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STRONGLY PROJECTABLE LATTICE ORDERED GROUPS

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Several theorems proved at first for archimedean vector lattices have turned out to be valid also for the more general case of archimedean lattice ordered groups. As an example we can mention here the theorem on the representation of archimedean vector lattices by means of real functions and its generalization for archimedean lattice ordered groups (cf., e.g., KANTOROVIČ, VULICH and PINSKER [11], Chap. XIII; BERNAU [2]).

The following theorem is well known (VEKSLER [13]; cf. also LUXEMBURG and ZAANEN [10], p. 137):

(A) *Let X be a vector lattice that is strongly projectable and σ-complete. Then X is complete.*

It is a natural question to ask whether each strongly projectable σ-complete lattice ordered group must be complete. In this note it will be shown that the answer is negative. The following affirmative result in this direction will be obtained for singular lattice ordered groups:

Theorem 1. *Let G be a singular lattice ordered group that is strongly projectable and σ-complete. Assume that G fulfils the condition*

(α) *each bounded set of singular elements of G has its supremum in G.*

Then G is complete.

Singular lattice ordered groups were investigated by IWASAWA [7], CONRAD and MCALLISTER [5] and by the author [8]. Let us remark that singular lattice ordered groups and vector lattices are in a certain sense on the opposite sides of the spectrum of lattice ordered groups. If G is a complete lattice ordered group, then $G = A \times B$, where A is the greatest singular convex l-subgroup of G, and B is the greatest convex l-subgroup of G that is a vector lattice. Every vector lattice is divisible; on the other hand, if A is singular, then for each $0 < a \in A$ there exists $a' \in A$ with $0 < a' \leqq a$ such that, if $n > 1$ is a positive integer, then the equation $nx = a'$ has no solution x in A.

If G is a complete lattice ordered group, then it is strongly projectable (this is a theorem of RIESZ; cf. [1], Chap. XIII) and obviously G is conditionally orthogonally complete. Every archimedean lattice ordered group that is signular and conditionally orthogonally complete must be complete [8]. Thus if G is a singular archimedean lattice ordered group, then the following conditions are equivalent:

(a) G is strongly projectable, σ-complete and fulfils (α);

(b) G is conditionally orthogonally complete;

(c) G is complete.

The examples given below show that for archimedean (even σ-complete) lattice ordered groups the strong projectability need not imply (b). The question whether (b) implies the strong projectability for archimedean l-groups remains open. (This question is closely related to a problem proposed by CONRAD [4]; cf. § 3 below.) In the case of vector lattices the answer to the question is affirmative (VEKSLER and GEJLER [14]).

A sequence $\{g_n\}$ of elements of a lattice ordered group G will be called a fundamental r_0-sequence, if there exists $0 < e \in G$ such that, for any positive integer n, we have $2^n|g_k - g_m| \leqq e$, whenever m, k are positive integers greather than n. The lattice ordered group G will be called r_0-complete if each fundamental r_0-sequence of elements of G is o-convergent. In § 4 it will be proved (Thm. 3) that if G is archimedean, projectable, conditionally orthogonally complete and r_0-complete, then it is complete. For a related result concerning vector lattices cf. VEKSLER and GEJLER [14], Thm. 4.

1. PRELIMINARIES

Let us recall some fundamental notions we shall need in the sequel. For the terminology, cf. BIRKHOFF [1], FUCHS [6] and CONRAD [3].

Let G be a lattice ordered group. The group operation and the lattice operations in G will be denoted by $+$ and by $\wedge$, $\vee$, respectively. Let $A \subset G$. We denote

$$A^\delta = \{g \in G : |g| \wedge |a| = 0 \text{ for each } a \in A\}\,,$$
$$A^{\delta\delta} = \{A^\delta\}^\delta\,.$$

The set A^δ is called a polar of G. The l-group G is said to be strongly projectable if each polar of G is a direct factor of G; this is equivalent with the following condition:

(β) If $0 < g \in G$, $A \subset G$, then $\sup\{b \in A^\delta : 0 \leqq b \leqq g\}$

exists in G.

G is called projectable, if for each element $a \in G$, the polar $\{a\}^\delta$ is a direct factor of G.

A subset $\emptyset \neq B \subset G$ is called disjoint (or orthogonal), if $0 < b$ for each $b \in B$, and $b_1 \wedge b_2 = 0$ for each pair of distinct elements $b_1, b_2 \in B$. The l-group G is called (conditionally) orthogonally complete if each (bounded) disjoint subset of G possesses its supremum in G. The l-group G is said to be complete (σ-complete) if each bounded subset (respectively, each bounded denumerable subset) of G has a supremum in G.

Let $0 \leqq s \in G$ and suppose that $x \wedge (s - x) = 0$ for each $x \in G$ with $0 \leqq x \leqq s$. Then s is said to be a singular element of G. It is easy to verify that $0 < s \in G$ is singular if and only if the interval $[0, s]$ is a Boolean algebra. The l-group G is called singular if for each $0 < g \in G$ there exists a singular element $s \in G$ with $0 < s \leqq g$. For any l-group G we denote by $S = S(G)$ the set of all singular elements of G. Then $S^{\delta\delta}$ is the greatest convex singular l-subgroup of G.

Let G be an archimedean lattice ordered group. We denote by $G^\wedge$ the Dedekind completion of G. Thus $G^\wedge$ is a complete l-group, G is an l-subgroup of $G^\wedge$ and each element of $G^\wedge$ is a supremum of a subset of G. For each $g \in G^\wedge$ there exists $h \in G$ with $g \leqq h$.

Let $A \subset G^\wedge$. We denote

$$A^\beta = \{g \in G^\wedge : |g| \wedge |a| = 0 \text{ for each } a \in A\}\,, \quad A^{\beta\beta} = (A^\beta)^\beta\,.$$

If $A = \{a\}$ is a singleton, we put $A^{\beta\beta} = [a]^0$. Because $G^\wedge$ is complete, it is strongly projectable and hence $A^{\beta\beta}$ is a direct factor of $G^\wedge$ for each $A \subset G^\wedge$. Thus $G^\wedge = A^\beta \times A^{\beta\beta}$ for each $A \subset G^\wedge$. The component of an element $f \in G^\wedge$ in A^β will be denoted by $f(A^\beta)$. If $A^\beta = [a]^0$, we denote $f(A^\beta) = f[a]^0$.

The set $A^{\beta\beta}$ is, in fact, the polar generated by the set A with respect to the lattice ordered group $G^\wedge$. If $0 \leqq s_i \in G^\wedge$ $(i = 1, 2, 3)$, $s_1 \wedge s_2 = 0$, $s_1 \vee s_2 = s_3$, then $[s_1]^0 \cap [s_2]^0 = \{0\}$ and $[s_3]^0$ is the least polar of $G^\wedge$ containing both $[s_1]^0$ and $[s_2]^0$ as subsets; since each $[s_i]^0$ is a direct factor of $G^\wedge$, we obtain

$$[s_3]^0 = [s_1]^0 \times [s_2]^0\,.$$

2. SINGULAR STRONGLY PROJECTABLE l-GROUPS

In this paragraph the proof of Thm. 1 will be given. Its idea is similar to that used in [9] (in [9] it was assumed that the lattice ordered group G is σ-complete and conditionally orthogonally complete).

Suppose that G is a σ-complete, singular and strongly projectable lattice ordered group. Further suppose that G fulfils the condition (α).

Let $0 < g \in G^\wedge$, $h \in G$, $h \geqq g$. There is $h_1 \in G$ such that $0 < h_1 \leqq g$. Thus there exists $0 < s \in S$ with $s \leqq g$. Denote

$$S_0 = \{s \in S : s \leqq g\}\,.$$

We have $S_0 \subset [0, h]$ and hence according to (α) there exists $s_0 = \sup S_0$ in G. The set S is a closed sublattice of G, thus $s_0 \in S$ and so $s_0 \in S_0$. We have already verified that $S_0 \neq \{0\}$. Since $G^\wedge$ is complete, it is archimedean. Therefore there exists a positive integer n_1 such that

(i) $(n_1 - 1) s_0 \leqq g$,

(ii) $n_1 s_0$ non $\leqq g$.

Let S'_1 be the set of all $s \in S_0$ such that $n_1 s \leqq g$. Then $S'_1 \neq \emptyset$ because $0 \in S'_1$, and S'_1 is bounded in G. Hence there exists $s'_1 = \sup S'_1$ in G. We have

$$n_1 s'_1 = n_1 \bigvee_{s \in S_1'} s = \bigvee_{s \in S_1'} n_1 s \leqq g \, ,$$

hence s'_1 is the greatest element of the set S'_1. Put $s_1 = s_0 - s'_1$. Then, since s_0 is singular, $s_1 \wedge s'_1 = 0$. Therefore $s_1 \vee s'_1 = s_1 + s'_1 = s_0$. Thus s'_1 is a relative complement of the element s_1 in the interval $[0, s_0]$. Hence if $0 < s \leqq s_1$, then $n_1 s$ non $\leqq g$.

Denote $g_1 = g[s_1]^0$. We have

$$(n_1 - 1) s_1 = (n_1 - 1) s_1[s_1]^0 \leqq g[s_1]^0 = g_1 \, .$$

Assume that $(n_1 - 1) s_1 < g_1$. Then there is $0 < s \in S$ with $s \leqq g_1 - (n_1 - 1) s_1$. Thus $s \in [s_1]^0$ and clearly $s \leqq s_0$. Therefore

$$s = s \wedge s_0 = s \wedge (s_1 \vee s'_1) = (s \wedge s_1) \vee (s \wedge s'_1) \, .$$

Because $[s_1]^0$ is a direct factor of $G^\wedge$, we have

$$G^\wedge = [s_1]^0 \times ([s_1]^0)^\beta \, . \tag{1}$$

Each direct factor of $G^\wedge$ is a convex l-subgroup of $G^\wedge$. From $s_1 \in [s_1]^0$ we infer that $s \wedge s_1 \in [s_1]^0$ and hence $(s \wedge s_1) [s_1]^0 = s \wedge s_1$. Moreover, since $s_1 \wedge s'_1 = 0$, we have $s'_1 \in ([s_1]^0)^\beta$ and thus $s \wedge s'_1 \in ([s_1]^0)^\beta$.

Therefore $(s \wedge s'_1) [s_1]^0 = 0$. Thus according to (1),

$$s = s[s_1]^0 = (s \wedge s_1) [s_1]^0 = s \wedge s_1 \, .$$

Hence $0 < s \leqq s_1$. This implies $n_1 s$ non $\leqq g$. If $n_1 s \leqq g_1$, then $n_1 s \leqq g$, which is a contradiction. Thus $n_1 s$ non $\leqq g_1$. On the other hand, we have

$$\begin{aligned} n_1 s &= (n_1 - 1) s + s \leqq (n_1 - 1) s_1 + s \leqq \\ &\leqq (n_1 - 1) s_1 + (g_1 - (n_1 - 1) s_1) = g_1 \, , \end{aligned}$$

which is a contradiction. Hence

$$g_1 = (n_1 - 1) s_1 \, .$$

Thus $g_1 \in G$. If $g = g_1$, then $g \in G$. Suppose that $g \neq g_1$. Then $g'_1 = g - g_1 > 0$. We have $g'_1, s'_1 \in ([s_1]^0)^\beta$, $g_1 \in [s_1]^0$, hence

$$s'_1 \wedge g_1 = 0\,, \quad g_1 \wedge g'_1 = 0\,.$$

Moreover, since $s'_1 \leqq s_0 \leqq g = g_1 \vee g'_1$, we get

$$s'_1 \wedge g'_1 = (s'_1 \wedge g_1) \vee (s'_1 \wedge g'_1) = s'_1 \wedge (g_1 \vee g'_1) = s'_1\,,$$

thus $s'_1 \leqq g'_1$. If $s \in S$, $s \leqq g'_1$, then $s \leqq s_0$ and $s \in ([s_1]^0)^\beta$, hence $s \wedge s_1 = 0$, therefore

$$s \wedge s'_1 = (s \wedge s'_1) \vee (s \wedge s_1) = s \wedge (s'_1 \vee s_1) = s \wedge s_0 = s\,,$$

hence $s \leqq s'_1$. Thus s'_1 is the greatest element of the set

$$S_1 = \{s \in S : s \leqq g'_1\}\,.$$

Now by the same method as we constructed $n_1, s_1, s'_1, g_1, g'_1$ corresponding to the pair (g, s_0), we can construct $n_2, s_2, s'_2, g_2, g'_2$ corresponding to the pair (g'_1, s'_1).

From $s_1 \wedge s'_1 = 0$, $s_1 \vee s'_1 = s_0$ it follows

$$[s_0]^0 = [s_1]^0 \times [s'_1]^0\,,$$

hence

(2) $$g[s_0]^0 = g[s_1]^0 + g[s'_1]^0\,.$$

The component of the element g in the direct factor $([s_0]^0)^\beta$ of $G^\wedge$ is the greatest element of the set

$$P = \{x \in ([s_0]^0)^\beta : 0 \leqq x \leqq g\}\,.$$

Assume that $0 < x \in P$. Then $x \wedge s_0 = 0$. There exists $0 < h_2 \in G$ with $h_2 \leqq g$ and hence there exists $0 < s \in S$ with $s \leqq h_2$. Thus $0 < s \leqq s_0 \wedge x$, which is a contradiction. Consequently, the component of g in $([s_0]^0)^\beta$ is 0 and hence $g \in [s_0]^0$. Thus $g = g[s_0]^0$. Hence from (2) we obtain

$$g = g_1 + g[s'_1]^0\,.$$

Therefore $g'_1 = g[s'_1]^0$.

From $n_1 s'_1 \leqq g$ we obtain

$$n_1 s'_1 = n_1 s'_1 [s'_1]^0 \leqq g[s'_1]^0 = g'_1\,.$$

Hence necessarily $n_1 < n_2$.

Analogously as above we have

$$g_2 = g'_1[s_2]^0 = (n_2 - 1)\, s_2 \in G\,,$$
$$g'_2 = g'_1 - g_2\,, \quad n_2 s'_2 \leqq g\,,$$

and s'_2 is a relative complement of s_2 in the interval $[0, s'_1]$. Moreover, if $0 < s \leqq s_0$, $n_2 s \leqq g$, then $s \leqq s'_2$.

By a straightforward induction we can verify that either

(a) there exist positive integers k, $n_1, \ldots, n_k$ and elements $s_1, \ldots, s_k \in S$ such that $g = (n_1 - 1) s_1 + \ldots + (n_k - 1) s_k$ (thus $g \in G$); or

(b) there exists a strictly increasing sequence of positive integers $\{n_k\}$ $(k = 1, 2, \ldots)$ and there exist elements $s_k, s'_k \in S$ $(k = 1, 2, \ldots)$ such that

(b_1) the system $\{s_k\}$ $(k = 1, 2, \ldots)$ is disjoint,

(b_2) $g_k = g[s_k]^0 = (n_k - 1) s_k$,

(b_3) $s_k \wedge s'_k = 0$, $s_k \vee s'_k = s'_{k-1}$ for $k > 1$,

(b_4) $n_k s'_k \leqq g$,

(b_5) if $s \in S_0$, $n_k s \leqq g$, then $s \leqq s'_k$.

Suppose that (b) is valid. Assume that there exists $0 < s \leqq s_0$ such that $s \wedge s_k = 0$ for $k = 1, 2, \ldots$ Then $s \leqq s'_k$ and hence by (b_4), $n_k s \leqq g$ for $k = 1, 2, \ldots$ As $G^\wedge$ is archimedean, this is a contradiction. Thus for each $0 < s \leqq s_0$ there exists a positive integer k with $s \wedge s_k > 0$.

We have $g_k = g[s_k]^0 \leqq g \leqq h$ for every positive integer k. By (b_2), $g_k \in G$. Since G is σ-complete, the element

$$g' = \bigvee g_k$$

exists in G and $g' \leqq g$.

Assume that $g' < g$. Then there is $0 < s \in S$ with $s \leqq g - g'$. Clearly $s \leqq s_0$, hence $0 < s^* = s \wedge s_k$ for a positive integer k. Thus

$$n_k s^* = (n_k - 1) s^* + s^* \leqq (n_k - 1) s_k + (g - g') \leqq g' + (g - g') = g .$$

Hence according to (b_5), $s^* \leqq s'_k$. Therefore $0 \leqq s^* \leqq s_k \wedge s'_k = 0$, which is a contradiction. This implies that $g' = g$. Hence $g \in G$.

Therefore $G^\wedge = G$ and so G is complete. Thus we have proved Thm. 1 that was formulated above.

Let us remark that for a singular lattice ordered group G the condition (α) is equivalent with the following condition:

(α') each bounded disjoint set of singular elements of G has a supremum in G.

Clearly (α) $\Rightarrow$ (α'). Assume that (α') is valid and let $\{s_i\}$ $(i \in I)$ be a set of singular elements of G, $g \in G$, $s_i \leqq g$ for each $i \in I$. The Axiom of Choice implies that in the interval $[0, g]$ there exists a set $\{f_j\}$ $(j \in J)$ of singular elements of G such that (i) $f_j \wedge f_k = 0$ whenever j and k are distinct elements of J, and (ii) for each singular element $0 < s \in [0, g]$ there is $j \in J$ with $s \wedge f_j > 0$. Then according to (α') the supremum $\bigvee_{j \in J} f_j = f_0$ exists in G, and because $S(G)$ is a closed sublattice of G, f_0 is singular. If $f_0 = 0$, then $s_i = 0$ for each $i \in I$, hence $\bigwedge s_i = 0$. Let $0 < f_0$. Then $[0, f_0]$ is a Boolean algebra and by (α'), each bounded disjoint subset of $[0, f_0]$ has a supremum in $[0, f_0]$. Hence the Boolean algebra $[0, f_0]$ is complete (cf. SIKORSKI [12], 20.1).

Let $i \in I$, $0 < s_i$. There exists $k \in J$ such that $f_k \wedge s_i > 0$, hence $f_0 \wedge s_i = e_i > 0$. Suppose that $e_i < s_i$ and denote $f' = s_i - e_i$. Then f' is singular and $0 < f' \leqq g$. Moreover, $f_0 \wedge f' = 0$, hence $f_j \wedge f' = 0$ for each $j \in J$. In view of (ii), this is a contradiction. Hence $s_i \in [0, f_0]$ for ach $i \in I$. Since $[0, f_0]$ is complete, there exists the supremum s_0 of the set $\{s_i\}_{i \in I}$ in $[0, f_0]$. Clearly $s_0 = \sup \{s_i\}_{i \in I}$ in G. Therefore (α) is valid.

3. EXAMPLES

A complete lattice ordered group is a vector lattice if and only if $S(G) = \{0\}$ (cf. CONRAD-MCALLISTER [5]). The following example shows that a σ-complete strongly projectable lattice ordered group G fulfilling the condition $S(G) = \{0\}$ need not be complete.

Example 1. Let G_0 be the set of all real functions defined on the interval $[0, 1]$. For $f, g \in G$ we put $f \leqq g$ if $f(x) \leqq g(x)$ is valid for each $x \in [0, 1]$. Then G_0 is an additive lattice ordered group. Let N be the set of all integers. Let G be the set of all elements $g \in G_0$ such that

$$\operatorname{card} \{x \in [0, 1] : g(x) \notin N\} \leqq \aleph_0 .$$

Then G is an l-subgroup of G_0. It is easy to verify that G_0 is complete and G is σ-complete. For each $0 < s \in G$ there exists $0 < x < s$ with $2x < s$, hence $0 < x \wedge (s - x)$. Thus $S(G) = \{0\}$.

Let $A \subset G$. Put

$$s(A) = \{x \in [0, 1] : f(x) \neq 0 \text{ for some } f \in A\} .$$

Then A^δ is the set of all $h \in G$ such that $h(x) = 0$ for each $x \in s(A)$. Let $0 < g \in G$. There is $g_1 \in G_0$ such that $g_1(x) = 0$ for each $x \in s(A)$ and $g_1(x) = g(x)$ for each $x \in [0, 1] \setminus s(A)$. Clearly $g_1 \in G$ and

$$g_1 = \sup \{f \in A^\delta : 0 \leqq f \leqq g\} .$$

Hence the l-group G is strongly projectable. Let r be a real, $r \notin N$. For each $x \in [0, 1]$ define $f_x \in G$ by $f_x(x) = r$, $f_x(j) = 0$ for $j \in [0, 1]$, $j \neq x$. The set $\{f_x : x \in [0, 1]\}$ is disjoint and bounded, $\sup \{f_x : x \in [0, 1]\}$ does not exist in G. Thus G fails to be conditionally orthogonally complete. Therefore G is not complete.

Let G be a singular lattice ordered group that is strongly projectable and σ-complete. Then G need not be complete.

Example 2. Let G_0 be as in Example 1. Let G be the set of all $g \in G_0$ such that (i) $g(x) \in N$ for each $x \in [0, 1]$, and (ii) the set

$$\sigma(g) = \{x \in [0, 1] : g(x) \text{ is odd}\}$$

has a cardinality less or equal to $\aleph_0$.

Then G is singular ($S(G)$ consists of all elements $g \in G$ such that, for each $x \in [0, 1]$, either $g(x) = 0$ or $g(x) = 1$); moreover, G is strongly projectable and σ-complete. But G is not conditionally orthogonally complete and hence G is not complete.

Let G be a singular l-group that is strongly projectable and fulfils the condition (α). Then G need not be complete.

Example 3. Let G_0 be the set of all real functions defined on the set N such that $f(x) \in N$ for each $x \in N$. Let $f^1 \in G_0$ with $f^1(x) = x$ for each $x \in N$. We denote by F_0 the set of all $f \in G_0$ such that $f(x) \in \{0, 1\}$ for each $x \in N$. Let G be the l-subgroup of G_0 generated by the set $\{f^1\} \cup F_0$. Then G is singular, strongly projectable and fulfils (α). Let N_1 be the set of all positive even integers. For each $n \in N_1$ let $f_n \in G_0$ such that $f_n(n) = \frac{1}{2}n$ and $f_n(x) = 0$ otherwise. Then $f_n \in G$, the set $\{f_n\}_{n \in N_1}$ is disjoint and bounded in G and it has no least upper bound in G. Hence G is not conditionally orthogonally complete; thus G fails to be complete.

Let $C(L)$, $C(P)$ and $C(SP)$ be the class of all lattice ordered groups that are respectively orthogonally complete, projectable and strongly projectable. Further, let $C(0) = C(L) \cap C(P)$. Let G be a lattice ordered group that is an l-subgroup of a cardinal product of linearly ordered groups. Let $X \in \{L, SP, 0\}$, $G_1 \in C(X)$ and let G^X be the intersection of all $H \in C(X)$, $H \subset G_1$ such that G is an l-subgroup of H and $S \cap G \neq \{0\}$ for each convex l-subgroup S of H. Then G^X belongs to $C(X)$ and it is called an X-hull of G (cf. Conrad [4]).

Conrad [4] proposed the problem whether the assertion

(i) $(G^L)^{SP} \subset G^0$

is valid for each archimedean lattice ordered group G. It was remarked above that the validity of the implication

(ii) G is conditionally orthogonally complete $\Rightarrow G$ is strongly projectable

for each archimedean lattice ordered group G is an open question.

If (ii) holds for each archimedean lattice ordered group, then also (i) is true for each archimedean lattice ordered group. In fact, assume that (ii) is valid for each archimedean lattice ordered group and let G be archimedean. Then (cf. [4], Thm. 2.6) $H = G^L$ is archimedean and clearly H is conditionally orthogonally complete. By (ii), H is strongly projectable and thus $H^{SP} = H$. Hence $H = (G^L)^{SP} \in C(L) \cap C(P)$; therefore $G^0 \subset H = G^L$. Clearly $G^L \subset G^0$ and thus $(G^L)^{SP} = G^L = G^0$. Hence (i) is valid.

4. r_0-COMPLETE LATTICE ORDERED GROUPS

Lemma 1. *Let G be a lattice ordered group that is archimedean, projectable and conditionally orthogonally complete. Let $0 < g_1 \in G^\wedge$, $0 < e_1 \in G$, $e_1 \leqq g_1$ and suppose that e_1 is a weak unit of the l-group $[g_1]^0$. Then there is $0 \leqq u_1 \in G$ such that $u_1 \leqq g_1 \leqq u_1 + e_1$.*

Proof. The assertions and the proofs of Lemmas 7–11 and those of Theorem 2 in [9] remain valid if the assumption of the σ-completeness of G is replaced by the weaker assumption that G is projectable; hence the assertion of Lemma 1 holds.

Lemma 2. ([9], Lemma 7.) *Let G be a conditionally orthogonally complete archimedean lattice ordered group, $0 < g_1 \in G^\wedge$. Then there is $e_1 \in G$ such that e_1 is a weak unit of the l-group $[g_1]^0$ and $e_1 \leqq g_1$.*

Lemma 3. ([9], Lemma 12.) *Let G be a conditionally orthogonally complete archimedean lattice ordered group and let $S(G) = \{0\}$, $0 < e_1 \in G$. Then there is $e_2 \in G$ such that e_2 is a weak unit of $[e_1]^0$ and $2e_2 \leqq e_1$.*

Let us remark that if e_1 is a weak unit of $[g_1]^0$, then clearly $[g_1]^0 = [e_1]^0$.

Theorem 2. *Let G be a lattice ordered group that is projectable, conditionally orthogonally complete and archimedean. Let $S(G) = \{0\}$, $0 < g_1 \in G$. Then there are elements $z_n, e_n \in G$ $(n = 1, 2, \ldots)$ such that for any positive integer n,*

(i) $0 \leqq z_n \leqq g_1 \leqq z_n + e_n$,

(ii) $0 \leqq 2e_{n+1} \leqq e_n$, $z_n \leqq z_{n+1}$, z_n, $e_n \in [g_1]^0$,

(iii) $\bigvee_{i=1}^{\infty} z_i = g_1$ *holds in* $G^\wedge$.

Proof. According to Lemma 2 there is $0 < e_1 \in G$ such that e_1 is a weak unit of the l-group $[g_1]^0$. Let u_1 have the same meaning as in Lemma 1. Put $z_1 = u_1$. We proceed by induction on n. Assume that we have defined, for a positive integer n, elements $u_1, \ldots, u_n, e_1, \ldots, e_n \in G \cap [g_1]^0$ such that

$$0 \leqq u_k\,, \quad 0 \leqq e_k\,, \quad u_1 + \ldots + u_n \leqq g_1 \leqq u_1 + \ldots + u_n + e_n$$

for $k = 1, \ldots, n$, and if $n > 1$, then

$$2e_k \leqq e_{k-1}$$

for $k = 2, \ldots, n$. Denote $z_n = u_1 + \ldots u_n$. Then $z_n \in G \cap [g_1]^0$. In the case $g_1 = z_n$ we put $u_k = e_k = 0$ for each $k > n$. Let $z_n < g_1$. By Lemma 2, there is $0 < e' \in G$ such that e' is a weak unit in $[g_1 - z_n]^0$ and $e' \leqq g_1 - z_n$. Clearly

$$[g_1 - z_n]^0 \subset [e_n]^0\,,$$

hence $e' \wedge e_n$ is a weak unit of $[g_1 - z_n]^0$. According to Lemma 3 there is $0 < e'_{n+1} \in$ $\in G$ with $2e'_{n+1} \leqq e_n$ such that e'_{n+1} is a weak unit in $[e_n]^0$. Denote

$$e_{n+1} = e'_{n+1} \wedge (g_1 - z_n)\,.$$

Then e_{n+1} is a weak unit of $[g_1 - z_n]^0$. Moreover, $e_{n+1} \in G \cap [g_1]^0$ and $2en_{+1} \leqq e_n$.

According to Lemma 1 there is $0 \leqq u_{n+1} \in G$ such that

$$u_{n+1} \leqq g_1 - z_n < u_{n+1} + e_{n+1} .$$

Denote $z_{n+1} = z_n + u_{n+1}$. Then $z_{n+1} \in G \cap [g_1]^0$ and

$$0 \leqq z_{n+1} \leqq g_1 \leqq z_{n+1} + e_{n+1} .$$

The set $\{z_n\}$ is bounded in $G^\wedge$, hence $\bar{z} = \bigvee z_n$ exists in $G^\wedge$ and $\bar{z} \leqq g_1$. Suppose that $0 < y = g_1 - \bar{z}$. According to (i) we have $y \leqq e_n$ and hence by (ii), $2^n y \leqq e_1$ is valid for every positive integer n. This is a contradiction, since $G^\wedge$ is archimedean. Thus $\bigvee z_n = g_1$ holds in $G^\wedge$.

Theorem 2'. *Let G be a lattice ordered group that is projectable, conditionally orthogonally complete and archimedean. Let $0 < g_1 \in G^\wedge$. Then there are elements $z_n, e_n \in G$ $(n = 1, 2, \ldots)$ such that the conditions* (i), (ii) *and* (iii) *of Thm.* 2 *are fulfilled.*

Proof. Put $A = (S(G))^{\delta\delta}$, $B = (S(G))^{\delta}$. According to Thm. 1 of [9] we have

$$G = A \times B$$

and both A and B are projectable, conditionally orthogonally complete and archimedean. Moreover, A is singular and hence by [8], A is complete. Clearly $S(B) = \{0\}$. Denote $g_2 = g_1(A)$, $g_3 = g_1(B)$. In the case $g_3 = 0$ we put $z_n = g_2$, $e_n = 0$ for every positive integer n. Let $g_3 > 0$. According to Thm. 2 there are elements z'_n, e'_n such that (i), (ii) and (iii) are valid if we replace g_1, z_n, e_n, G by g_3, z'_n, e'_n, B. Put

$$z_n = g_2 + z'_n , \quad e'_n = e_n .$$

Then the conditions (i), (ii) and (iii) are valid.

Theorem 3. *Let H be an l-group that is projectable, conditionally orthogonally complete and archimedean. Assume that each fundamental r_0-sequence $\{h_i\}$ of H with $0 \leqq h_i \leqq h_{i+1}$ $(i = 1, 2, \ldots)$ is o-convergent in H. Then H is complete.*

Proof. Put $G = (S(H))^{\delta}$ (the symbol δ is considered with respect to the l-group H). According to Theorem 1 of [9], G is a direct factor of H, hence G is archimedean, projectable and conditionally orthogonally complete. Clearly $S(G) = \{0\}$. If $\{h_i\}$ is a fundamental r_0-sequence in H, $\{h_i\} \subset G$, then $\{h_i\}$ is a fundamental r_0-sequence in G.

Let g_1, z_i, e_i have the same meaning as in Thm. 2. Let i, k, m be positive integers, $i < k \leqq m$. Then

$$z_i \leqq z_k \leqq z_m \leqq g_1 \leqq z_i + e_i .$$

Hence

$$|z_k - z_m| \leqq e_i , \quad 2^{i-1} e_i \leqq e_1$$

holds. This implies that $\{z_i\}$ $(i = 2, 3, 4, \ldots)$ is a fundamental r_0-sequence in G, thus it is a fundamenal r_0-sequence in H and hence, according to the assumption, the sequence $\{z_i\}$ o-converges to an element z of H. Obviously $z \in G$. Since $z_i \leqq z_{i+1}$ for $i = 1, 2, \ldots$, we have $z = \bigvee z_i$. Now by Thm. 2 we obtain that $z = g_1$. Thus G is complete.

We have $H = (S(H))^{\delta\delta} \times G$. The l-group $A = (S(H))^{\delta\delta}$ is singular, archimedean and conditionally orthogonally complete. Therefore by [8], A is complete. Hence the l-group H is complete as well.

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Added in proof. Bernau (J. London Math. Soc. (2) *12* (1976), 320—322) proved that each archimedean orthogonally complete lattice ordered group is projectable, and Rotkovič (Czech. Math. J., to appear) shoved that each archimedean conditionally orthogonally complete lattice ordered group is projectable. Hence in Thms. 2, 2′ and 3 the assumption of the projectability can be omitted.