S. M. Shamim Imdadi; M. Rama Mohana Rao

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TOPOLOGICAL DYNAMICS AND THE EXTENSION OF LYAPUNOV'S METHOD

S. M. SHAMIM IMDADI, Tripoli, and M. RAMA MOHANA RAO, Kanpur

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1. It has long been realized that the theory of topological dynamics has direct and important applications to autonomous differential equations, but this theory is not well developed as a powerful technique in applications to nonautonomous differential equations. However, quite recently G. R. SELL [3] has shown that there is a way of viewing the solutions of nonautonomous differential equations as a dynamical system. This view point is very general and includes all differential equations satisfying only the weakest hypotheses. In particular, he has shown that the solutions of every *admissible differential equation* $x' = f(x, t)$, defined on $W \times R$, can be viewed as *a local dynamical system* π defined on the phase space $W \times F^*_{co}$, where W is an open set in R^n and F^*_{co} is the "hull" of f.

Our aim in this paper is to extend the phase space of the local dynamical system π by defining the prolongational set D^*_f, and investigate sufficient conditions for stability of solutions of the given differential equation and the corresponding set of limiting equations. This would generalize most of the results of G. R. Sell in [3] and [4], as the phase space in our case contains $W \times F^*_{co}$.

Section 2 deals with preliminaries and basic lemmas. Section 3 is devoted for the extension of local dynamical system π on $W \times F^*_{co}$ to one on $W \times D^*_f$. In Section 4, we obtain sufficient conditions to establish a number of results on stability of solutions of the given differential equation and the corresponding set of limiting equations in terms of Lyapunov functions. A simple example is constructed to illustrate the results.

2. Let W be an open set in R^n, the euclidean n-space. Let $R^+ = [0, \infty)$ and $R^- = (-\infty, 0]$. For any $x \in R^n$, $|x| = \sum_{i=1}^{n} |x_i|$. We shall say that a function $f : W \times R \to R^n$ is *admissible* if (i) f is continuous, and (ii) the solutions of the differential equation

$$x' = f(x, t) \tag{2.1}$$

are unique. By the second condition we mean that given any point (x_0, t_0) in $W \times R$, there is precisely one solution ϕ of (2.1) that satisfies $\phi(t_0) = x_0$. Let $C = C(W \times R, R^n)$ denote the set of all continuous functions f defined on $W \times R$ with values in R^n. It is evident that if $f \in C$ is an admissible function, then every translate f_τ of f, where $f_\tau(x, t) = f(x, t + \tau)$, is an admissible function. Now let $C^* = C^*(W \times R, R^n)$ denote the class of all admissible functions f defined on $W \times R$ with values in R^n. It is known [3, Theorem 1] that the mapping $\pi^* : C \times R \to C$, defined by $\pi^*(f, \tau) = f_\tau$, is a dynamical system on C, when C has the compact open topology. Therefore, it is evident that $\pi^* : C^* \times R \to C^*$, the restriction of π^* on $C^* \subset C$ is also a dynamical system on C^*. Let ϱ be a metric which generates the compact open topology on C.

Let $f \in C^*$. Define *positive prolongational set* D_f^+ *and positive prolongational limit set* J_f^+ of f as follows:

$D_f^+ = \{g \in C$: there is a sequence $\{f_n\}$ in C^* and a sequence $\{t_n\}$ in R^+ such that $f_n \to f$ and $\pi^*(f_n, t_n) \to g\}$,

$J_f^+ = \{g \in C$: there is a sequence $\{f_n\}$ in C^* and a sequence $\{t_n\}$ in R^+ such that $f_n \to f$, $t_n \to +\infty$, and $\pi^*(f_n, t_n) \to g\}$.

Similarly, we define *negative prolongational set* D_f^- *and negative prolongational limit set* J_f^- of f by considering the sequence $\{t_n\}$ in R^-. Let $D_f^* = D_f^+ \cup D_f^-$.

We say that the function $f \in C^*$ is D_f^**-regular* if and only if every $g \in D_f^*$ is admissible.

Remark 2.1. It is clear that, for any $f \in C^*$, the ω-limit set $\Omega_f^* \subset J_f^+$ and $D_f^+ = F \cup J_f^+$, where $F = \{f_\tau : \tau \in R^+\}$. Further, if f is D_f^*-regular, then f is *regular* [3].

We need the following lemmas in our subsequent discussion.

Lemma 2.1. *Given* $h \in J_f^+$ *and* $\varepsilon > 0$, *there exist a function* $g \in C^*$ *and a real number* $\tau = \tau(\varepsilon) > 0$ *such that* $\varrho(f, g) < \varepsilon$ *and* $\varrho(h, g_\tau) < \varepsilon$.

Proof. Let $h \in J_f^+$ and $\varepsilon > 0$ be given. From the definition of J_f^+, there exist sequences $\{f_n\} \in C^*$, $\{t_n\} \in R^+$, such that $f_n \to f$, $t_n \to \infty$, and $\pi^*(f_n, t_n) \to h$. Therefore, it follows that $\varrho(f, f_n) < \varepsilon$ for all $n \geqq N_1(\varepsilon)$, and $\varrho(h, \pi^*(f_n, t_n)) < \varepsilon$ for all $n \geqq N_2(\varepsilon)$. Let $N = \max(N_1, N_2)$. Choose $f_N = g$ and $t_N = \tau$, Then, obviously $g \in C^*$, $\varrho(f, g) < \varepsilon$, and $\varrho(h, g_\tau) < \varepsilon$. This completes the proof.

Lemma 2.2. *If* $f \in C^*$, *then* (i) J_f^+ *is closed and invariant subset of* C, *and* (ii) D_f^+ *is closed and positively invariant subset of* C.

Proof. (i) Let $\{g_n\}$ be a sequence in J_f^+ with $g_n \to g$. We prove that $g \in J_f^+$. By definition, for each integer K, there are sequences $\{f_n^K\} \in C^*$ and $\{t_n^K\} \in R^+$ such that $f_n^K \to f$, $t_n^K \to +\infty$, and $\pi^*(f_n^K, t_n^K) \to g_K$. We may assume by taking subsequences if necessary that $t_n^K > K$, $\varrho(f_n^K, f) \leqq 1/K$, and $\varrho(\pi^*(f_n^K, t_n^K), g_K) \leqq 1/K$ for $n \geqq K$. Now consider the sequences $\{f_n^n\}, \{t_n^n\}$. Obviously $\{f_n^n\} \in C^*$ and $f_n^n \to f$, $t_n^n \to +\infty$. Hence

$$\varrho(g, \pi^*(f_n^n, t_n^n)) \leqq \varrho(g, g_n) + \varrho(g_n, \pi^*(f_n^n, t_n^n)) \leqq \varrho(g, g_n) + \frac{1}{n}.$$

Since $g_n \to g$, we have $\pi^*(f_n^n, t_n^n) \to g$. Thus $g \in J_f^+$. This shows that J_f^+ is closed in C.

To prove that J_f^+ is invariant, let $g \in J_f^+$ and $\tau \in R$ be given. Then, there exist sequences $\{f_n\} \in C^*$ and $\{t_n\} \in R^+$ such that $f_n \to f$, $t_n \to +\infty$, and $\pi^*(f_n, t_n) \to g$. Now consider the sequence $\{t_n + \tau\}$. Clearly $t_n + \tau \to +\infty$, and $\pi^*(f_n, t_n + \tau) = \pi^*(\pi^*(f_n, t_n), \tau) \to \pi^*(g, \tau)$. Since $f_n \to f$, we have $\pi^*(g, \tau) \in J_f^+$. As $\tau \in R$ is arbitrary, it is celar that J_f^+ is invariant.

The proof of (ii) is similar and hence omitted.

Remark 2.2. Analogous results hold for D_f^- and J_f^-.

3. Let $f \in C^*$ and $F = \{f_\tau : \tau \in R\}$. For each point $p = (x, g)$ in $X = W \times F$, let $I_p = I_{(x,g)}$ be the maximal interval of definition of the solution $\phi(x, g, t)$ of $x' = g(x, t)$ that satisfies $\phi(x, g, 0) = x$. Let $S = \{(x, g; t) = (p; t) \in X \times R : t \in I_p\}$, and define $\pi : S \to X$ by

$$\pi(x, g; t) = (\phi(x, g, t), g_t). \tag{3.1}$$

It has been proved by G. R. Sell [3] that π defined by (3.1) is a local dynamical system on $X = W \times F$.

Theorem 3.1. *Let $f \in C^*(W \times R, R^n)$ be a function that satisfies a local Lipschitz condition in x, where the Lipschitz constant is independent of t. Then f is D_f^*-regular.*

The Lipschitz condition stated above means that for every compact subset K of W, there is a positive constant c such that

$$|f(x, t) - f(y, t)| \leqq c|x - y| \quad (x \in K,\ y \in K,\ t \in R).$$

Proof. Let K be a compact set in W and let f_τ be any translate of f. Then there is a positive constant c such that

$$|f_\tau(x, t) - f_\tau(y, t)| = |f(x, t + \tau) - f(y, t + \tau)| \leqq c|x - y|,$$

for all x and y in K and all t in R. In other words, every translate of f satisfies the Lipschitz condition with the same Lipschitz constant. Now let $h \in J_f^+$. Then by Lemma 2.1, given $\varepsilon > 0$ there exist a function $g \in C^*$ and a real number $\tau = \tau(\varepsilon) > 0$ such that $\varrho(f, g) < \varepsilon$ and $\varrho(h, g_\tau) < \varepsilon$.

Let I be any compact set in R and let $I' = \{t + \tau : t \in I\}$. Then $M = K \times I$ and $M' = K \times I'$ are compact sets in $W \times R$. Since the metric ϱ generates the topology of uniform convergence on compact sets (this topology is the same as the compact open topology on $C(W \times R, R^n)$ [1, pp. 186 and 230]), there is a nonnegative function $l(M'; \varepsilon)$ such that $l(M'; \varepsilon) \to 0$ as $\varepsilon \to 0$, and

$$\sup\{|f(x, t) - g(x, t)| : (x, t) \in M'\} \leqq l(M'; \varepsilon),$$

whenever $f \in C(W \times R, R^n)$ and $\varrho(f, g) < \varepsilon$. (The function l depends on f, which is fixed for our argument). Thus if $\varrho(f, g) < \varepsilon$, then

$$|g(x, t) - g(y, t)| \leqq |g(x, t) - f(x, t)| + |f(x, t) - f(y, t)| + \\ + |f(y, t) - g(y, t)| \leqq 2l(M'; \varepsilon) + c|x - y|,$$

whenever (x, t) and (y, t) are in M'. This implies that

$$|g_\tau(x, t) - g_\tau(y, t)| \leqq 2l(M'; \varepsilon) + c|x - y|,$$

whenver (x, t) and (y, t) are in M.

Now since $\varrho(h, g_\tau) < \varepsilon$, we can again construct a nonnegative function $l'(M; \varepsilon)$, such that $l'(M; \varepsilon) \to 0$ as $\varepsilon \to 0$, and

$$\sup\{|h(x, t) - g_\tau(x, t)| : (x, t) \in M\} \leqq l'(M; \varepsilon).$$

Therefore, we have

$$|h(x, t) - h(y, t)| \leqq |h(x, t) - g_\tau(x, t)| + |g_\tau(x, t) - g_\tau(y, t)| + \\ + |g_\tau(y, t) - h(y, t)| \leqq 2l'(M; \varepsilon) + 2l(M'; \varepsilon) + c|x - y|,$$

whenever (x, t) and (y, t) are in M. If we let $\varepsilon \to 0$ we get

$$|h(x, t) - h(y, t)| \leqq c|x - y| \quad (x \in K,\ y \in K,\ t \in I).$$

However, since I is arbitrary we get

$$|h(x, t) - h(y, t)| \leqq c|x - y| \quad (x \in K,\ y \in K,\ t \in R).$$

Similarly, we can prove that every function h in J_f^- satisfies the Lipschitz condition with the same Lipschitz constant as f. This completes the proof.

Theorem 3.2. *Let $f \in C^*$ be a D_f^*-regular function. Then the local dynamical system π on $W \times F$ defined by (3.1), can be extended to a local dynamical system on $W \times D_f^*$. The extension is given by (3.1).*

Proof. Since every $h \in D_f^*$ is admissible, we can define the extension π by (3.1). It then follows that π satisfies all the properties [3, p. 247] for local dynamical system.

4. Let $f \in C^*$. If the positive prolongational limit set J_f^+ is non-empty, then we say that the set of *limitng equations* for (2.1) is the set of all differential equations of the form

$$x' = f^*(x, t)\,, \tag{4.1}$$

where $f^* \in J_f^+$.

In our subsequent discussion, we suppose that $x(t) = x(x_0, f, t)$ and $y(t) = y(y_0, f^*, t)$ are any two solutions of (2.1) and (4.1) with $x(t_0) = x_0$ and $y(t_0) = y_0$, existing for all $t \in R^+$. We shall write $d(x, y)$ for $|x - y|$. Let a function $V(t, x, y) \geqq 0$ be defined and continuous on the product space $R^+ \times W \times W$, and suppose that it satisfies local Lipschitz condition in x and y. Following YOSHIZAWA [5], we define the function

$$D^+ V(t, x, y) = \overline{\lim_{h \to 0^+}} \frac{1}{h} \left[V(t + h,\ x + h f(x, t),\right.$$
$$\left. y + h f^*(y, t)) - V(t, x, y)\right].$$

With respect to these functions we state the following lemma whose proof can be found in [2].

Lemma 4.1. *Let the function $g(t, r)$ be defined and continuous on $R^+ \times R^+$. Suppose further that $D^+ V(t, x, y) \leqq g(t, V(t, x, y))$.*

Let $r(t)$ be the maximal solution of the differential equation

$$r' = g(t, r)\,, \quad r(t_0) = r_0 \geqq 0\,. \tag{4.2}$$

If $x(t)$ and $y(t)$ are any two solutions of (2.1) and (4.1) such that $V(t_0, x_0, y_0) \leqq r_0$, then $V(t, x(t), y(t)) \leqq r(t)$, $t \geqq t_0$.

In order to unify our results on stability, we list the following conditions:

(C_1) *For each $\varepsilon > 0$ and $t_0 \geqq 0$, there exists a positive function $\delta = \delta(\varepsilon)$ such that if $d(x_0, y_0) \leqq \delta$, then $d(x(t), y(t)) < \varepsilon$ for all $t \geqq t_0$.*

(C_2) *For each $\eta > 0$ and $t_0 \geqq 0$, there exist positive numbers δ_0 and $T = T(\eta)$ such that $d(x(t), y(t)) < \eta$ for $t \geqq t_0 + T$, whenever $d(x_0, y_0) \leqq \delta_0$.*

(C_3) *The conditions (C_1) and (C_2) hold simultaneously.*

Remark 4.1. Corresponding to the definitions above, if we say that the scalar differential equation (4.2) has the property (C_1S), we mean that the following condition is satisfied:

(C_1S) *Given $\varepsilon > 0$ and $t_0 \geqq 0$ there exists a positive function $\delta = \delta(\varepsilon)$ such that the inequality $r_0 \leqq \delta$ implies $r(t) < \varepsilon$ for all $t \geqq t_0$.*

Conditions (C_2S) and (C_3S) may be reformulated similarly.

Remark 4.2. We assume hereafter that the solutions $r(t)$ of (4.2) are non-negative, for $t \geqq t_0$, so as to ensure that $g(t, r(t))$ is defined. Such a requirement is clearly satisfied if we assume that $g(t, 0) = 0$ for all t.

Further, we assume that

(A) *The function* $b(r)$ *is continuous and nonincreasing in* r, $b(r) > 0$ *for* $r > 0$, *and* $b(d(x, y)) \leqq V(t, x, y)$.

Now we have the following main result on stability of solutions of (2.1) and (4.1).

Theorem 4.1. *Let the assumptions of Lemma* 4.1 *hold, together with* (A). *Suppose further that the scalar differential equation* (4.2) *satisfies one of the conditions* (C_1S), (C_2S) *and* (C_3S), *then the differential systems* (2.1) *and* (4.1) *satisfy the corresponding one of the conditions* (C_1), (C_2) *and* (C_3).

The proof is similar to the proof of Theorem 4 in [2] and hence omitted.

Remark 4.3. Similar to Theorem 4.1, some theorems are given in [2], [5] and [6] for two entirely different differential systems.

Example 4.1. ($n = 1$). Consider the differential equation

$$x' = f(t, x), \tag{4.3}$$

where $f(t, x) = -x - x^5 e^{-2t}$. Choose $f_n(t, x) = -x - x^5 e^{-2t} - x^3 t/n$, and $\{t_n\} = \{n\}$. Then, we have $\pi^*(f_n, t_n) = -x - x^5 e^{-2(t+n)} - x^3(t + n)/n$. Therefore, $f^*(t, x) = -x - x^3$, where $f^* \in J_f^+$. Let $V(t, x, y) = x^2 + y^2$. Then the condition (A) is clearly satisfied. After a little computation, we obtain $D^+ V(t, x, y) \leqq \leqq -2\, V(t, x, y)$.

If we set $g(t, r) = -2r$, we see that the scalar differential equation (4.2) satisfies conditions (C_1S) and (C_2S). From Theorem 4.1, we conclude therefore that the solutions of the differential equation (4.3) and the corresponding limiting equation satisfy the conditions (C_1) and (C_2) and hence the condition (C_3).

References

[1] *J. L. Kelley:* General Topology, Van Nostrand, Princeton, N. J. 1955.

[2] *V. Lakshmikantham:* Differential Systems and Extension of Lyapunov's Method. Michigan Math. J. *9* (1962), 311—320.

[3] *G. R. Sell:* Non-autonomous Differential Equations and Topological Dynamics I. The Basic Theory, Trans. Amer. Math. Soc. *127* (1967), 241—262.

[4] *G. R. Sell:* Non-autonomous Differential Equations and Topological Dynamics II. Limiting Equations, Trans. Math. Soc. *127* (1967), 263—283.

[5] *T. Yoshizawa:* Lyapunov's Function and Boundedness of Solutions, Funkcial. Ekvac. *2* (1959), 95—142.

[6] *T. Yoshizawa:* Stability and Boundedness of Systems, Arch. Rational Mech. Anal. *6* (1960), 409—421.

Authors' address: S. M. Shamim Imdadi, Department of Mathematics, Faculty of Sciences, P. O. Box 656, University of Tripoli, Libya; M. Rama Mohana Rao, Department of Mathematics, Indian Institute of Technology, Kanpur — 208016 India.