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Czechoslovak Mathematical Journal, Vol. 26 (1976), No. 3, 477–505

Persistent URL: http://dml.cz/dmlcz/101421

Czechoslovak Mathematical Journal, 26 (101) 1976, Praha

ON EXTENSIONS OF PARTIAL x-OPERATORS

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(Received December 17, 1974)

Dedicated to the Memory of Professor WOLFGANG KRULL

In this paper we introduce the notion of a *partial x-operator of a semigroup*, which is a generalization of notions from the ideal theory due to KRULL, PRÜFER, LORENZEN and lately to AUBERT (s. Section 5). The main result is *Theorem on x-extension* (3.3.4) concerning the existence of an extension of a partial x-operator to an x-operator (e.g. *x-extension*) and describing the finest and the coarsest ones. Here, when describing the finest x-extension, it is necessary to use the transfinite induction (3.10.6).

In Section 4 we introduce some applications of Theorem on x-extension. Especially necessary and sufficient conditions are given when an x-operator of a semigroup may be extended to an x-operator of its total quotient semigroup and the finest and the coarsest x-extensions are described (4.9).

1. FUNDAMENTAL CONCEPTS

1.1. Algebraic concepts. By a *semigroup* we understand a non-empty set with a binary commutative and associative operation.

Let $G = (G, \cdot)$ be a semigroup. For $A \subseteq G$, $B \subseteq G$, $b \in G$ we use the usual notation:

$$A . B = \{g_1 . g_2 : g_1 \in A,\ g_2 \in B\}\ , \quad A . b = b . A = A . \{b\}\ ,$$

$$A : B = \{g \in G : g . B \subseteq A\}\ , \quad A : b = A : \{b\}\ .$$

If the semigroup G contains an identity element, we shall denote it by 1_G.

An element $0 \in G$ is said to be *zero of the semigroup* G if it holds:

$$g \in G \Rightarrow g . 0 = 0\ .$$

The semigroup G with zero 0 will be called a *(trivial) group with zero* if $(G - \{0\}, \cdot)$ is a (trivial) group.

The element $g \in G$ is called *regular* if it holds:

$$a \in G\,, \quad b \in G\,, \quad a \,.\, g = b \,.\, g \Rightarrow a = b\,.$$

The semigroup $G^* = (G^*, \cdot)$ of all fractions (a/b) $(a \in G,\ b \in G,\ b$ is regular) with the usual multiplication and equality is called the *total quotient semigroup of* (*the semigroup*) G; in case G contains no regular element, we shall consider (by convention) G to be its own total quotient semigroup ($G = G^*$).

In case all elements of G are regular, the semigroup G^* is a group – the *quotient group of* (*the semigroup*) G.

We shall call a subset A of G^* *fractionary* (or *bounded*) if there exists a regular element $g \in G^*$ such that $g \,.\, A \subseteq G$. The element g is called a *multiplier for* A. In case G contains no regular element, then we consider each subset of $G^* = G$ to be fractionary.

1.2. Topological concepts. Let P be a set. The system of all subsets of the set P will be denoted by 2^P.

A mapping z of the system 2^P into $2^P(A \to A_z)$ will be called a *general closure operator of* (*the set*) P if it holds:

1° $A \subseteq P \Rightarrow A \subseteq A_z$,
2° $A \subseteq B \subseteq P \Rightarrow A_z \subseteq B_z$.

If it holds moreover:

3° $A \subseteq P \Rightarrow A_z = A_{zz}$,
z will be called a *closure operator of* (*the set*) P.

For general closure operators z_1, z_2 of P we put as usual $z_1 \leqq z_2$ if $A_{z_1} \subseteq A_{z_2}$ for each $A \subseteq P$ and we say that z_1 is *finer* than z_2 or that z_2 is *coarser* than z_1. The relation $\leqq$ is an ordering of the set of all general closure operators of P. The least (largest) element of this ordered set is the closure operator $u(v)$ of P defined by:

$$A \subseteq P \Rightarrow A_u = A\,, \quad A_v = P\,.$$

The closure operator $u(v)$ will be called the *finest* (*coarsest*) *closure operator of* (*the set*) P.

Let z be a general closure operator of P. The finest closure operator of P from the set of closure operators of P coarser than z will be called *the modification of* z.

We define to each ordinal $\xi > 0$ a general closure operator z_ξ of P by transfinite induction: for $M \subseteq P$ we put $M_{z_1} = M_z$ and for an ordinal $\xi = \eta + 1 > 1$ we put $M_{z_\xi} = (M_{z_\eta})_z$ while for a limit ordinal ξ we put $M_{z_\xi} = \bigcup M_{z_\eta}$ $(0 < \eta < \xi)$.

Evidently,

1.2.1. *There is an ordinal* $\zeta > 0$ *such that* z_ζ *is the modification of* z.

If $\emptyset_z = \emptyset$, then the pair (P, z) is a *topological space* in the sense of ČECH's paper [2] from the year 1937. The idea of the construction of z_ξ is due to HAUSDORFF ([2], 6.5). The following notion of *neighborhood* as well as the statement 1.2.2 are taken over from [2] (2.1 and 2.1.4).

A set $U \subseteq P$ is said to be a *z-neighborhood* of p $(p \in P)$ if $p \notin (P - U)_z$. The following assertion is evident.

1.2.2. If $p \in P$, $M \subseteq P$, *then* $p \in M_z$ *if and only if* $U \cap M \neq \emptyset$ *for every z-neighborhood* U *of* p.

1.3. Convention. *In the whole paper* $S = (S, \cdot)$ *will denote a semigroup.*

If I is a set and for each $\iota \in I$ it holds $A_\iota \subseteq S$, then for $I = \emptyset$ we put:

$$\bigcap A_\iota(\iota \in I) = S\,, \quad \bigcup A_\iota(\iota \in I) = \emptyset\,.$$

2. PARTIAL x-OPERATOR

2.1. Definition. Let $\mathcal{Y} \subseteq 2^S$. A mapping y of the set $\mathcal{Y}$ into the set 2^S $(A \to A_y)$ is said to be a *partial x-operator of (the semigroup)* S if it holds:

1° $A \in \mathcal{Y} \Rightarrow A \subseteq A_y$,

2° $A \in \mathcal{Y},\ B \in \mathcal{Y},\ B \subseteq A_y \Rightarrow B_y \subseteq A_y$,

3° $A \in \mathcal{Y},\ B \in \mathcal{Y},\ a \in S,\ a \,.\, B \subseteq A_y \Rightarrow a \,.\, B_y \subseteq A_y$.

We shall call the set $\mathcal{Y}$ the *domain of* y. If the domain of a partial x-opetator y of S is the set 2^S, then the mapping y is said to be an *x-operator of (the semigroup)* S. Then y *is* evidently *a closure operator of the set* S.

2.2. Remark. a) If an x-operator y of S fulfils also the condition $S \,.\, B_y \subseteq B_y$ for each $B \subseteq S$, we get the notion of an *x-operation* studied by AUBERT ([1]) (s. 5.5.1), which JOHNSON and LEDIAEV ([5]) call an *x-operator* (in case the semigroup S contains an identity element).

b) *If the semigroup* S *contains an identity element, then* 3° *implies* 2°, *evidently.*

2.3. Definition. Let x be a closure operator of S.

a) For $A \subseteq S$, $B \subseteq S$ we put $A \circ B = (A \,.\, B)_x$. Then $(2^S, \circ)$ *is a commutative groupoid.* We denote the system of all sets M_x $(M \subseteq S)$ by $\mathfrak{J}(S) = \mathfrak{J}(S, x)$. Then $(\mathfrak{J}(S), \circ) = (\mathfrak{J}(S, x), \circ)$ is a subgroupoid of the groupoid $(2^S, \circ)$.

b) We say that the operation $\cdot$ on the semigroup S is *weakly continuous* if for each $a \in S$, $b \in S$ and x-neighborhood V of $a \,.\, b$ there exists an x-neighborhood U of a such that $U \,.\, b \subseteq V$.

2.4. Theorem. *Let x be a closure operator of S. Then the following statements are equivalent*:

(a) *x is an x-operator of the semigroup S,*

(b) *the operation $\cdot$ on the semigroup S is weakly continuous,*

(c) $A \subseteq S$, $A_\iota \subseteq S$ *for each* $\iota \in I$ *implies* $A \,.\, [\bigcup A_\iota(\iota \in I)]_x \subseteq [\bigcup A \,.\, A_\iota(\iota \in I)]_x$,

(d) $A \subseteq S$, $A_\iota \subseteq S$ *for each* $\iota \in I$ *implies* $A \circ [\bigcup A_\iota(\iota \in I)]_x = [\bigcup A \circ A_\iota(\iota \in I)]_x$.

Proof. I. Let (a) hold, let $a \in S$, $b \in S$ and let V be an x-neighborhood of $a \,.\, b$. We put $C = \{s \in S : s \,.\, b \in (S - V)_x\}$. It holds $b \,.\, C \subseteq (S - V)_x$ and according to 2.1, 3° $b \,.\, C_x \subseteq (S - V)_x$, hence $a \notin C_x$. It follows that $U = S - C$ is an x-neighborhood of a and $U \,.\, b \subseteq S - (S - V)_x \subseteq V$. Therefore (a) implies (b).

II. Let (b) hold, let $A \subseteq S$, $A_\iota \subseteq S$ for each $\iota \in I$ and let $a \in A.\, [\bigcup A_\iota(\iota \in I)]_x$. Then there exist $b \in A$ and $c \in [\bigcup A_\iota(\iota \in I)]_x$ such that $a = b \,.\, c$. Let V be an x-neighborhood of a. Then there exists an x-neighborhood U of c such that $U \,.\, b \subseteq V$. According to 1.2.2 there exists $d \in U \cap [\bigcup A_\iota(\iota \in I)]$. Then $d \,.\, b \in V \cap [\bigcup A \,.\, A_\iota(\iota \in I)]$ and 1.2.2 implies that $a \in [\bigcup A \,.\, A_\iota(\iota \in I)]_x$. Consequently, (c) holds.

III. The equivalence (c) $\Leftrightarrow$ (d) and the implication (c) $\Rightarrow$ (a) can be proved easily.

Thus, Theorem 2.4 is proved.

2.5. Remark. For a closure operator x of S *the axiom* 3° *in* 2.1 *is equivalent to the property*:

(1) $a \in S$, $B \subseteq S \Rightarrow a \,.\, B_x \subseteq (a \,.\, B)_x$,

which is equivalent to the axiom:

(2) $A \subseteq S$, $B \subseteq S \Rightarrow A \,.\, B_x \subseteq (A \,.\, B)_x$.

Aubert ([1]) calls this axiom the *continuity axiom* and gives some of its equivalent forms which we shall use the following ones ([1], Theorems 1 and 3):

(3) $A \subseteq S$, $B \subseteq S \Rightarrow A \circ B = A_x \circ B_x$,

(4) $A \subseteq S$, $B \subseteq S \Rightarrow (A_x : B)_x = A_x : B$.

If the set I in (c) and (d) of 2.4 is a two-element set, we get further equivalent formulas of this axiom given in [1] (Theorem 1).

From (3) of 2.5 or directly from (2) of 2.5 similarly as in the proof of Theorem 2 ([1]), it follows:

2.6. Proposition. *Let x be an x-operator of the semigroup S. Then the groupoids* $(2^S, \circ)$ *and* $(\mathfrak{I}(S), \circ)$ *are semigroups.*

2.7. Definition. Let $\mathscr{Y} \subseteq 2^S$ and let y be a mapping of $\mathscr{Y}$ into 2^S. Then we put:

$$E(y) = \{s \in S : A \in \mathscr{Y} \Rightarrow s \,.\, A_y \subseteq A_y\}\,.$$

Evidently, the following Propositions 2.8–2.10 hold:

2.8. Proposition. *Let $\mathscr{Y}_i \subseteq 2^S$ and let y_i be a mapping of $\mathscr{Y}_i$ into 2^S $(i = 1, 2)$. If for each $B \in \mathscr{Y}_2$ there exists $A \in \mathscr{Y}_1$ such that $A_{y_1} = B_{y_2}$, then $E(y_1) \subseteq E(y_2)$.*

In particular: if z is a general closure operator of S and x is a closure operator of S coarser than z, then $E(z) \subseteq E(x)$.

2.9. Proposition. *If x is a closure operator of S, then $E(x) = \{s \in S : t \in S \Rightarrow \Rightarrow s \,.\, t \in \{t\}_x\}$.*

2.10. Proposition. *Let $\mathscr{Y} \subseteq 2^S$ and let y be a mapping of $\mathscr{Y}$ into 2^S. Then it holds:*

$$a \in E(y)\,, \quad b \in E(y) \Rightarrow a \,.\, b \in E(y)\,.$$

In particular: if $E(y) \neq \emptyset$, then $E(y)$ is a subsemigroup of the semigroup S.

2.11. Proposition. *Let y be a partial x-operator of S with a domain $\mathscr{Y}$. Then it holds:*

$$A \in \mathscr{Y}\,, \quad A \subseteq E(y) \Rightarrow A_y \subseteq E(y)\,.$$

In particular: if $E(y) \in \mathscr{Y}$, then $[E(y)]_y = E(y)$ and for $A \subseteq E(y)$, $B \subseteq E(y)$, $A \,.\, B \in \mathscr{Y}$ it is $(A \,.\, B)_y \subseteq E(y)$.

Therefore, for an x-operator x of S it holds:

$$[E(x)]_x = E(x)\,; \quad A \subseteq E(x)\,, \quad B \subseteq E(x) \Rightarrow A \circ B \subseteq E(x)\,.$$

Proof. For $A \in \mathscr{Y}$, $A \subseteq E(y)$ and for $B \in \mathscr{Y}$ we have $A \,.\, B_y \subseteq B_y$ (by definition), hence $b \,.\, A \subseteq B_y$ for $b \in B_y$. It follows that $b \,.\, A_y \subseteq B_y$, therefore $A_y \subseteq E(y)$.

2.12. Proposition. *Let x be an x-operator of S. Then the following statements are equivalent:*

(a) *the semigroup $(\mathfrak{I}(S), \circ)$ contains an identity element,*

(b) $s \in S \Rightarrow s \in [s \,.\, E(x)]_x$.

If (a) *holds, then $1_{\mathfrak{I}(S)} = E(x)$.*

Proof. I. Let (a) hold and let $E \in \mathfrak{I}(S)$ be the identity element of $(\mathfrak{I}(S), \circ)$. Then $E(x) \,.\, E \subseteq E$, hence $E(x) = E(x) \circ E \subseteq E$. On the other hand, $E \,.\, A_x \subseteq E \circ A_x = A_x$ for each $A \subseteq S$, therefore $E \subseteq E(x)$ (by the definition of $E(x)$). Thus $1_{\mathfrak{I}(S)} = E = E(x)$. For $s \in S$ we get $s \in \{s\}_x = \{s\}_x \circ E = [s \,.\, E]_x$ by 2.5(3). Consequently, (b) holds.

II. If (b) holds, then for $I \in \mathfrak{I}(S)$ we have $I \supseteq [E(x) . I]_x \supseteq I$, whence $I = [E(x) . . I]_x = E(x) \circ I$. Q.E.D.

2.13. Proposition. *Let x be an x-operator of the semigroup S with identity. Then $\{1_S\}_x = E(x)$ and $\{1_S\}_x$ is the identity element of the semigroup $(\mathfrak{I}(S), \circ)$. If the element $a \in S$ has an inverse, then $a . A_x = (a . A)_x$ for $A \subseteq S$ and in particular: $\{a\}_x = a . E(x)$.*

Proof. According to 2.5(3) $A_x = (A . 1_S)_x = A_x \circ \{1_S\}_x$ for each $A \subseteq S$, hence $\{1_S\}_x$ is the identity element of $\mathfrak{I}(S)$ and 2.12 implies $\{1_S\}_x = E(x)$.

It holds $a . A_x \subseteq (a . A)_x$ for $a \in S$ with an inverse $a^{-1} \in S$ and $A \subseteq S$, hence $A_x \subseteq \subseteq a^{-1} . (a . A)_x \subseteq (a^{-1} . a . A)_x = A_x$. Therefore $A_x = a^{-1} . (a . A)_x$.

2.14. Proposition. *Let x be an x-operator of S and let the semigroup $(\mathfrak{I}(S), \circ)$ contain an identity element. If the element $I \in \mathfrak{I}(S)$ has an inverse $I^{-1} \in \mathfrak{I}(S)$, then $I^{-1} = E(x) : I$.*

Proof. From $I^{-1} . I \subseteq I^{-1} \circ I = E(x)$ it follows that $I^{-1} \subseteq E(x) : I$. Since $(E(x) : : I) . I \subseteq E(x)$, it holds $(E(x) : I) \circ I \subseteq E(x)$, whence $E(x) = I^{-1} \circ I \subseteq (E(x) : I) \circ I \subseteq \subseteq E(x)$, therefore $I^{-1} \circ I = (E(x) : I) \circ I$, hence $I^{-1} = (E(x) \circ I) \circ E(x) = E(x) : I$, since by 2.5(4) $E(x) : I \in \mathfrak{I}(S)$.

2.15. Proposition. *Let x be an x-operator of S. Then the following statements are equivalent:*

(a) *the semigroup $(2^S, \circ)$ contains an identity element,*

(b) *x is the finest closure operator of the set S and the semigroup S contains an identity element.*

If (a) *and* (b) *hold, then* $E(x) = \{1_S\} = 1_{2^S}$.

Proof. If (b) holds, then clearly $E(x) = \{1_S\}$ is the identity element of $(2^S, \circ)$.

Let $E \in 2^S$ be the identity element of $(2^S, \circ)$. For $A \subseteq S$ we have $A = A \circ E = = (A . E)_x$, whence $A_x = (A . E)_{xx} = (A . E)_x = A$.

Evidently, there exists $e \in E$. For $s \in S$ we get $e . s \in E \circ \{s\} = \{s\}$, hence $e . s = s$. Thus, the element e is the identity element of the semigroup S.

2.16. Proposition. *Let x be an x-operator of S. Then the following statements hold:*

(A) *$\emptyset_x$ is the zero of the semigroups $(2^S, \circ)$ and $(\mathfrak{I}(S), \circ)$.*

(B) *The following statements are equivalent:*

  (a) *$(\mathfrak{I}(S), \circ)$ is a trivial group,*

  (b) *$(\mathfrak{I}(S), \circ)$ is a group,*

  (c) *x is the coarsest closure operator of the set S.*

(C) *The semigroup $(2^S, \circ)$ is not a group.*

Proof. Clearly, $\emptyset_x$ is the zero of $(2^S, \circ)$ and therefore also the zero of $(\mathfrak{J}(S), \circ)$. Since $S \neq \emptyset$, the semigroup $(2^S, \circ)$ cannot be a group by 2.15. The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) in (B) are evident.

Let (b) in (B) hold. Let $A \in \mathfrak{J}(S)$ be the inverse of $\emptyset_x$ in the group $(\mathfrak{J}(S), \circ)$. Then $S = S \circ \emptyset_x \circ A = (S \,.\, \emptyset)_x \circ A = \emptyset_x \circ A \subseteq \emptyset_x \circ S = \emptyset_x$, whence $\emptyset_x = S$. Consequently (c) in (B) holds.

2.17. Proposition. *Let x be an x-operator of S. Then it holds*:

(A) *The following statements are equivalent*:

(a) $(\mathfrak{J}(S), \circ)$ *is a trivial group with zero,*

(b) $(\mathfrak{J}(S), \circ)$ *is a group with zero,*

(c) $S \,.\, S \nsubseteq \emptyset_x$; $A \subseteq S$, $A \nsubseteq \emptyset_x \Rightarrow A_x = S$.

(B) *The following statements are equivalent*:

(a) $(2^S, \circ)$ *is trivial group with zero,*

(b) $(2^S, \circ)$ *is a group with zero,*

(c) $S = \{1_S\}$ *and x is the finest closure operator of S.*

Proof. I. The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) are in both cases (A) and (B) evident.

II. Let $(\mathfrak{J}(S), \circ)$ be a group with zero. Then clearly $\emptyset_x \neq S$, hence $S \in \mathfrak{J}(S) - \{\emptyset_x\}$, whence we get $S \,.\, S \nsubseteq \emptyset_x$.

Let E be the identity element of $\mathfrak{J}(S)$ (according to 2.12 $E = \mathrm{E}(x)$). For $s \in S - \emptyset_x$ we put $B = \{s^n : n \text{ positive integer}\}$. Let $C \in \mathfrak{J}(S)$ be the inverse of B_x in $\mathfrak{J}(S)$. Then we have $s \in B_x = C \circ B_x \circ B_x = C \circ (B \,.\, B)_x \subseteq C \circ B_x = E$, hence $E = S$.

For $A \subseteq S$, $A \nsubseteq \emptyset_x$ let $D \in \mathfrak{J}(S)$ denote the inverse of A_x in $\mathfrak{J}(S)$. Then $A_x = A_x \circ S \supseteq$ $\supseteq A_x \circ D = S$, thus $A_x = S$.

The implication (b) $\Rightarrow$ (c) in (A) holds.

III. If $(2^S, \circ)$ is a group with zero, then according to 2.15 x is the finest closure operator of S, the semigroup S contains the identity element and $1_{2^S} = \{1_S\}$. It follows that $S \circ S = S$, hence $S = 1_{2^S} = \{1_S\}$. Q.E.D.

Necessary and sufficient conditions for a closure operator x of the set S with the property (c) in 2.17 (A) to be an x-operator of the semigroup S, can be derived from the following proposition, which is easy to verify.

2.18. Proposition. *Let $M \subseteq S$ and let a closure operator x of S be defined in the following way*: $A \subseteq M \Rightarrow A_x = M$; $A \nsubseteq M$, $A \subseteq S \Rightarrow A_x = S$. *Then the following statements are equivalent*:

(a) *x is an x-operator of the semigroup S,*

(b) $S \,.\, M \subseteq M$; $a \in S$, $b \in S - M$, $a \,.\, b \in M \Rightarrow a \,.\, S \subseteq M$.

3. x-EXTENSIONS OF A PARTIAL x-OPERATOR

3.1. Lemma. *Let z be a general closure operator of S with the property:*

$$a \in S, \quad A \subseteq S \Rightarrow a \,.\, A_z \subseteq (a \,.\, A)_z \,.$$

Then the modification of z is an x-operator of S.

Proof. Let η be an ordinal greater than 1 and let the following implication hold for each ordinal $1 \leqq \xi < \eta$:

$$b \in S, \; B \subseteq S \Rightarrow b \,.\, B_{z_\xi} \subseteq (b \,.\, B)_{z_\xi} \,.$$

Let $a \in S$, $A \subseteq S$. If η is a limit ordinal, then $a \,.\, A_{z_\eta} = a \,.\, \bigcup A_{z_\xi}$ $(1 \leqq \xi < \eta) = \bigcup a \,.\, A_{z_\xi}$ $(1 \leqq \xi < \eta) \subseteq \bigcup (a \,.\, A)_{z_\xi}$ $(1 \leqq \xi < \eta) = (a \,.\, A)_{z_\eta}$. If there exists an ordinal number α such that $\eta = \alpha + 1$, then $a \,.\, A_{z_\eta} = a \,.\, (A_{z_\alpha})_z \subseteq (a \,.\, A_{z_\alpha})_z \subseteq [(a \,.\, A)_{z_\alpha}]_z = (a \,.\, A)_{z_\eta}$.

Now Lemma follows from 1.2.1.

3.2. Definition. Let $\mathscr{Y} \subseteq 2^S$, let y be a mapping of $\mathscr{Y}$ into 2^S and let x be a mapping of 2^S into 2^S. Then we call x *an extension of y (in the set S)* if $B \in \mathscr{Y}$ implies $B_y = B_x$.

If an x-operator x of the semigroup S is an extension of y in S, then we call x an *x-extension of y (in the semigroup S)*.

3.3. Let $\mathscr{Y} \subseteq 2^S$ and let y be a mapping of $\mathscr{Y}$ into 2^S: For $A \subseteq S$ we put:

(1) $A_z = A \cup \bigcup B_y(B \in \mathscr{Y},\; B \subseteq A) \cup \bigcup s \,.\, B_y \; (s \in S,\; B \in \mathscr{Y},\; s \,.\, B \subseteq A)$,

(2) $A_v = \bigcap B_y(B \in \mathscr{Y},\; B_y \supseteq A) \cap \bigcap (B_y : s) \, (s \in S,\; B \in \mathscr{Y},\; B_y \supseteq A \,.\, s)$.

Clearly, the following assertion holds:

3.3.1. *z is a general closure operator of S, which satisfies:*

(a) $a \in S,\; A \subseteq S \Rightarrow a \,.\, A_z \subseteq (a \,.\, A)_z$,

(b) $B \in \mathscr{Y} \Rightarrow B_y \subseteq B_z$.

3.3.2. *v is an x-operator of S, for which it holds:* $B \in \mathscr{Y} \Rightarrow B_v \subseteq B_y$.

Proof. Evidently, v is a closure operator of S and $B_v \subseteq B_y$ for $B \in \mathscr{Y}$.

Let $a \in S$, $A \subseteq S$. If $B \in \mathscr{Y}$, $B_y \supseteq a \,.\, A$, then $B_y : a \supseteq A_v$, hence $B_y \supseteq a \,.\, A_v$. If $B \in \mathscr{Y}$, $s \in S$, $B_y \supseteq a \,.\, A \,.\, s$, then $B_y : s \,.\, a \supseteq A_v$, hence $B_y : s \supseteq a \,.\, A_v$. It follows that $a \,.\, A_v \subseteq (a \,.\, A)_v$.

We shall denote by u the modification of the general closure operator z of S.

From 3.3.1 and Lemma 3.1 we obtain the following assertion:

3.3.3. *u is an x-operator of S.*

3.3.4. Main Theorem (Theorem on x-extension). *The following statements are equivalent*:

(a) *y is a partial x-operator of the semigroup S,*

(b) $B \in \mathscr{Y} \Rightarrow B_y = B_z = B_{zz}$,

(c) *u is an extension of y in S,*

(d) *v is an extension of y in S,*

(e) *there exists an x-extension of y in the semigroup S.*

If (a)–(e) *hold, then $u(v)$ is the finest (coarsest) x-operator of the semigroup S, which is an extension of y.*

Proof. Clearly, (b) $\Rightarrow$ (c) $\Rightarrow$ (e) $\Rightarrow$ (a) and (d) $\Rightarrow$ (e). Let us suppose that (a) holds.

I. Let $B \in \mathscr{Y}$. According to 3.3.1 (b), $B_y \subseteq B_z$. Let $b \in B_z$. If $b \in B$, then $b \in B_y$. Let $C \in \mathscr{Y}$. If $C \subseteq B$ and $b \in C_y$, then $C_y \subseteq B_y$ and hence $b \in B_y$. If there exists $s \in S$ such that $s \,.\, C \subseteq B$ and $b \in s \,.\, C_y$, then $s \,.\, C_y \subseteq B_y$ and therefore $b \in B_y$. Thus $B_z \subseteq B_y$, whence $B_y = B_z \subseteq B_{zz}$.

Let $b \in B_{zz}$ and let $C \in \mathscr{Y}$. If $b \in C_y$ and $C \subseteq B_z$, then $C_y \subseteq B_y$, whence $b \in B_y$. If there exists $s \in S$ such that $s \,.\, C \subseteq B_z$ and $b \in s \,.\, C_y$, then $s \,.\, C \subseteq B_y$, hence $s \,.\, C_y \subseteq B_y$ and conseqently $b \in B_y$. Thus $B_{zz} \subseteq B_y$.

Therefore (a) implies (b).

II. Let $B \in \mathscr{Y}$. According to 3.3.2, $B_v \subseteq B_y$. For $C \in \mathscr{Y}$, $C_y \supseteq B$ we have $C_y \supseteq B_y$. For $C \in \mathscr{Y}$, $s \in S$, $C_y \supseteq B \,.\, s$ we obtain $C_y \supseteq B_y \,.\, s$, hence $C_y : s \supseteq B_y$. It follows that $B_v \supseteq B_y$.

Thus (a) implies (d).

III. Let w be an x-extension of y in S and let $A \subseteq S$.

Let $B \in \mathscr{Y}$. If $B \subseteq A$, then $B_y = B_w \subseteq A_w$. If there exists $s \in S$ such that $s \,.\, B \subseteq A$, then $s \,.\, B_y = s \,.\, B_w \subseteq A_w$. Therefore $A_z \subseteq A_w$, which implies $u \leqq w$.

If $B_y \supseteq A$, then $B_y = B_w = B_{yw} \supseteq A_w$. If there exists $s \in S$ such that $B_y \supseteq A \,.\, s$, then $B_y \supseteq A_w \,.\, s$, hence $B_y : s \supseteq A_w$. Therefore $w \leqq v$.

The proof is complete.

3.4. Remark. a) We can omit neither the equality $B_y = B_z$ nor the equality $B_z = B_{zz}$ in 3.3.4(2).

aa) Let us put $\mathscr{Y} = \{S\}$, $S_y = \emptyset$. Then the mapping $y : \mathscr{Y} \to 2^S$ is not a partial x-operator of S. For $A \subseteq S$ we have $A_z = A$, therefore $A_z = A_{zz}$, but $S_y \neq S_z$.

ab) Let the semigroup S have at least three different elements a, b, c and let $s_1 \,.\, s_2 = c$ hold for each $s_1 \in S$, $s_2 \in S$. Let us put $\mathscr{Y} = \{\{a\}, \{a, b\}\}$, $\{a\}_y = \{a, b\}$, $\{a, b\}_y = \{a, b, c\}$. The mapping $y : \mathscr{Y} \to 2^S$ is not a partial x-operator of S. It holds $\{a\}_z = \{a, b\} = \{a\}_y$, $\{a, b\}_z = \{a, b, c\} = \{a, b\}_y$, but $\{a\}_{zz} = \{a, b\}_z = \{a, b, c\} \neq \{a\}_z$.

b) For different ordinals $\eta_1, \eta_2 > 0$ there always exist a semigroup S and a partial x-operator y of S such that $z_{\eta_1} \neq z_{\eta_2}$ (s. 3.10.6).

c) For $A \subseteq S$ and $B \in \mathcal{Y}$ it holds:

$$B_y : (B_y : A) = \bigcap (B_y : s)\,(s \in S,\ B_y \supseteq A \,.\, s)\,.$$

If the semigroup S contains an identity element, then $A \subseteq S$ satisfies:

$$A_z = A \cup \bigcup s \,.\, B_y (s \in S,\ B \in \mathcal{Y},\ s \,.\, B \subseteq A)\,,$$
$$A_v = \bigcap (B_y : s)\,(s \in S,\ B \in \mathcal{Y},\ B_y \supseteq A \,.\, s) = \bigcap [B_y : (B_y : A)]\,(B \in \mathcal{Y})\,.$$

The following two propositions, 3.6 and 3.7 give necessary and sufficient conditions when the formulas for z and v can be simplified in another way. Before formulating these propositions we introduce a lemma which follows from 3.3.4 and 2.13. It can be proved also directly.

3.5. Lemma. *Let S contain an identity element and let y be a partial x-operator of S with the domain $\mathcal{Y}$. Then for any element $a \in S$ which has an inverse, it holds:*

$$A \in \mathcal{Y}\,, \quad a \,.\, A \in \mathcal{Y} \Rightarrow a \,.\, A_y = (a \,.\, A)_y\,.$$

3.6. Proposition. *Let $\mathcal{Y} \subseteq 2^S$, let y be a mapping of $\mathcal{Y}$ into 2^S and let z be the general closure operator of S defined by the formula (1). Then the following statements are equivalent:*

(a) $B \in \mathcal{Y}$, $b \in B_y$, $s \in S$, $s \,.\, b \notin s \,.\, B \Rightarrow$ *there exists* $D \in \mathcal{Y}$ *such that* $D \subseteq s \,.\, B$ *and* $s \,.\, b \in D_y$,

(b) $A \subseteq S \Rightarrow A_z = A \cup \bigcup B_y (B \in \mathcal{Y},\ B \subseteq A)$.

Proof. I. Let (a) hold and let $A \subseteq S$, $s \in S$, $B \in \mathcal{Y}$, $s \,.\, B \subseteq A$, $c \in s \,.\, B_y$. Then there exists $b \in B_y$ such that $c = s \,.\, b$. If $s \,.\, b \in s \,.\, B$, then $c \in A$. If $s \,.\, b \notin s \,.\, B$, then there exists $D \in \mathcal{Y}$ such that $D \subseteq s \,.\, B$ and $s \,.\, b \notin D_y$. Hence $c \in D_y$, $D \in \mathcal{Y}$ and $D \subseteq A$. Thus (b) holds.

II. Let (b) hold and let $B \in \mathcal{Y}$, $b \in B_y$, $s \in S$, $s \,.\, b \notin s \,.\, B$. Let us put $A = s \,.\, B$. Then $s \,.\, b \in s \,.\, B_y \subseteq A_z$. Hence there exists $D \in \mathcal{Y}$ such that $D \subseteq A$, $s \,.\, b \in D_y$. Therefore (a) holds.

3.6.1. Corollary. *Let y be a partial x-operator of S with the domain $\mathcal{Y}$ satisfying*

$$s \in S\,, \quad B \in \mathcal{Y} \Rightarrow s \,.\, B \in \mathcal{Y}\,.$$

Then the general closure operator z is given by the formula:

$$A \subseteq S \Rightarrow A_z = A \cup \bigcup B_y (B \in \mathcal{Y},\ B \subseteq A)\,.$$

3.7. Proposition. *Let* $\mathscr{Y} \subseteq 2^S$, *let* y *be a mapping of* $\mathscr{Y}$ *into* 2^S *and let* v *be the closure operator of* S *defined by the formula* (2). *Then the following statements are equivalent*:

(a) $B \in \mathscr{Y}$, $s \in S$, $d \in S$, $d \, . \, s \notin B_y \Rightarrow$ *there exists* $D \in \mathscr{Y}$ *such that* $d \notin D_y$ *and* $D_y \supseteq$ $\supseteq B_y : s$,

(b) $A \subseteq S \Rightarrow A_v = \bigcap B_y(B \in \mathscr{Y},\ B_y \supseteq A)$.

Proof. I. Let (a) hold and let $A \subseteq S$, $s \in S$, $B \in \mathscr{Y}$, $B_y \supseteq A \, . \, s$, $d \in \bigcap C_y\ (C_y \supseteq A)$. If $d \notin B_y : s$, then $d \, . \, s \notin B_y$, hence there exists $D \in \mathscr{Y}$ such that $d \notin D_y$ and $D_y \supseteq B_y : s$. Since $B_y : s \supseteq A$, we obtain a contradiction.

II. Let (b) hold and let $B \in \mathscr{Y}$, $s \in S$, $d \in S$, $d \, . \, s \notin B_y$. Let us put $A = B_y : s$. Since $A \, . \, s \subseteq B_y$, it holds $A_v \subseteq B_y : s$, hence $A = A_v$. It follows that there exists $D \in \mathscr{Y}$ such that $D_y \supseteq A$ and $d \notin D_y$.

3.7.1. Corollary. *Let* S *be a group and let* y *be a partial x-operator of* S *with the domain* $\mathscr{Y}$ *and with the property*:

$$s \in S, \quad B \in \mathscr{Y} \Rightarrow s \, . \, B \in \mathscr{Y} \, .$$

Then the closure operator v *is given by the formula*:

$$A \subseteq S \Rightarrow A_v = \bigcap B_y(B \in \mathscr{Y},\ B_y \supseteq A) \, .$$

Proof. For $B \in \mathscr{Y}$, $s \in S$, $d \in S$, $d \, . \, s \notin B_y$ we put $D = s^{-1} \, . \, B$. Then $D \in \mathscr{Y}$ and according to 3.5 $D_y = s^{-1} \, . \, B_y$, which implies the assertion.

3.7.2. Let S be a group with zero 0 and let $\mathscr{Y} \subseteq 2^S$. A mapping y of the set $\mathscr{Y}$ into 2^S is called an *α-mapping* if the following conditions are fulfilled:

1° There exists $C \in \mathscr{Y}$ such that $0 \notin C$,

2° $D \in \mathscr{Y}$, $0 \in D \Rightarrow D_y = S$,

3° $D \in \mathscr{Y}$, $0 \notin D \Rightarrow D_y = S - \{0\}$.

Evidently, then *y is a partial x-operator of* S and for $\emptyset \neq A \subseteq S$ we have:

$$A_v = \bigcap B_y(B \in \mathscr{Y},\ B_y \supseteq A) = \begin{cases} S & \text{in case} \quad 0 \in A \, , \\ S - \{0\} & \text{in case} \quad 0 \notin A \, . \end{cases}$$

For the empty set we obtain $\emptyset_v = \emptyset$.

3.7.3. Corollary. *Let* S *be a group with zero, let* y *be a partial x-operator of* S *with the domain* $\mathscr{Y}$ *which is not an α-mapping and let the following implication hold*:

$$s \in S, \quad B \in \mathscr{Y} \Rightarrow s \, . \, B \in \mathscr{Y} \, .$$

Then the closure operator v is given by the formula:

$$A \subseteq S \Rightarrow A_v = \bigcap B_y(B \in \mathscr{Y},\ B_y \supseteq A)\,.$$

Proof. Let 0 be the zero of S and let $B \in \mathscr{Y}$, $s \in S$, $d \in S$, $d\,.\,s \notin B_y$. If $s \neq 0$, we put $D = s^{-1}\,.\,B$. Then $D \in \mathscr{Y}$ and $D_y = s^{-1}\,.\,B_y$ follows from 3.5 whence $D_y \supseteq B_y : s$ and $d \notin D_y$.

Let $s = 0$. Then $0 \notin B_y$ and therefore $B_y : s = \emptyset$. Let us suppose that $d \in D_y$ for each $D \in \mathscr{Y}$. Then $d \neq s$. For $c \in S - \{0\}$ and $D \in \mathscr{Y}$ we get $c\,.\,d^{-1}\,.\,D \in \mathscr{Y}$ and from 3.5 it follows that $d \in (d\,.\,c^{-1}\,.\,D)_y = d\,.\,c^{-1}\,.\,D_y$, thus $c \in D_y$ and therefore $S - \{0\} \subseteq D_y$. If $0 \in D$, then evidently $D_y = S$. If $0 \notin D$, then $D \subseteq B_y = S - \{0\}$, hence $D_y = S - \{0\}$. Then y is an α-mapping, which is a contradiction.

3.8. Remark. The mappings z and v defined by (1) and (2) have not generally the form 3.6(b) and 3.7(b) even in case of y being a partial x-operator of S.

Example. Let S be a group which contains at least three different elements a, b, e, where $e = 1_S$ and $a^2 = e$. Put $\mathscr{Y} = \{\{a\}\}$, $\{a\}_y = \{e, a\}$. Then y is a partial x-operator of S. For $A = \{b\}$ we obtain $A_z = A_v = \{b, ab\}$, but

$$A \cup \bigcup B_y(B \in \mathscr{Y},\ B \subseteq A) = \{b\} \quad \text{and} \quad \bigcap B_y(B \in \mathscr{Y},\ B_y \supseteq A) = S\,.$$

(S. also the example in 5.5.4.)

3.9. Let y be a partial x-operator of S with the domain $\mathscr{Y}$. Let z, u, v have the same meaning as in 3.3. Thus by 3.3.4, $u(v)$ is the finest (coarsest) x-operator of S, which is an extension of y.

3.9.1. $\mathrm{E}(z) \subseteq \mathrm{E}(u) \subseteq \mathrm{E}(y) = \mathrm{E}(v)$.

Proof. According to 2.8, $\mathrm{E}(z) \subseteq \mathrm{E}(u) \subseteq \mathrm{E}(v) \subseteq \mathrm{E}(y)$. Let $r \in \mathrm{E}(y)$, $A \subseteq S$, $B \in \mathscr{Y}$. For $B_y \supseteq A$ we get $r.\,A_v \subseteq r\,.\,B_y \subseteq B_y$. If $s \in S$ and $B_y \supseteq A\,.\,s$, then $r\,.\,s\,.\,A_v \subseteq r\,.\,B_y \subseteq B_y$, whence $r\,.\,A_v \subseteq B_y : s$. Therefore $r\,.\,A_v \subseteq A_v$ from wbere we obtain $r \in \mathrm{E}(v)$. The assertion is proved.

For $A \subseteq S$ we put:

$$(3) \qquad A_p = A_z \cup A\,.\,\mathrm{E}(y) = A_z \cup A\,.\,\mathrm{E}(v) = A_z \cup A_z\,.\,\mathrm{E}(y) = A_z \cup A_z\,.\,\mathrm{E}(v)\,.$$

3.9.2. *p is a general closure operator of S with the following properties*:

(a) $a \in S,\ A \subseteq S \Rightarrow a\,.\,A_p \subseteq (a\,.\,A)_p$,
(b) $z \leqq p \leqq v$,
(c) $B \in \mathscr{Y} \Rightarrow B_p = B_y$,
(d) $\mathrm{E}(y) = \mathrm{E}(p)$.

Proof. Obviously, p is a general closure operator of S. For $a \in S$, $A \subseteq S$ we get by 3.3.1 (a) $a . A_p = a . A_z \cup a . A . E(y) \subseteq (a . A)_z \cup a . A . E(y) = (a . A)_p$.

Evidently $z \leqq p$ and for $A \subseteq S$ we obtain $A_p = A_z \cup A . E(v) \subseteq A_v \cup A_v . E(v) \subseteq A_v$.

From 3.3.4(b) and from thedefinition 2.7 of $E(y)$ (or from 3.3.4(b) and (d) and the previous property (b)) the property (c) follows.

For $A \subseteq S$ we have $E(y) . A_p = E(y) . A_z \cup E(y) . E(y) . A_z \subseteq A_p$ according to 2.10. This implies $E(y) \subseteq E(p)$. Since $p \leqq v$, we obtain from 2.8 $E(p) \subseteq E(v)$.

3.9.3. Theorem. *Let w denote the modification of p. Then w is the finest x-operator x of S, which is an extension of y in S with the property* $E(x) = E(y)$. *The coarsest one of such x-operators of S is the x-operator v.*

Proof. From 3.9.2(b) we get $p \leqq w \leqq v$, whence by 2.8 we obtain $E(p) \subseteq E(w) \subseteq$ $\subseteq E(v)$. 3.9.1 and 3.9.2(d) then imply $E(w) = E(y)$. From 3.9.2(c) and 3.3.4(d) we get $B_y = B_p \subseteq B_w \subseteq B_v = B_y$ for $B \in \mathscr{Y}$, whence by Lemma 3.1 and 3.9.2(a) we obtain that w is an x-extension of y.

Let x be an x-estension of y in S with the property $E(x) = E(y)$. Then for $A \subseteq S$ we have $A_x \supseteq A_x . E(x) \supseteq A . E(y)$ and since $z \leqq x$, we obtain $A_p \subseteq A_x$, whence $w \leqq x$.

The proof is complete.

3.9.4. Proposition. *Let S contain an identity element and let* $\{1_S\} \in \mathscr{Y}$. *Then* $E(x) = E(y) = \{1_S\}_x = \{1_S\}_y$ *for any x-extension x of y in S.*

Proof. By 2.13 and 3.9.1 we get $E(x) = \{1_S\}_x = \{1_S\}_y = \{1_S\}_v = E(v) = E(y)$.

3.9.5. Remark. a) Generally, $E(u) = E(y)$ does not hold. If $\mathscr{Y} = \emptyset$, e.g., then $A_u = A$ for each $A \subseteq S$, hence $E(u) = \{1_S\}$ if the semigroup S has an identity element and $E(u) = \emptyset$ in the opposite case, while $E(y) = S$.

Also in Example 3.10 (by 2.13) $E(u) = \{1_S\} \neq E(y)$.

b) For different ordinals $\eta_1, \eta_2 > 0$ there always exist a semigroup S and a partial x-operator of S such that $p_{\eta_1} \neq p_{\eta_2}$ (s. 3.10.6).

3.10. Example. Let $\alpha, \xi, \eta, \xi_1, \xi_2, \xi_3, \eta_1, \eta_2$ denote ordinal numbers. We denote

$$S = \{[\xi, 0], [\xi, 1] : \xi \leqq \alpha\}$$

and for $\xi_1 \leqq \alpha$, $\xi_2 \leqq \alpha$ we put

$$[\xi_1, 0] . [\xi_2, 1] = [\xi_2. 1] . [\xi_1, 0] = [\xi_1, 0] , \quad [\xi_1, \varepsilon] . [\xi_2, \varepsilon] = [\xi_3, \varepsilon],$$

where $\varepsilon = 0$ or $\varepsilon = 1$ and $\xi_3 = \min \{\xi_1, \xi_2\}$.

Then $S = (S, \cdot)$ *is a semigroup with an identity element and* $1_S = [\alpha, 1]$.

For $\eta < \alpha$ we put:

$$B_\eta = \begin{cases} \{[\xi, 0], [\xi, 1] : \xi \leqq \eta\} & \text{in case } \eta \text{ is isolated}, \\ \{[\xi, 0], [\xi_1 1] : \xi < \eta\} & \text{in case } \eta \text{ is limit}, \end{cases}$$

$(B_\eta)_y = B_\eta \cup \{[\eta, 1], [\eta + 1,1]\}$.

The system of all B_η $(\eta < \alpha)$ is denoted by $\mathscr{Y}$.

Obviously, the following assertion holds:

3.10.1. $E(y) = \{[\xi, 1] : \xi \leqq \alpha\} \cup \{[0, 0]\}$.

3.10.2. *The mapping $y : \mathscr{Y} \to 2^S$ is a partial x-operator.*

Proof. The properties 1° and 2° in 2.1 are evident. Let $\xi \leqq \alpha$, $\eta < \alpha$, $\eta' < \alpha$. Then $[\xi, 0] . B_\eta = [\xi, 0] . (B_\eta)_y$ and in the case $[\xi, 1] . B_\eta \subseteq (B_{\eta'})_y$ we have $\eta \leqq \eta'$, thus $[\xi, 1] . (B_\eta)_y \subseteq (B_\eta)_y \subseteq (B_{\eta'})_y$. Therefore the condition 3° in 2.1 holds.

For $\eta \leqq \alpha$ let us put:

$$A_\eta = \begin{cases} \{[\xi, 0] : \xi \leqq \alpha\} \cup \{[\xi, 1] : \xi \leqq \eta\} & \text{for } \eta \text{ isolated}, \\ \{[\xi, 0] : \xi \leqq \alpha\} \cup \{[\xi, 1] : \xi < \eta\} & \text{for } \eta \text{ limit}. \end{cases}$$

Let us denote the set A_0 by A. The mappings z and p are given by the formulas (1) and (3).

3.10.3. $\eta < \alpha \Rightarrow (A_\eta)_z = A_{\eta+1}$.

Proof. From the relation $B_\eta \subseteq A_\eta$ we obtain $A_{\eta+1} \subseteq (A_\eta)_z$. Let $a \in (A_\eta)_z - A_{\eta+1}$. Then $a = [\xi, 1]$, where $\eta + 1 < \xi \leqq \alpha$. Then there exist $\xi_1 \leqq \alpha$ and $\xi_2 < \alpha$ such that $[\xi_1, 1] . B_{\xi_2} \subseteq A_\eta$ and $a \in [\xi_1, 1] . (B_{\xi_2})_y$. Hence it follows that $[\xi, 1] = [\xi_1, 1] . . [\xi_2, 1]$ or $[\xi, 1] = [\xi_1, 1] . [\xi_2 + 1,1]$. Hence $\xi \leqq \xi_1$, $\xi \leqq \xi_2 + 1$. Then we get $\eta + 1 < \xi_1$, $\eta + 1 \leqq \xi_2$, whence $[\eta + 1,1] \in B_{\xi_2}$, hence $[\eta + 1,1] = [\xi_1, 1] . . [\eta + 1,1] \in [\xi_1, 1] . B_{\xi_2} \subseteq A_\eta$, which is a contradiction.

The following assertion evidently holds:

3.10.4. $\eta \leqq \alpha \Rightarrow E(y) . A_\eta = A_\eta$.

3.10.5. $0 < \eta \leqq \alpha \Rightarrow A_{z_\eta} = A_{p_\eta} = A_\eta$.

Proof. This assertion is proved by transfinite induction and by virtue of 3.10.3 and 3.10.4:

For $\eta = 1$ we have $A_{z_1} = (A_0)_z = A_1$, $A_{p_1} = A_z \cup A . E(y) = A_1$.

Let the assertion hold for each ξ $(1 \leqq \xi < \eta \leqq \alpha)$.

If η is isolated, then $\eta = \xi + 1$ and $A_{z_\eta} = (A_{z_\xi})_z = (A_\xi)_z = A_{\xi+1} = A_\eta$, $A_{p_\eta} = = (A_{p_\xi})_p = (A_\xi)_p = (A_\xi)_z \cup A_\xi . E(y) = A_{\xi+1} \cup A_\xi = A_\eta$.

For limit η we get $A_{z_\eta} = \bigcup A_{z_\xi}$ $(1 \leqq \xi < \eta) = \bigcup A_\xi$ $(1 \leqq \xi < \eta) = A_\eta$ and analogously we obtain $A_{p_\eta} = A_\eta$.

From 3.10.5 it follows directly:

3.10.6. *Let $\eta_1 \neq \eta_2$, $0 < \eta_1 \leqq \alpha$, $0 < \eta_2 \leqq \alpha$. Then $z_{\eta_1} \neq z_{\eta_2}$, $p_{\eta_1} \neq p_{\eta_2}$.*

4. APPLICATIONS OF THEOREM ON x-EXTENSION

If we put in 3.3 $\mathscr{Y} = \emptyset$ and $y = \emptyset$, then the following proposition follows from 3.3.4, which we can also see directly:

4.1. Proposition. *The finest (coarsest) closure operator of the set S is the finest (coarsest) x-operator of the semigroup S.*

4.2. Proposition. *Let $M \subseteq S$. Then there exists an x-operator x of S such that $M_x = M$. The finest one of such x-operators is the finest closure operator of the set S while the coarsest of them is the mapping $v : 2^S \to 2^S$ defined by the formulas:*

$$A \subseteq M \Rightarrow A_v = [M : (M : A)] \cap M\,,$$

$$A \subseteq S\,, \quad A \nsubseteq M \Rightarrow A_v = M : (M : A)\,.$$

Proof. If we set in 3.3 $\mathscr{Y} = \{M\}$ and $M_y = M$, we obtain the proposition (s. 3.4c)).

4.3. Proposition. *Let $M \subseteq S$. Then the following statements are equivalent:*

(a) *there exists an x-operator x of the semigroup S such that $M = \emptyset_x$,*

(b) $S \,.\, M \subseteq M$.

If (a) *and* (b) *hold, then the finest (coarsest) x-operator of S with the property given in* (a) *is the mapping $u(v)$: $2^S \to 2^S$ defined in the following way:*

$$A \subseteq S \Rightarrow A_u = A \cup M\,,$$

$$A \subseteq M \Rightarrow A_v = M\,,$$

$$A \nsubseteq M\,, \quad A \subseteq S \Rightarrow A_v = M : (M : A)\,.$$

Proof. In 3.3 we set $\mathscr{Y} = \{\emptyset\}$ and $\emptyset_y = M$.

4.4. Proposition. *Let $\mathscr{Y}$ be the system of all non-empty subsets of the set S and let y be a partial x-operator of the semigroup S with the domain $\mathscr{Y}$. Let $u(v)$ be the finest (coarsest) x-operator of S which is an extension of y in S and let $M = \bigcap B_y$ $(\emptyset \neq B \subseteq S)$.*

Then it holds:

$\emptyset_u = \emptyset$; $S \,.\, M \subseteq M \Rightarrow \emptyset_v = M$; $S \,.\, M \nsubseteq M \Rightarrow \emptyset_v = \emptyset$;

$E(u) = E(v) = E(y)$.

If x is an x-extension of y in S, then $x = u$ or $x = v$.

Proof. From 3.3.4 it follows that $\emptyset_u = \emptyset$. Let $M \neq \emptyset$ (in the case $M = \emptyset$ the assertion holds). Then $M \in \mathscr{Y}$ and $M \subseteq B_y$ for each $B \in \mathscr{Y}$, whence $M_y = M$.

If x is an x-extension of y in S and $\emptyset \neq \emptyset_x$, then $\emptyset_x \in \mathscr{Y}$ and $\emptyset_x \subseteq M_x = M \subseteq \subseteq \emptyset_{xy} = \emptyset_{xx} = \emptyset_x$. Thus $\emptyset_x = M$.

If $S \,.\, M \subseteq M$, then $M_y : s = M : s \supseteq M$ for each $s \in S$ and by 3.3.4, $\emptyset_v = M$. If $S \,.\, M \nsubseteq M$, then there exists $s \in S$ such that $s \,.\, M \nsubseteq M$, hence $M \nsubseteq M : s$ and by 3.3.4, $M \neq \emptyset_v$. Therefore $\emptyset_v = \emptyset$.

The equalities $E(u) = E(v) = E(y)$ follow from 2.9 and 3.9.1.

4.5. Proposition. *Let $M \subseteq S$. Then the following statements are equivalent*:

(a) *there exists an x-operator x of S with the property* $E(x) = M$,

(b) $M \,.\, M \subseteq M$ *and the set M contains all elements $s \in S$ with the following property*: $t \in S \Rightarrow s \,.\, t = t$ *or there exists* $m_t \in M$ *such that* $s \,.\, t = m_t \,.\, t$.

If (a) *holds, then the finest x-operator of S with the property given in* (a) *is the closure operator u of S defined by the formula*:

$$A \subseteq S \Rightarrow A_u = A \,.\, M \cup A \,.$$

If $M : M = M$, then (a) *holds and the coarsest x-operator of S wtih the property given in* (a) *is the closure operator v of S defined by the formula*:

$$A \subseteq S \Rightarrow A_v = M : (M : A) \,.$$

Proof. I. Let (a) hold. According to 2.10, $M \,.\, M \subseteq M$. If $s \in S$ has the property given in (b), then for $t \in S$ we get $s \,.\, t = t \in \{t\}_x$ or $s \,.\, t = t \,.\, m_t \in t \,.\, E(x) \subseteq \{t\}_x$. By 2.9, $s \in M$. Thus (a) implies (b).

II. Let (b) hold. For $A \subseteq S$ let us put $A_u = A \,.\, M \cup A$. Using 2.9 we can see directly that u is the finest x-operator of S with the property $E(u) = M$.

III. Let $M : M = M$. Then (b) holds and according to II (a) holds, too. For $A \subseteq M$ we have $M : (M : A) \subseteq M$ and by 4.2, v defined by the formula $A_v = M : (M : A)$ $(A \subseteq S)$ is the coarsest x-operator of S with the property $M_v = M$. For $s \in E(v)$ we have $s \,.\, M \subseteq M$, hence $s \in M$. If $A \subseteq S$, then $M \,.\, [M : (M : A)] \,.\, (M : A) \subseteq M \,.\, M \subseteq \subseteq M$, whence $M \,.\, A_v \subseteq M : (M : A) = A_v$, therefore $M \subseteq E(v)$. Hence we obtain $M = E(v)$ and from 2.11 it follows that v is the coarsest x-operator of S with the property given in (a).

4.6. Remark. If (b) holds in 4.5, then in general the coarsest x-operator of S with the property given in 4.5(a) need not exist:

Example. Let $s_0 \in S$ exist such that it holds:

$$s_1 \in S\,,\quad s_2 \in S \Rightarrow s_1 \,.\, s_2 = s_0\,.$$

The following two assertions are evident.

4.6.1. *A closure operator x of the set S is an x-operator of the semigroup S if and only if $\emptyset_x \neq \emptyset \Rightarrow s_0 \in \emptyset_x$.*

4.6.2. *If for an x-operator x of S there exists $\emptyset \neq A \subseteq S$ such that $s_0 \notin A_x$, then* $E(x) = \emptyset$. *In the opposite case* $E(x) = S$.

For each $\emptyset \neq B \subseteq S - \{s_0\}$ we put:

$$\emptyset_{v(B)} = \emptyset\,;\quad \emptyset \neq A \subseteq B \Rightarrow A_{v(B)} = B\,;\quad A \nsubseteq B\,,\quad A \subseteq S \Rightarrow A_{v(B)} = S\,.$$

Then $v(B)$ is a closure operator of S and by 4.6.1, $v(B)$ is an x-operator of S. From 4.6.2 it follows that $E(v(B)) = \emptyset$. If x is an x-operator of S such that $E(x) = \emptyset$, then by 4.6.2 there exists $\emptyset \neq B = B_x \subseteq S - \{s_0\}$. Since $\emptyset_x = \emptyset$ according to 2.11, we have $v(B) \geqq x$.

If we denote by $\mathcal{O}$ the set of all x-operators x of S with the property $E(x) = \emptyset$, then it holds:

4.6.3. (a) $\emptyset \neq B \subseteq S - \{s_0\} = v(B)$ *is a maximal element of the ordered set* $(\mathcal{O}, \leqq)$,

(b) $x \in \mathcal{O} \Rightarrow$ *there exists* $\emptyset \neq B \subseteq S - \{s_0\}$ *such that* $v(B) \geqq x$.

4.7. Proposition. *Let $M \subseteq S$. Then the following statements are equivalent*:

(a) *there exists an x-operator x of the semigroup S such that M is the identity element of the semigroup* $(\mathfrak{I}(S, x), \circ)$,

(b) $M : M = M$.

If (a) *holds, then the coarsest x-operator of S with the property mentioned in* (a) *is the closure operator v of S defined by the formula*:

$$A \subseteq S \Rightarrow A_v = M : (M : A)\,.$$

The finest one such x-operators of S is the modification of the general closure operator z of S defined by the formula:

$$A \subseteq S \Rightarrow A_z = A \,.\, M \cup A \cup (A : M)\,.$$

Proof. I. If (a) holds, then by 2.12, $E(x) = M$ and for $s \in S$, $s \,.\, M \subseteq M$ we have $s \in \{s\}_x = \{s\}_x \circ M = (s \,.\, M)_x \subseteq M_x = M$, hence $M : M \subseteq M$. By 4.5, $M \,.\, M \subseteq M$, therefore $M \subseteq M : M$. Consequently, $M : M = M$.

II. Let $M : M = M$. For $A \subseteq S$ let us put $A_v = M : (M : A)$. Then 4.5 implies that v is the coarsest x-operator of S with the property $E(v) = M$. For $s \in S$ we get $s \,.\, M \,.\, (M : s \,.\, M) \subseteq M$, hence $s \,.\, (M : s \,.\, M) \subseteq M : M = M$. Hence it follows that $s \in M : (M : s \,.\, M) = (s \,.\, M)_v$. According to 2.12, M is the identity element of the semigroup $(\mathfrak{I}(S, v), \circ)$.

III. Clearly, the mapping $z : 2^S \to 2^S$ given in the proposition is a general closure operator of the set S. For $a \in S$, $A \subseteq S$ we obtain $a \,.\, A_z = a \,.\, A \,.\, M \cup a \,.\, A \cup a \,.\, (A : M) \subseteq a \,.\, A \,.\, M \cup a \,.\, A \cup (a \,.\, A : M) = (a \,.\, A)_z$. By Lemma 3.1 the modification u of z is an x-operator of S.

Let $M : M = M$. Then $M_u = M$, whence $E(u) \subseteq M : M = M$ follows. For $A \subseteq S$ we have $M \,.\, A_u \subseteq A_{uz} = A_u$, hence $M \subseteq E(u)$, which implies $M = E(u)$. For $s \in S$ we get $s \in (s \,.\, M : M)$, therefore $s \in (s \,.\, E(u))_u$ and according to 2.12, M is the identity element of the semigroup $(\mathfrak{I}(S, u), \circ)$.

If x is an x-operator of S such that M is the identity element of the semigroup $(\mathfrak{I}(S, x), \circ)$, then for $A \subseteq S$, $s \in A : M$, according to 2.12, it holds $s \in (s \,.\, M)_x \subseteq A_x$. Since $A \,.\, M \subseteq A_x \circ M = A_x$, we get $z \leqq x$, thus $u \leqq x$.

The proof is complete.

4.8. Remark. The general closure operator z defined in 4.7 is generally not a closure operator. Moreover, it holds:

4.8.1. *There exist a semigroup S and a subset $M \subseteq S$ such that $M : M = M$ and for the general closure operator z defined in 4.7 it holds: if $0 < \eta_1 \leqq \omega$, $0 < \eta_2 \leqq \omega$ are different ordinal numbers, then $z_{\eta_1} \neq z_{\eta_2}$.*

Proof. Put $S = \{m_1, m_2, \ldots\} \cup \{a_0, a_1, \ldots\}$ and $m_i \,.\, m_j = m_{i+j}$, $a_k \,.\, a_l = a_0$, $a_k \,.\, m_i = m_i \,.\, a_k = a_{k-i}$ for $k \geqq i$ and $a_k \,.\, m_i = m_i \,.\, a_k = a_0$ for $k \leqq i$ $(k, l = 0, 1, \ldots;\ i, j = 1, 2, \ldots)$. Then $(S, \cdot)$ is a semigroup and for $M = \{m_1, m_2, \ldots\}$ we have $M : M = M$. For a non-negative integer n we put $A_n = \{a_0, a_1, \ldots, a_n\}$. Clearly, $A_n : M = A_{n+1}$ and for $n \geqq 1$, $A_n \,.\, M = A_{n-1}$. By mathematical induction it follows that $(A_0)_{z_n} = A_n$ for any positive integer n. Consequently, the proof is complete.

However, the general closure operator z_η does not increase any more for ordinal numbers greater than ω. Indeed, the following assertion holds:

4.8.2. *Let $M \subseteq S$. Then the general closure operator z defined in 4.7 satisfies $z_\omega = z_{\omega+1}$.*

Proof. Let $A \subseteq S$. Then $A_{z_{\omega+1}} = A_{z_\omega} . M \cup A_{z_\omega} \cup (A_z : M) \supseteq A_{z_\omega}$. Conversely, $A_{z_\omega} . M = (\bigcup_{i=1}^{\infty} A_{z_i}) . M = \bigcup_{i=1}^{\infty} (A_{z_i} . M) \subseteq \bigcup_{i=1}^{\infty} A_{z_{i+1}} = A_{z_\omega}$ and $A_{z_\omega} : M = (\bigcup_{i=1}^{\infty} A_{z_i}) : M = \bigcup_{i=1}^{\infty} (A_{z_i} : M) \subseteq \bigcup_{i=1}^{\infty} A_{z_{i+1}} = A_{z_\omega}$. Thus $A_{z_{\omega+1}} = A_{z_\omega}$.

4.9. Proposition. *Let $S^* = (S^*, \cdot)$ be the total quotient semigroup of the semigroup S and let y be an x-operator of the semigroup S regarded as a mapping of 2^S into 2^{S^*}.*[1]) *Then the following statements are equivalent:*

(a) *there exists an x-extension of y in the semigroup S^*,*

(b) *for any regular element $a \in S$ and for $b \in S$, $A \subseteq S$, $B \subseteq S$ it holds:*

$$b . B \subseteq a . A_y \Rightarrow b . B_y \subseteq a . A_y ,$$

(c) *for any regular element $a \in S$ and for $A \subseteq S$ it holds $a . A_y = (a . A)_y$.*

Let x be an x-extension of y in the semigroup S^*. If $r \in S$ is regular, then $E(x) = r^{-1} . \{r\}_y$ and $E(x)$ is the identity element of the semigroup $(\mathfrak{I}(S^*, x), \circ)$. For a fractionary subset $A \subseteq S^*$ with a multiplier a it holds $A_x = a^{-1} . (a . A)_y$.

If (a)–(c) hold, then the finest (coarsest) x-operator of S^*, which is an extension of y in the semigroup S^*, is the modification of the general closure operator z of S^* (x-operator v of S^*), where it holds for $A \subseteq S^*$:

$$A_z = A \cup (A \cap S)_y \cup \bigcup r^{-1} . (r . A \cap S)_y \; (r \in S \text{ regular}) ;$$

if $S^* = S$, then $A_v = A_y$ and if $S^* \neq S$, then

$A_v = \bigcap [(s . A)_y : s] \; (s \in S, \; s . A \subseteq S)$ [2])

$= a^{-1} . (a . A)_y$ *for a fractionary set A with a multiplier a*

$= \bigcap [(s . A)_y : s] \; (s \in S$ *is not regular*, $s . A \subseteq S)$ [2]) *in case A is not a fractionary set.*

Proof. I. According to Main Theorem 3.3.4 the statements (a) and (b) are equivalent. Evidently, (c) also implies (b). The implication (a) = (c) follows from 2.13.

II. Let x be an x-extension of y in the semigroup S^*. Then for a regular element $r \in S$, 2.13 implies $r^{-1} . \{r\}_y = r^{-1} . \{r\}_x = \{r^{-1} . r\}_x = \{1_{S^*}\}_x = E(x)$ and $E(x)$ is the identity element of the semigroup $(\mathfrak{I}(S^*, x), \circ)$. For a fractionary subset $A \subseteq S^*$ with a multiplier a we get $(a . A)_y = (a . A)_x = a . A_x$ according to 2.13 and since $a . A \subseteq S$. Consequently $A_x = a^{-1} . (a . A)_y$.

Furthermore, let (a)–(c) hold and let $A \subseteq S^*$ and $S \neq S^*$.

III. Let z denote the general closure operator of S^* mentioned in this proposition. If $B \subseteq S$, $B \subseteq A$, then $B_y \subseteq (A \cap S)_y$. Hence $\bigcup B_y (B \subseteq S, \; B \subseteq A) = (A \cap S)_y$.

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[1]) y is a mapping of 2^S into 2^S and if i is the identity embedding of 2^S into 2^{S^*}, then we consider y to be the mapping $i \circ y$.

[2]) The operation: is considered in the semigroup S^*.

Let $B \subseteq S$, $s \in S^*$, $s . B \subseteq A$. Then there exists $a \in S$ and a regular element $b \in S$ such that $s = a/b$. Then $s . B_y = b^{-1} . a . B_y \subseteq b^{-1} . (a . B)_y \subseteq b^{-1} . (b . A \cap S)_y$. Hence $\bigcup s . B_y(s \in S^*, B \subseteq S, s . B \subseteq A) = \bigcup r^{-1} . (r . A \cap S)_y$ $(r \in S$ regular$)$.

Then we obtain from 3.3.4 that the modification of z is the finest x-operator of S^*, which is an extension of y in S^*.

IV. The semigroup S^* has an identity element, therefore, by 3.4.c) the coarsest x-operator v of S^*, which is an extension of y in S^*, satisfies $A_v = \bigcap(B_y : s)$ $(s \in S^*, B \subseteq S, B_y \supseteq A . s) = \bigcap[(s . A)_y : s]$ $(s \in S^*, s . A \subseteq S) = \bigcap[(s . A)_y : s]$ $(s \in S, s . A \subseteq S)$, since for $a \in S$, $b \in S$ regular we have $a/b . A \subseteq S$, which impliex $(a . A)_y : a \subseteq (a/b . A)_y : a/b$. Then the given formula for v follows in case A is not fractionary. The proposition is proved.

4.10. Problem. *Is the general closure operator z defined in 4.9 a closure operator or does there even exist a semigroup S (if need be with the cancellation law) such that $z_{\eta_1} \neq z_{\eta_2}$ for different ordinal numbers $\eta_1 > 0$, $\eta_2 > 0$?*

4.11. Example. Let $R = (R, +, \cdot)$ be a commutative ring, $T = (T, +, \cdot)$ its total quotient ring. For $\emptyset \neq M \subseteq R$ let M_y denote the ideal of the ring R generated by the set M. Let $\mathcal{Y}$ denote the system of all non-empty subsets of R. Then y is a partial x-operator of the semigroup $(R, \cdot)$ with the domain $\mathcal{Y}$. By 4.4 there exist just two x-operators y_1, y_2 of $(R, \cdot)$ which are extensions of y. Here $\emptyset_{y_1} = \emptyset$ and $\emptyset_{y_2} = \{0_R\}$.

By 4.9 there exist x-operators of $(T, \cdot)$, which are extensions of y_1 and y_2 in $(T, \cdot)$, respectively. The finest (coarsest) ones of such operators are denoted by u_1 and u_2 $(v_1$ and $v_2)$, respectively.

Let $\emptyset \neq M \subseteq T$. If M is fractionary with a multiplier m, then by 4.9, $M_{u_1} = M_{u_2} = M_{v_1} = M_{v_2} = m^{-1} . (m . M)_y$, which is the fractional ideal of the ring $(R, +, \cdot)$ generated by the set M. If M is not fractionary, then by 4.9, $M_{v_1} = M_{v_2} = \bigcap[(s . M)_y : s]$ $(s \in R$ is not regular, $s . M \subseteq R)$. In case $(R, +, \cdot)$ is an integral domain, we have $M_{v_1} = M_{v_2} = T$. For x-operators u_1, u_2 of $(T, \cdot)$, $M_{u_1} = M_{u_2}$ is the R-submodule of the R-module T generated by the set M.

Evidently, $E(u_1) = E(u_2) = E(v_1) = E(v_2)$ is the fractional R-ideal generated by $\{1_T\}$ in case $R \neq T$. Otherwise, this set is equal to $R = T$.

5. VARIOUS SYSTEMS OF IDEALS CONSIDERED AS PARTIAL x-OPERATORS

5.1. Krull (1924). 5.1.1. Let $\mathfrak{B} = (\mathfrak{B}, ., \leqq)$ be a semigroup with an operation $\cdot$ and an ordering $\leqq$, where the ordered set $(\mathfrak{B}, \leqq)$ is a conditionally complete lattice[3]) with a least element $\mathfrak{o}$.

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[3]) The ordered set $(\mathfrak{B}, \leqq)$ is said to be a *conditionally complete lattice* if it is a lattice and each of its non-empty bounded subsets has an infimum and a supremum. If, moreover, $(\mathfrak{B}, \leqq)$ has a least element then each of its non-empty subsets has an infimum.

We call $\mathfrak{B} = (\mathfrak{B}, \leqq)$ a $\mathfrak{K}$-*system of ideals* if it holds:

(1) $\mathfrak{a} \in \mathfrak{B},\ \emptyset \neq \mathfrak{M} \subseteq \mathfrak{B} \Rightarrow \mathfrak{a} \,.\, \inf \mathfrak{M} = \inf \mathfrak{a} \,.\, \mathfrak{M}$.

From (1) it follows:

5.1.2. *For a $\mathfrak{K}$-system of ideals $\mathfrak{B} = (\mathfrak{B}, \cdot, \leqq)$ and $\mathfrak{a} \in \mathfrak{B}$, $\mathfrak{b} \in \mathfrak{B}$, $\mathfrak{c} \in \mathfrak{B}$ it holds*:

$$\mathfrak{a} \leqq \mathfrak{b} \Rightarrow \mathfrak{a} \,.\, \mathfrak{c} \leqq \mathfrak{b} \,.\, \mathfrak{c} \,.$$

From 2.4 and 2.6 we obtain

5.1.3. *Let x be an x-operator of the semigroup S. Then $(\mathfrak{J}(S), \circ, \supseteq)$ is a $\mathfrak{K}$-system of ideals.*

5.1.4. Let $\mathfrak{B} = (\mathfrak{B}, \cdot, \leqq)$ be a $\mathfrak{K}$-system of ideals. For $\emptyset \neq \mathfrak{M} \subseteq \mathfrak{B}$ we put

$$\mathfrak{M}_y = \{\mathfrak{m} \in \mathfrak{B} : \mathfrak{m} \geqq \inf \mathfrak{M}\} \,.$$

From (1) and 5.1.2 we conclude:

y is a partial x-operator of the semigroup $(\mathfrak{B}, \cdot)$, its domain is the system of all non-empty subsets of the set $\mathfrak{B}$.

5.1.5. Let $\mathfrak{B} = (\mathfrak{B}, \cdot, \leqq)$ be a $\mathfrak{K}$-system of ideals. For $\emptyset \neq \mathfrak{M} \subseteq \mathfrak{B}$ let us put $\mathfrak{M}_u = \mathfrak{M}_v = \mathfrak{M}_y$. Further, let us put $\emptyset_u = \emptyset$ and in case $(\mathfrak{B}, \leqq)$ has a largest element $\mathfrak{b}$ with the property $\mathfrak{a} \,.\, \mathfrak{b} = \mathfrak{b}$ for each $\mathfrak{a} \in \mathfrak{B}$ we put $\emptyset_v = \{\mathfrak{b}\}$. In the opposite case we put $\emptyset_v = \emptyset$.

Then 4.4 implies:

u, v are the only x-operators of the semigroup $(\mathfrak{B}, \cdot)$, which are extensions of y in $\mathfrak{B}$. Furthermore, $E(u) = E(v) = E(y)$ holds.

5.1.6. *Let $\mathfrak{B} = (\mathfrak{B}, \cdot, \leqq)$ be a $\mathfrak{K}$-system of ideals. Then the following statements are equivalent*:

(a) $E(y) = \mathfrak{B}$,

(b) $\mathfrak{a} \in \mathfrak{B},\ \mathfrak{b} \in \mathfrak{B} \Rightarrow \mathfrak{a} \,.\, \mathfrak{b} \geqq \sup \{\mathfrak{a}, \mathfrak{b}\}$.

Proof. I. Let $E(y) = \mathfrak{B}$, $\mathfrak{a} \in \mathfrak{B}$, $\mathfrak{b} \in \mathfrak{B}$. Then $\{\mathfrak{a} \,.\, \mathfrak{b}\}_y = \{\{\mathfrak{a}\}_y \,.\, \{\mathfrak{b}\}_y\}_y \subseteq \{\mathfrak{a}\}_y \cap \cap \{\mathfrak{b}\}_y = \{\sup \{\mathfrak{a}, \mathfrak{b}\}\}_y$, which implies $\mathfrak{a} \,.\, \mathfrak{b} \geqq \sup \{\mathfrak{a}, \mathfrak{b}\}$.

II. Let (b) hold and let $\mathfrak{a} \in \mathfrak{B}$, $\mathfrak{b} \in \mathfrak{B}$. Then $\mathfrak{b} \,.\, \{\mathfrak{a}\}_y \subseteq \{\mathfrak{a}, \mathfrak{b}\}_y \subseteq \{\sup \{\mathfrak{a}, \mathfrak{b}\}\}_y \subseteq \subseteq \{\mathfrak{a}\}_y$. Thus $\mathfrak{b} \in E(y)$, whence $E(y) = \mathfrak{B}$.

5.1.7. The system of ideals introduced and studied by KRULL in the paper [6] is the $\mathfrak{K}$-system of ideals with the property (b) in 5.1.6.

5.2. Prüfer (1932). Let $\mathfrak{G}$ denote an integral domain, $\mathfrak{K} = (\mathfrak{K}, +, \cdot)$ its quotient field, $\mathscr{Y}$ the system of all non-empty finite subsets of the set $\mathfrak{K}$ and y a mapping of $\mathscr{Y}$ into $2^{\mathfrak{K}}$.

Let us introduce the following properties of y:

(1) $A \subseteq A_y$,

(2) $B \subseteq A_y \Rightarrow B_y \subseteq A_y$,

(3) $\{a\}_y = a \,.\, \mathfrak{G}$,

(4) $a \in A_y \Rightarrow a \,.\, b \in (b \,.\, A)_y$,

(5) $a + b \in \{a, b\}_y$,

where $A \in \mathscr{Y}$, $B \in \mathscr{Y}$, $a \in \mathfrak{K}$, $b \in \mathfrak{K}$.

5.2.1. PRÜFER in [10] introduced and studied the system of sets $\{A_y : A \in \mathscr{Y}\}$, where y had the properties (1)–(5). Here by finite sets Prüfer obviously means the finite and non-empty sets (s. 5.4.1).

5.2.2. *The following statements are equivalent*:

(a) *y is a partial x-operator of the semigroup* $(\mathfrak{K}, \cdot)$,

(b) (1), (2) *and* (4) *hold.*

Proof. Let (b) hold and let $a \in \mathfrak{K}$, $A \in \mathscr{Y}$, $B \in \mathscr{Y}$, $a \,.\, B \subseteq A_y$. Given $b \in B_y$, then according to (4) and (2) $a \,.\, b \in (a \,.\, B)_y \subseteq A_y$, hence $a \,.\, B_y \subseteq A_y$.

If y is a partial x-operator of $(\mathfrak{K}, \cdot)$, then (1) and (2) hold evidently. For $A \in \mathscr{Y}$, $b \in \mathfrak{K}$ we have $b \,.\, A \in \mathscr{Y}$ and $b \,.\, A \subseteq (b \,.\, A)_y$, hence $b \,.\, A_y \subseteq (b \,.\, A)_y$ from which (4) follows.

From 3.9.4 the following assertion follows.

5.2.3. *If y is a partial x-operator of the semigroup* $(\mathfrak{K}, \cdot)$, *then* $\mathrm{E}(x) = \mathrm{E}(y) = \{1_{\mathfrak{K}}\}_x = \{1_{\mathfrak{K}}\}_y$ *for any x-extension x of y in* $\mathfrak{K}$.

5.2.4. *Let y be a partial x-operator of the semigroup* $(\mathfrak{K}, \cdot)$.

(A) *In case* $\mathfrak{G} \neq \mathfrak{K}$ *the following statements are equivalent*:

(a) $\mathrm{E}(y) = \mathfrak{G}$,

(b) $\{1_{\mathfrak{K}}\}_y = \mathfrak{G}$,

(c) (3) *holds.*

(B) *In case* $\mathfrak{G} = \mathfrak{K}$ *the following statements are equivalent*:

(a) $\mathrm{E}(y) = \mathfrak{G}$, $\{0_{\mathfrak{K}}\}_y = \{0_{\mathfrak{K}}\}$,

(b) $\{1_{\mathfrak{K}}\}_y = \mathfrak{G}$, $\{0_{\mathfrak{K}}\}_y = \{0_{\mathfrak{K}}\}$,

(c) (3) *holds.*

Proof. By 3.3.4 there exists an x-extension x of y and by 5.2.3, $E(x) = E(y) = \{1_{\mathfrak{K}}\}_y$. If $\{0_{\mathfrak{K}}\}_y \neq \{0_{\mathfrak{K}}\}$, then $\{0_{\mathfrak{K}}\}_y = \mathfrak{K}$. Thus by 2.13, we obtain the assertion.

5.2.5. *Let y be a partial x-operator of the semigroup $(\mathfrak{K}, \cdot)$. Then the finest (coarsest) x-operator of $(\mathfrak{K}, \cdot)$, which is an extension of y in $\mathfrak{K}$, is the mapping $u(v)$ of the system $2^{\mathfrak{K}}$ into $2^{\mathfrak{K}}$ defined for $A \subseteq \mathfrak{K}$ by the formula:*

$$A_u = \bigcup B_y(B \in \mathscr{Y},\ B \subseteq A)\,;$$

in the case that y is not an α-mapping, it holds

$$A_v = \bigcap B_y(B \in \mathscr{Y},\ B_y \supseteq A)\,;$$

in the case that y is an α-mapping, it holds

$$A_v = \begin{cases} \mathfrak{K} \ \textit{for} \ \ 0_{\mathfrak{K}} \in A\,, \\ \mathfrak{K} - \{0_{\mathfrak{K}}\} \ \ \textit{for} \ \ 0_{\mathfrak{K}} \notin A\,, \quad A \neq \emptyset\,, \\ \emptyset \ \ \textit{for} \ \ A = \emptyset\,. \end{cases}$$

Proof. The formula for the x-operator u follows from 3.6.1 and the formulas for v follow from 3.7.2 and 3.7.3.

5.3. Krull (1935). Let D denote an integral domain and $L = (L, +, \cdot)$ its quotient field. Let $\mathscr{Y}_1$ be the set of all non-empty fractional ideals of D and y_1 a mapping of $\mathscr{Y}_1$ into 2^L such that A_{y_1} is a fractional ideal of D for each $A \in \mathscr{Y}_1$.

Let us denote the properties of y_1 as follows:

(1) $A \subseteq A_{y_1}$,

(2) $A \subseteq B \Rightarrow A_{y_1} \subseteq B_{y_1}$,

(3) $(A_{y_1})_{y_1} = A_{y_1}$,

(4) $(a \,.\, A)_{y_1} = a \,.\, A_{y_1}$,

(5) $(a)_{y_1} = (a)$,

where $a \in L$, $A \in \mathscr{Y}_1$, $B \in \mathscr{Y}_1$ and (a) denotes the fractional ideal of D generated by the element a.

5.3.1. Krull in his book "*Idealtheorie*" ([7]) paragraph 43 introduced (1)–(5) as axioms (for an integrally closed integral domain D) with further two axioms: $(A_{y_1} + B_{y_1})_{y_1} = (A + B)_{y_1}$, $(A_{y_1} \,.\, B_{y_1})_{y_1} = (A \,.\, B)_{y_1}$ ($A \,.\, B$ denotes the ideal product), which follow from the former ones. The mapping $y_1(A \to A_{y_1})$ is denoted by $'(A \to A')$ and called *'-operation* (*'-Operation*). Krull studies this '-operation in detail in his paper [8].

In GILMER's treatise "*Multiplicative Ideal Theory*" ([3]) D need not be integrally closed and the set $\mathcal{Y}_1$ does not contain the zero ideal. The mapping y_1 is called a *$*$-operation on D* and references to the literature concerning this notion are given in the paper.

Evidently, it holds:

5.3.2. *The mapping y_1 is a partial x-operator of the semigroup $(L, \cdot)$ if and only if (1)-(4) hold.*

Further, let $\mathcal{Y}$ denote the system of all non-empty fractionary subsets of L and for $M \in \mathcal{Y}$, let M_{y_2} denote the fractional ideal of D generated by the set M. The mapping $y_2 : \mathcal{Y} - 2^L$ is a partial x-operator of $(L, \cdot)$.

For $M \in \mathcal{Y}$ we set $M_y = (M_{y_2})_{y_1}$. Then y is a mapping of $\mathcal{Y}$ into 2^L and evidently the first part of the following assertion holds. The other part follows from the formula (2) in 3.3.

5.3.3. *A mapping y is a partial x-operator of $(L, \cdot)$ if and only if y_1 is a partial x-operator of $(L, \cdot)$. In this case the coarsest x-operator of $(L, \cdot)$, which is an extension of y in L, is then the coarsest x-operator of $(L, \cdot)$, which is an extension of y_1.*

5.3.4. *If y is a partial x-operator of $(L, \cdot)$, then $\mathrm{E}(x) = \mathrm{E}(y) = \mathrm{E}(y_1) = \{1_L\}_x = = \{1_L\}_y = (1_L)_{y_1}$ for any x-extension x of y in $(L, \cdot)$.*

Proof. By 5.3.3 the coarsest x-operator v of $(L, \cdot)$, which is an extension of y, is the coarsest x-operator of $(L, \cdot)$, which is an extension of y_1. By 3.9.1 we have $\mathrm{E}(y) = \mathrm{E}(v) = \mathrm{E}(y_1)$ and by 3.9.4, $\mathrm{E}(x) = \mathrm{E}(y) = \{1_L\}_x = \{1_L\}_y$ for any x-extension x of y. Since $\{1_L\}_y = (\{1_L\}_{y_2})_{y_1} = (1_L)_{y_1}$, the proof is complete.

5.3.5. *Let y_1 be a partial operator of $(L, \cdot)$ and let $D \neq L$. Then the following assertions are equivalent:*

(a) $\mathrm{E}(y_1) = D$,

(b) (5) *holds.*

Proof. If (5) holds, then $(1_L)_{y_1} = D$ and by 5.3.4, $\mathrm{E}(y_1) = D$.

If $\mathrm{E}(y_1) = D$, then according to 5.3.4 $\mathrm{E}(x) = D$ for any x-extension x of y and from 2.13, $(a)_{y_1} = \{a\}_y = \{a\}_x = a \,.\, D = (a)$ for each $a \in L - \{0_L\}$. If there exists $b \in L - \{0_L\}$ such that $b \in (0_L)_{y_1}$, then $L \,.\, b \subseteq (0_L)_{y_1}$, hence $L = (0_L)_{y_1} \subseteq D$, which is a contradiction. Thus $(0_L)_{y_1} = (0_L)$.

5.3.6. *Let y be a partial x-operator of $(L, \cdot)$. Then the finest (coarsest) x-operator of $(L, \cdot)$, which is an extension of y, y_1 is the modification of the general closure operator z, z_1 of L, respectively (the mapping v of the system 2^L into 2^L), defined* *for* $A \subseteq L$ *by the formula*:

$$A_z = \bigcup B_y(B \in \mathscr{Y},\ B \subseteq A)\,, \quad A_{z_1} = A \cup \bigcup B_{y_1}(B \in \mathscr{Y}_1,\ B \subseteq A)$$

for $D \neq L$,

$$A_v = \begin{cases} A_y & \text{for} \quad A \in \mathscr{Y}\,, \\ L & \text{for} \quad A \notin \mathscr{Y}\,, \quad A \neq \emptyset\,, \\ (0) & \text{for} \quad A = \emptyset\,, \end{cases}$$

for $D = L$,

$$A_v = \begin{cases} L & \text{for} \quad \emptyset \neq A \neq \{0\}\,, \\ (0)_y & \text{for} \quad A = \emptyset \quad \text{or} \quad A = \{0\}\,. \end{cases}$$

Proof. This assertion follows from 3.6.1 and 3.7.3.

5.4. Lorenzen (1939). Let $\mathfrak{g}$ be a semigroup with an identity element in which the cancellation law holds and let $\mathfrak{G} = (\mathfrak{G}, \cdot)$ be its quotient group.

Let us denote by $\mathscr{A}(\mathscr{B})$ the system of all finite, non-empty (fractionary, non-empty) subsets of the set $\mathfrak{G}$ and let $a(b)$ be a mapping of $\mathscr{A}(\mathscr{B})$ into $2^{\mathfrak{G}}$.

Further, let y denote a or b, let $\mathscr{Y}$ denote $\mathscr{A}$ or $\mathscr{B}$ and let us denote by (1)–(4) the following properties of y:

(1) $A \subseteq A_y$,

(2) $B \subseteq A_y \Rightarrow B_y \subseteq A_y$,

(3) $\{a\}_y = a\,.\,\mathfrak{g}$,

(4) $a\,.\,A_y = (a\,.\,A)_y$,

where $A \in \mathscr{Y}$, $B \in \mathscr{Y}$ and $a \in \mathfrak{G}$.

5.4.1. Lorenzen in [9] introduced and studied the system of sets $\mathfrak{J} = \{A_y : A \in \mathscr{Y}\}$, where y has the properties (1)–(4). He denotes the mapping y by r and in case $y = a$ he calls $\mathfrak{J}$ the *r-system of ideals* (das *r-Idealsystem*) while in case $y = b$ $\mathfrak{J}$ is called the *total r-system of ideals* (das *totale r-Idealsystem*). JAFFARD in his book "*Les Systèmes d'Idéaux*" [4] studies equivalent systems of ideals.

In Lorenzen's paper [9] the author does not say explicitly that $\emptyset \notin \mathscr{Y}$ but from the context we can conclude that the empty set is not considered an element of the system $\mathscr{Y}$. If $\emptyset \in \mathscr{A}$, then for $\mathfrak{g} \neq \mathfrak{G}$ we have $\emptyset_a = \emptyset$ (if $d \in \emptyset_a$, then for each $g \in \mathfrak{G}$ we get $g\,.\,d \in g\,.\,\emptyset_a = \emptyset_a$, thus $\emptyset_a = \mathfrak{G}$). But then the notion "*r-closed*" ($A_a : A_a = \mathfrak{g}$ for each $A \in \mathscr{A}$, Definition 2 [9]) is never fulfilled for $\mathfrak{g} \neq \mathfrak{G}$ since $\emptyset_a : \emptyset_a = \emptyset : \emptyset = \mathfrak{G}$. Similarly in the case $\emptyset \in \mathscr{B}$ the notion "*total r-closed*" ($B_b : B_b = \mathfrak{g}$ for each $B \in \mathscr{B}$, Definition 4 [9]) is never fulfilled for $\mathfrak{g} \neq \mathfrak{G}$.

For the same reason we can see that also Jaffard in [4] and Prüfer in [10] mean the finite non-empty sets when saying finite sets.

From 2.13 it easily follows:

5.4.2. *The mapping y is a partial x-operator of $\mathfrak{G}$ if and only if* (1), (2) *and* (4) *hold.*

From 3.9.4 we obtain:

5.4.3. *If y is a partial x-operator of $\mathfrak{G}$, then* $E(x) = E(y) = \{1_{\mathfrak{G}}\}_x = \{1_{\mathfrak{G}}\}_y$ *for any x-extension x of y in $\mathfrak{G}$.*

This together with 2.13 implies:

5.4.4. *If y is a partial x-operator, then* $E(y) = \mathfrak{g}$ *if and only if* (3) *holds.*

From 3.6.1 and 3.7.1 we get:

5.4.5. *Let y be a partial x-operator of $\mathfrak{G}$. Then the finest (coarsest) x-operator of $\mathfrak{G}$, which is an extension of y, is the modification u of the general closure operator z of $\mathfrak{G}$ (the mapping v of $2^{\mathfrak{G}}$ into $2^{\mathfrak{G}}$) defined for $A \subseteq \mathfrak{G}$ by:*

$$A_z = \bigcup B_y(B \in \mathscr{Y},\ B \subseteq A),\quad A_v = \bigcap B_y(B \in \mathscr{Y},\ B_y \supseteq A).$$

For $y = a$ and $\mathscr{Y} = \mathscr{A}$ the general closure operator z is a closure operator of $\mathfrak{G}$ (hence it equals its modification u).

5.4.6. *Let b be a partial x-operator of $\mathfrak{G}$ for which* (3) *holds. Then the coarsest x-operator of $\mathfrak{G}$, which is an extension of b, is the mapping v of $2^{\mathfrak{G}}$ into $2^{\mathfrak{G}}$ defined for $A \subseteq \mathfrak{G}$*

$$A_v = \begin{cases} A_b & \text{for } A \in \mathscr{B}, \\ \mathfrak{G} & \text{for } \emptyset \neq A \notin \mathscr{B}, \\ \emptyset & \text{for } A = \emptyset \text{ in case } \mathfrak{g} \neq \mathfrak{G}, \\ \mathfrak{G} & \text{in case } \mathfrak{g} = \mathfrak{G}. \end{cases}$$

Proof. Since (3) holds, we have $\mathfrak{g}_b = \mathfrak{g}$. It follows that $B_b \in \mathscr{B}$ for $B \in \mathscr{B}$ and thus by 5.4.5 we get $A_v = \mathfrak{G}$ for $\emptyset \neq A \notin \mathscr{B}$. If $g \in \emptyset_v$, then $h \,.\, g \in h \,.\, \emptyset_v \subseteq \emptyset_v$ for each $h \in \mathfrak{G}$, hence $\emptyset_v = \mathfrak{G}$, which is possible only if $\mathfrak{g} = \mathfrak{G}$.

5.4.7. Let a be a partial x-operator of $\mathfrak{G}$ fulfilling (3) and let u, v be x-operators of $\mathfrak{G}$ defined in 5.4.5 for $y = a$, $\mathscr{Y} = \mathscr{A}$. Let $u_1(v_1)$ be the mapping $u(v)$ restricted to the system $\mathscr{B}$. Then u_1 and v_1 have the properties (1)–(4) and a mapping b of $\mathscr{B}$ into $2^{\mathfrak{G}}$ fulfilling (1)–(4) and extending the mapping a satisfies

$$B_{u_1} \subseteq B_b \subseteq B_{v_1}$$

for $B \in \mathscr{B}$.

Setting $a = r$ Lorenzen ([9]) denotes u_1 by the symbol r_s and v_1 by the symbol r_v.

5.4.8. Let us put $\mathscr{C} = \{\{g\} : g \in \mathfrak{G}\}, \{g\}_c = g \,.\, \mathfrak{g} = (g)$ for $g \in \mathfrak{G}$. Then c is a partial x-operator of $\mathfrak{G}$ with the domain $\mathscr{C}$. Then from 3.6.1 and 3.7.1 we obtain:

the finest (coarsest) x-operator of $\mathfrak{G}$, *which is an extension of c, is the mapping* $u(v)$ *of* $2^{\mathfrak{G}}$ *into* $2^{\mathfrak{G}}$ *defined for* $A \subseteq \mathfrak{G}$:

$$A_u = \bigcup(a)\,(a \in A) = A \,.\, \mathfrak{g},$$
$$A_v = \bigcap(a)\,(a \in \mathfrak{G},\ (a) \supseteq A) = \mathfrak{g} : (\mathfrak{g} : A).$$

Now 3.9.4 implies:

For any x-extension x of c in $\mathfrak{G}$, *it holds* $\mathrm{E}(x) = \mathfrak{g}$.

Restriction of $u(v)$ to $\mathscr{A}$ or $\mathscr{B}$ is usually denoted by $s(v)$ (s. Lorenzen [9], Jaffard [4]).

5.5. Aubert (1962). Let x be a mapping of 2^S into 2^S. The following properties of x let be denoted by (1)–(3″):

(1) $A \subseteq A_x$,

(2) $A \subseteq B_x \Rightarrow A_x \subseteq B_x$,

(3) $A \,.\, B_x \subseteq B_x \cap (A \,.\, B)_x$,

(3′) $A \,.\, B_x \subseteq B_x$,

(3″) $A \,.\, B_x \subseteq (A \,.\, B)_x$,

where $A \subseteq S$, $B \subseteq S$.

5.5.1. Aubert in [1] defined and studied the mapping x fulfilling (1)–(3). ((3) is equivalent to the conjunction of (3′) and (3″).) Then he says that a *system of x-ideals* or shortly an *x-system in* S is defined. He calls the axiom (3″) the *continuity axiom* (s. 2.5).

In Jaffard's book [4] (1960) in Appendix (Appendice – Les x-Idéaux), axioms equivalent (except an unimportant exception – the mapping $A \to A_x$ concerns only non-empty sets, s. 4.4) to those of Aubert are introduced.

Clearly, it holds:

5.5.2. *x is an x-operator of* S *if and only if* (1), (2) *and* (3″) *hold.*

From Definition 2.7 we get:

5.5.3. $\mathrm{E}(x) = S$ *if and only if* (3′) *holds.*

5.5.4. If $\mathscr{F}$ is the system of all subsets of S and y is a mapping of $\mathscr{F}$ into 2^S, then

y is a partial x-operator of S *if and only if* $A \in \mathscr{F}$, $B \in \mathscr{F}$ *satisfy*:

$$A \subseteq A_y\,; \quad A \subseteq B_y \Rightarrow A_y \subseteq B_y\,; \quad A \,.\, B_y \subseteq (A \,.\, B)_y\,.$$

Further, we have

$E(y) = S$ *if and only if* $A \,.\, B_y \subseteq B_y$ *for* $A \in \mathscr{F}$, $B \in \mathscr{F}$.

If (1)–(3) hold for y and for $A \in \mathscr{F}$, $B \in \mathscr{F}$ (therefore, y is a partial x-operator of S satisfying $E(y) = S$), then Aubert speaks about a *finite x-system.*

From 3.6.1, 3.9.1 and 2.8 we get:

If y is a partial x-operator of S, then the finest x-operator of S, which is an extension of y in S, is the mapping u of 2^S into 2^S given for $A \subseteq S$ by the formula:

$$A_u = \bigcup B_y(B \in \mathscr{F},\ B \subseteq A)\,.$$

For any x-extension x of y in S it holds $E(x) = E(y)$.

In case $E(y) = S$ Aubert calls the x-system defined by u a *finite x-system.*

For the coarsest x-operator v of S, which is an extension of y (in case y is a partial x-operator), the formula $A_v = \bigcap B_y(B \in \mathscr{F},\ B_y \supseteq A)$ $(A \subseteq S)$ does not hold in general even if $E(y) = S$.

Example. Let S be an infinite set, 0, α different elements of S. We put $s_1 \,.\, s_2 = 0$ for each $s_1 \in S$, $s_2 \in S$, $[s_1, s_2] \neq [\alpha, \alpha]$, and $\alpha \,.\, \alpha = \alpha$. Then $(S, \cdot)$ is a semigroup. For any finite subset A of S we put $A_y = A \cup \{0\}$. The mapping y is a partial x-operator of S with the domain $\mathscr{Y}$ of all finite subsets of S and evidently $E(y) = S$. We set $B = \{0\}$, $s = d = \alpha$. Then $B_y = B \not\ni d \,.\, s$ and $B_y : s = \{0\} : \alpha = S - \{\alpha\}$. By 3.7 there exists $A \subseteq S$ such that $A_v \neq \bigcap B_y(B \in \mathscr{Y},\ B_y \supseteq A)$, where v is the coarsest x-operator of S extending y.

5.5.5. Let S^* be the total quotient semigroup of S, $\mathscr{Y}$ the system of all fractionary subsets of S^* and y a mapping of $\mathscr{Y}$ into 2^{S^*}.

We have:

y is a partial x-operator of S^* if and only if for each $A \in \mathscr{Y}$, $B \in \mathscr{Y}$ and $a \in S^*$ the following implication holds:

(4) $A \subseteq A_y$; $A \subseteq B_y \Rightarrow A_y \subseteq B_y$; $a \,.\, B_y \subseteq (a \,.\, B)_y$.

If the semigroup S has an identity element, if (4) holds and if $S_y = S$, $S \,.\, B_y \subseteq B_y(B \in \mathscr{Y})$, then Aubert ([1], paragraph 14) speaks about a *fractionary x-system in S* (*or in S^**). The given properties of y for the semigroup S with an identity element are equivalent to the property that y is a partial x-operator of S^* and $E(y) = S$.

In case y is a partial x-operator of S^* and $S_y = S$, 4.9 yields the finest (coarsest) x-operator of S^*, which is an extension of y.

References

[1] *K. E. Aubert:* Theory of x-ideals, Acta Mathematica, vol. *107* (1962), 1—52.

[2] *E. Čech:* Topological spaces, in: Topological papers of Eduard Čech, Czechoslovak Academy of Sciences, Prague 1968, Nr. 28. Translated from Czech: Topologické prostory, Čas. Pěst. Mat., *66* (1937), D225—D264.

[3] *R. W. Gilmer:* Multiplicative Ideal Theory, Queen's Papers in Pure and Applied Mathematics, No. 12, Queen's University, Kingston, Ontario 1968.

[4] *P. Jaffard:* Les Systèmes d'Idéaux, Dunod, Paris 1960.

[5] *E. W. Johnson* and *J. P. Lediaev:* Quasi-noetherian x-systems, Acta Math. Acad. Scient. Hungaricae, tom. 21 (3—4), (1970), 335—339.

[6] *W. Krull:* Axiomatische Begründung der allgemeinen Idealtheorie, Sitzungsberichte der Physikalisch-medizinischen Sozietät zu Erlangen, B. *56* (1924), 47—63.

[7] *W. Krull:* Idealtheorie, Springer-Verlag, Berlin 1935.

[8] *W. Krull:* Beiträge zur Arithmetik kommutativer Integritätsbereiche, Math. Zeitschrift *41* (1936), 545—577.

[9] *P. Lorenzen:* Abstrakte Begründung der multiplikativen Idealtheorie, Math. Zeitschrift, *45* (1939), 533—553.

[10] *H. Prüfer:* Untersuchungen über Teilbarkeitseigenschaften in Körpern, Journal für die reine und angewandte Mathematik, B. *168* (1932), 1—36.

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