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WEAK STABILITY OF MULTIVALUED DIFFERENTIAL EQUATIONS

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I. Introduction. In this paper we give a version of weak stability theorem for multivalued differential equations of the form

$$(1) \qquad \dot{x} \in F(t, x)$$

where F is an upper-semicontinuous mapping.

II. Notation and basic definitions. Let R^n be the Euclidean n-dimensional space, o its zero element, $\varrho(x, y)$ the Euclidean distance from x to y, $\|x\| = \varrho(o, x)$. If A is a subset of R^n, let $\varrho(x, A) = \inf_{y \in A} \varrho(x, y)$, $\varrho^*(A, B) = \sup_{x \in A} \varrho(x, B)$ and $U(A, \varepsilon) = \{x \in R^n \mid \varrho(x, A) < \varepsilon\}$. The set of all compact, convex and nonempty subsets of R^n is denoted by X^n, the set of all subsets of R^n by Ω^n. Given $S \subset R^m$, then a mapping $F : S \to X^n$ is upper-semicontinuous on S if for every $y \in S$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $F(U(y, \delta)) \subset U(F(y), \varepsilon)$. Let $I = \langle 0, T \rangle$, $T > 0$. A mapping $F : I \to X^n$ is Borel measurable if its graph $\{(t, x) \in R^{n+1} \mid x \in F(t)\}$ is a Borel subset of $I \times R^n$. Let $F : R^m \to X^n$ be upper-semicontinuous, let $f : I \to R^m$ be continuous. Then the composition $F \circ f : I \to X^n$ is upper-semicontinuous and therefore its graph is closed in $I \times R^n$ — see KURATOWSKI [3] p. 187, Theorem 7 and p. 184, Theorem 1. Let $A_1, A_2, \ldots$ be subsets of R^n. Then $x \in \underline{\lim}\, A_k$ if for every $\varepsilon > 0$ there is a value n such that $U(x, \varepsilon) \cap A_k \neq \emptyset$ for every k, $k \geqq n$. See Kuratowski [3]. Let $I = \langle 0, T \rangle$, $T > 0$ and let $F(t) \in \Omega^n$, $F(t)$ nonempty a.e. in I. Let Φ be the set of all point-valued, integrable functions f, $f : I \to R^n$ with the property $f(t) \in F(t)$ a.e. in I. Following AUMANN [1] we define

$$\int_I F(t)\, \mathrm{d}t = \left\{ \int_I f(t)\, \mathrm{d}t \,\middle|\, f \in \Phi \right\}.$$

This integral has the following properties (see Aumann [1]):

i) if F is Borel measurable and integrally bounded (i.e. if there exists an integrable function h, $h : R \to R$ such that $\|x\| \leqq h(t)$ for all x and almost all t in I such that $x \in F(t)$) then $\int_I F(t)\,dt$ is nonempty;

ii) if $F(t)$ is closed for almost all t in I and if $F(\cdot)$ is integrally bounded, then $\int_I F(t)\,dt$ is compact;

iii) if $F_1, F_2, \ldots$ is a sequence of set-valued functions that are all Borel measurable and bounded by the same integrable point-valued function, then

$$\int_I \underline{\lim}\, F_k(t)\,dt \subset \underline{\lim} \int_I F_k(t)\,dt\,.$$

When proving the weak stability theorem we shall use the standard definition of limit of transfinite sequences of real numbers, see Sierpiński [5] p. 390.

III. Multivalued differential equations. We shall investigate multivalued differential equations of the form (1), under the following assumptions on the right-hand side F:

a) the mapping F is upper-semicontinuous on R^{n+1};

b) there exists a constant m such that for each $x \in R^n$ and almost all t in R and each $y \in F(t, x)$ it is $\|y\| \leqq m$;

c) $o \in F(t, o)$ for almost all t in R.

A function $x(\cdot)$ is called a solution of equation (1) on the interval $I = \langle T_1, T_2 \rangle$, $T_1 < T_2$, if it is absolutely continuous on I and if $\dot{x}(t) \in F(t, x(t))$ a. e. in I. It is well-known that if F, $F : I \times R^n \to X^m$ satisfies the assumptions a) and b) then for each $(t_0, x_0) \in R^{n+1}$ there exists a solution $x(\cdot, t_0, x_0)$ of equation (1) on the interval $\langle t_0, \infty)$ satisfying the initial condition $x(t_0, t_0, x_0) = x_0$. See Olech [4]. The assumption c) implies that o is a solution of equation (1) on $(-\infty, +\infty)$. This solution will be called trivial. We shall use the following lemma (proved in Cellina's work [2] p. 534):

Lemma 1. *Let $F : R^{m+1} \to X^n$ be upper-semicontinuous. Then a continuous function $x(\cdot)$ is a solution of (1) on the interval I if and only if for each pair $t_1, t_2 \in I$, $t_1 < t_2$ it holds*

$$x(t_2) - x(t_1) \in \int_{t_1}^{t_2} F(s, x(s))\,ds\,. \tag{2}$$

IV. Weak stability. Definition 1. The trivial solution of equation (1) is called *weakly stable* if for each t_0 and $\varepsilon > 0$ there is a $\delta > 0$ such that for each $y \in U(o, \delta)$ there exists a solution $x(\cdot, t_0, y)$ satisfying $\|x(t, t_0, y)\| \leqq \varepsilon$ for all t, $t \geqq t_0$.

Definition 2. A function $V: R^{n+1} \to R$ satisfying the conditions

i) V is lower-semicontinuous on R^{n+1}, i.e.

$$\varliminf_{(\tau,y)\to(t,x)} V(\tau, y) \geqq V(t, x) \quad \text{for every} \quad (t, x) \in R^{n+1} ;$$

ii)

$$\inf_{\xi\in F(t,x)} \varliminf_{\substack{\tau\searrow 0\\ \|y\|\to 0}} \frac{V(t + \tau, x + \tau . \xi + \tau . y) - V(t, x)}{\tau} \leqq 0 \quad \text{for all} \quad (t, x) \in R^{n+1} ,$$

is called a *weak Liapunov function* for equation (1).

Theorem. *Let V be a weak Liapunov function for equation (1). Then for every initial condition (t_0, x_0) there exists a solution $x(\cdot, t_0, x_0)$ of equation (1) defined on $\langle t_0, +\infty)$ such that*

$$V(t, x(t, t_0, x_0)) \leqq V(t_0, x_0) \quad \text{for each } t , \quad t \in \langle t_0, +\infty) .$$

Corollary. *If V is a weak Liapunov function for (1) and if V satisfies the additional conditions*

iii) *$V(t, o) = o$ and $V(t, \cdot)$ is continuous at o for each t;*

iv) *V is positively definite, i.e. there exists a function $\varphi : \langle 0, \infty) \to \langle 0, \infty)$ such that $\varphi(r) > 0$ for every $r > 0$ and $V(t, x) \geqq \varphi(\|x\|)$ for all $(t, x) \in R^{n+1}$,*

then the trivial solution is weakly stable.

The proof follows from the above theorem by a standard argument.

Proof of the theorem. Let an initial condition (t_0, x_0) and a positive number T, $T > t_0$, be given. We shall construct a solution $x(\cdot, t_0, x_0)$ of (1) with the property

$$V(t_0, x_0) \geqq V(\eta, x(\eta, t_0, x_0) \quad \text{for each } \eta , \quad \eta \in \langle t_0, T\rangle .$$

Let $\varepsilon > 0$ be given. By the assumption ii) in Definition 2, for each $(t, x) \in R^{n+1}$ there exists a positive number τ and vectors ξ, $\xi \in F(t, x)$ and y such that the inequalities

$$\tau \leqq \varepsilon , \quad \|y\| \leqq \varepsilon \quad \text{and} \quad V(t + \tau, x + \tau(\xi + y)) - V(t, x) \leqq \varepsilon . \tau \tag{3}$$

are satisfied simultaneously. Let $\tau(t, x) = \sup \{\tau |$ there are ξ and y such that $\xi \in F(t, x)$ and (3) is valid$\}$. Since $F(t, x)$ is compact and V is lower-semicontinuous we obtain that there exist $\xi(t, x)$ and $y(t, x)$ such that

$$\xi(t, x) \in F(t, x) \tag{4}$$

and such that the inequalities (3) are valid with $\tau = \tau(t, x)$, $\xi = \xi(t, x)$ and $y = y(t, x)$.

Now let ω_1 be the ordinal number of the set of all ordinal numbers of classes one and two, which is ordered according to their magnitude (i.e. ω_1 is the first non-denumerable ordinal number) and let $\tau^*(t, x) = \min(\tau(t, x), T - t)$. We shall need the following definition:

Definition 3. Let α be an ordinal number, $\alpha < \omega_1$, and let $W(\alpha)$ be the set of all ordinal numbers which are smaller then α. Then a mapping

$$Z\colon W(\alpha) \cup \{\alpha\} \to \langle t_0, T\rangle \times R^n$$

is called *a generalized sequence* (with the length $\omega(Z) = \alpha$). The mapping Z: $W(\omega_1) \to \langle t_0, T\rangle \times R^n$ is called a generalized sequence with the ength ω_1.

Definition 4. Let $\mathcal{M}$ be the set of all generalized sequences Z with the property $\omega(Z) < \omega_1$, satisfying the following conditions:

1) for $\beta = 0$ it is $Z(\beta) = (t_0, x_0)$;
2) if β is an ordinal number of the 1-st kind, $\beta > 0$ and $Z(\beta - 1) = (t_{\beta-1}, x_{\beta-1})$, $Z(\beta) = (t_\beta, x_\beta)$ then $t_\beta = t_{\beta-1} + \tau^*(t_{\beta-1}, x_{\beta-1})$ and $x_\beta = x_{\beta-1} + \tau^*(t_{\beta-1}, x_{\beta-1}) \cdot (\xi(t_{\beta-1}, x_{\beta-1}) + y(t_{\beta-1}, x_{\beta-1}))$;
3) if β is a number of the 2-nd kind and $Z(\beta) = (t_\beta, x_\beta)$, $Z(\gamma) = (t_\gamma, x_\gamma)$ for $\gamma < \beta$ then $t_\beta = \sup_{\gamma<\beta} t_\gamma$ and $x_\beta = \lim_{\gamma\to\beta} x_\gamma$;
4) if $\beta < \gamma \leqq \omega(Z)$ and $Z(\beta) = (t_\beta, x_\beta)$, $Z(\gamma) = (t_\gamma, x_\gamma)$ then $t_\beta < t_\gamma \leqq T$.

For $Z_1, Z_2 \in \mathcal{M}$ we define $Z_1 \lesssim Z_2$ if $\omega(Z_1) \leqq \omega(Z_2)$ and if $Z_1(\beta) = Z_2(\beta)$ for each β, $0 \leqq \beta \leqq \omega(Z_1)$.

It is easy to see that $Z_1 \lesssim Z_1$ holds, $Z_1 \lesssim Z_2$, $Z_2 \lesssim Z_1$ implies $Z_1 = Z_2$, and $Z_1 \lesssim Z_2$, $Z_2 \lesssim Z_3$ implies $Z_1 \lesssim Z_3$, i.e. the relation $\lesssim$ is a partial order relation.

If a set $\mathcal{M}_1$, $\mathcal{M}_1 \subset \mathcal{M}$ is such that for each $Z_1, Z_2 \in \mathcal{M}_1$ it is either $Z_1 \lesssim Z_2$ and/or $Z_2 \lesssim Z_1$ we shall say that the relation $\lesssim$ is a simple ordering on the set $\mathcal{M}_1$.

By means of the generalized sequences we shall construct an "approximate" solution $x_\varepsilon^Z(\cdot)$ of equation (1) on the interval $\langle t_0, t_{\omega(Z)})$ in the following way. Let $Z \in \mathcal{M}$ and let $\varphi_\varepsilon^Z(\cdot)$ be such that if α is an ordinal number, $Z(\alpha) = (t_\alpha, x_\alpha)$, $t_\alpha < T$ and $t \in \langle t_\alpha, t_{\alpha+1})$, then $\varphi_\varepsilon^Z(t) = \xi(t_\alpha, x_\alpha) + y(t_\alpha, x_\alpha)$. It is easy to see that such a function $\varphi_\varepsilon^Z(\cdot)$ exists for every Z, $Z \in \mathcal{M}$ and that $\varphi_\varepsilon^Z(\cdot)$ is defined, bounded, and Borel measurable on the interval $\langle t_0, t_{\omega(Z)})$. By means of $\varphi_\varepsilon^Z(\cdot)$ we define the "approximate" solution $x_\varepsilon^Z(\cdot)$ on $\langle t_0, t_{\omega(Z)})$ by

$$x_\varepsilon^Z(t) = x_0 + \int_{t_0}^{t} \varphi_\varepsilon^Z(\tau)\, d\tau\,.$$

Hence, the function $x_\varepsilon^Z(\cdot)$ is absolutely continuous on $\langle t_0, t_{\omega(Z)})$ and it is easy to see that $x_\varepsilon^Z(t_\alpha) = x_\alpha$.

It is not difficult to prove that if $\mathcal{M}_1$ is a simply ordered set, $\mathcal{M}_1 \subset \mathcal{M}$, then there exists a Z^* in $\mathcal{M}$ satisfying $Z \leqq Z^*$ for all $Z \in \mathcal{M}_1$ and $\omega(Z^*) < \omega_1$. Applying Zorn's lemma to $\mathcal{M}$ shows that there exists a maximal element Z in $\mathcal{M}$ such that $\omega(Z) = \eta < \omega_1$ and $Z(\eta) = (t_\eta, x_\eta)$. If there were $Z(\eta) = (t_\eta, x_\eta)$, $t_\eta < T$, it would follow that there exists $Z_1 \in \mathcal{M}$, $Z \leqq Z_1$, $Z \neq Z_1$, in contradiction to the maximality of Z. Hence $t_\eta = T$.

Since ε is arbitrary, it follows that for every positive integer n there exists a generalized sequence Z_n such that both functions $\varphi^{Z_n}_{1/n}(\cdot)$ and $x^{Z_n}_{1/n}(\cdot)$ are defined on $\langle t_0, T\rangle$. Let us write $\varphi_n(\cdot)$ and $x_n(\cdot)$ instead of $\varphi^{Z_n}_{1/n}(\cdot)$ and $x^{Z_n}_{1/n}(\cdot)$, respectively. Using the assumption b) concerning the right hand side of equation (1) we conclude that the functions $x_n(\cdot)$, $n = 1, 2, \dots$ are equibounded and equicontinuous. Hence, there exists a subsequence (let us denote it again by $\{x_n(\cdot)\}$) which is uniformly convergent on $\langle t_0, T\rangle$. We shall show that $x^*(t) = \lim_{n\to\infty} x_n(t)$ is a solution of equation (1) on $\langle t_0, T\rangle$. In virtue of Lemma 1 it is sufficient to prove that the relation

$$x^*(t_2) - x^*(t_1) \in \int_{t_1}^{t_2} F(t, x^*(t)\, \mathrm{d}t \tag{5}$$

holds for every t_1, t_2, $t_0 \leqq t_1 \leqq t_2 \leqq T$. To prove (5) we shall follow Cellina's idea, see [2]. The set $\int_{t_1}^{t_2} F(t, x^*(t))\, \mathrm{d}t$ is compact, hence it is sufficient to prove

$$\varrho\left(x_n(t_2) - x_n(t_1)\,,\ \int_{t_1}^{t_2} F(s, x^*(s))\, \mathrm{d}s\right) \to 0\ .$$

for $n \to \infty$ or, equivalently,

$$\varrho\left(\int_{t_1}^{t_2} \varphi_n(s)\, \mathrm{d}s\,,\ \int_{t_1}^{t_2} F(s, x^*(s))\, \mathrm{d}s\right) \to 0$$

for $n \to \infty$.

Let us denote $\{z \mid z = \varphi_n(t) - v,\ v \in F(t, x^*(t))\}$ by $\varphi_n(t) - F(t, x^*(t))$. Then we have (as a consequence of the definition of Aumann's integral)

$$\varrho\left(\int_{t_1}^{t_2} \varphi_n(s)\, \mathrm{d}s\,,\ \int_{t_1}^{t_2} F(t, x^*(s))\, \mathrm{d}s\right) = \varrho\left(o,\ \int_{t_1}^{t_2} (\varphi_n(s) - F(s, x^*(s)))\, \mathrm{d}s\right).$$

Now let $\tau \in \langle t_1, t_2\rangle$ be fixed. The mapping $F(\cdot)$ is upper-semicontinuous, hence, for every $\varepsilon_1 > 0$ there exists a $\delta > 0$ such that

$$F(\tau, U(x^*(\tau), \delta)) \subset U(F(\tau, x^*(\tau)), \varepsilon_1)\,. \tag{6}$$

As a consequence of the definition of $x_n(\cdot)$ we have that for every positive integer n there exists an α such that $\tau \in \langle t^n_\alpha, t^n_{\alpha+1})$, where t^n_α, $t^n_{\alpha+1}$ are such that $Z_n(\alpha) = (t^n_\alpha, x_n(t^n_\alpha))$, $Z_n(\alpha + 1) = (t^n_{\alpha+1}, x_n(t^n_{\alpha+1}))$ and (3) implies that $t^n_{\alpha+1} - t^n_\alpha < 1/n$. The above argument together with the assumption b) concerning the right hand side of equation (1) yields

$$\|x_n(\tau) - x_n(t_\alpha^n)\| = \left\| \int_{t_\alpha^n}^{\tau} \varphi_n(s)\, \mathrm{d}s \right\| \leqq \frac{1}{n}\left(m + \frac{1}{n}\right).$$

It follows that there exists a positive integer n_0 such that for every n, $n \geqq n_0$ the inequalities

$$\frac{1}{n} < \frac{\delta}{2}, \quad \|x_n(\tau) - x_n(t_\alpha^n)\| \leqq \frac{\delta}{4}, \quad \|x_n(\tau) - x^*(\tau)\| \leqq \frac{\delta}{4}$$

hold simultaneously. Hence, $\|x_n(t_\alpha^n) - x^*(\tau)\| \leqq \delta/2$ and we have

$$\varphi_n(\tau) = \varphi_n(t_\alpha^n) \in U\left(F(t_\alpha^n, x_n(t_\alpha^n)), \frac{1}{n}\right) \subset$$

$$U\left(F(U(\tau, x^*(\tau)), \delta), \frac{1}{n}\right) \subset U\left(F(\tau, x^*(\tau)), \varepsilon_1 + \frac{1}{n}\right).$$

Consequently $o \in \underline{\lim}\, (\varphi_n(\tau) - F(\tau, x^*(\tau)))$ and using the property iii) of Aumann's integral we have

$$o \in \int_{t_1}^{t_2} \underline{\lim}\, (\varphi_n(s) - F(s, x^*(s)))\, \mathrm{d}s \subset \underline{\lim} \int_{t_1}^{t_2} (\varphi_n(s) - F(s, x(s)))\, \mathrm{d}s\,.$$

(It is easy to prove that the functions $\varphi_n(\cdot) - F(\cdot, x^*(\cdot))$, $n = 1, 2, \ldots$ are Borel measurable.) So we have

$$\varrho\left(o, \int_{t_1}^{t_2} (\varphi_n(s) - F(s, x^*(s)))\, \mathrm{d}s\right) \to 0 \quad \text{for} \quad n \to \infty$$

and hence, $x^*(\cdot)$ is a solution of equation (1) on $\langle t_0, T\rangle$. Let t be fixed, $t \in \langle t_0, T\rangle$. From the construction of $x_n(\cdot)$ and from (3) it follows that in every neighbourhood $U(t, 1/n)$ there exists a point t_α^n such that $(t_\alpha^n, x_\alpha^n) = Z_n(\alpha)$ and

$$V(t_\alpha^n, x_n(t_\alpha^n)) - V(t_0, x_0) \leqq \frac{1}{n}\, T\,.$$

But $x_n(\cdot) \to x^*(\cdot)$ uniformly and $t_\alpha^n \to t$, hence $x_n(t_\alpha^n) \to x^*(t)$ and the lower-semicontinuity of V implies

$$V(t, x^*(t)) - V(t_0, x_0) \leqq \underline{\lim}_{n\to\infty} \left(V(t_\alpha^n, x_n(t_\alpha^n)) - V(t_0, x_0)\right) \leqq \underline{\lim}_{n\to\infty} \frac{1}{n}\, T = 0\,.$$

The solution $x^*(\cdot)$ can be constructed by the same method on the intervals $\langle T, 2T\rangle$, $\langle 2T, 3T\rangle, \ldots$ which completes the proof.

Note. For a similar theorem concerning ordinary differential equations with continuous right-hand sides see YORKE [6].

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