12).

References

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