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Czechoslovak Mathematical Journal, Vol. 26 (1976), No. 3, 411–423,424–437

Persistent URL: http://dml.cz/dmlcz/101416

Czechoslovak Mathematical Journal, 26 (101) 1976, Praha

LYAPUNOV STABILITY AND STABILITY AT CONSTANTLY ACTING DISTURBANCES OF AN ABSTRACT DIFFERENTIAL EQUATION OF THE SECOND ORDER

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(Received October 25, 1974)

INTRODUCTION

The aim of this paper is the investigation of stability of a solution of the equation

$$u''(t) + A\,u(t) = F(t, u(t)) \tag{1}$$

where $u(t)$ (for $t \in R^+ \cap \mathscr{D}(u)$ being fixed, $R^+ = [0, \infty)$) is an element of a real Hilbert space H of real functions $h = h(x)$, defined on a subset Ω of a Euclidean space E_n. We shall denote by $\|\cdot\|$ a norm in the space H. Suppose that A is a linear, selfadjoint, strictly positive operator with domain of definition $\mathscr{D}(A) \subset H$ and with the spectral resolution of identity $E(s)$. If we denote $\delta = \inf \sigma(A)$, it is $\delta > 0$.

We define a function $f(A)$ of the operator A as follows: $f(A)\,x = \int_\delta^\infty f(s)\,\mathrm{d}E(s)\,x$ for $x \in \{x \in H \mid \int_\delta^\infty |f(s)|^2\,\mathrm{d}\|E(s)\,x\|^2 < \infty\} \equiv \mathscr{D}(f(A))$ and for continuous functions $f(s)$ mapping the interval $[\delta, \infty)$ to the complex plane.

Put

$$\mathscr{U} = \bigcup_{t_0 \geqq 0} \{u : [t_0, \infty) \equiv \mathscr{D}(u) \to H \mid u \in C^2(\mathscr{D}(u), H) \cap$$
$$\cap\, C^1(\mathscr{D}(u), \mathscr{D}(A^{1/2})) \cap C(\mathscr{D}(u), \mathscr{D}(A))\}\,,$$

where the norm in $\mathscr{D}(A^\alpha)$ is $\|x\|_{\mathscr{D}(A^\alpha)} = \|A^\alpha x\|$.

Denote

$$|||u(t)||| = \|(u(t), u'(t))\|_{\mathscr{D}(A) \times \mathscr{D}(A^{1/2})} =$$
$$= [\|A\,u(t)\|^2 + \|A^{1/2}\,u'(t)\|^2]^{1/2} \quad \text{for} \quad u \in \mathscr{U}\,, \quad t \in \mathscr{D}(u)\,.$$

The function on the right hand side of the equation (1) is defined on the set $\mathscr{D}(F) = \bigcup_{u \in \mathscr{U}} \{(t, u(t), u'(t)) \mid t \in \mathscr{D}(u)\} \subset R^+ \times \mathscr{D}(A) \times \mathscr{D}(A^{1/2})$. In a brief symbolic form we write $F(t, u(t))$ instead of $F(t, u(t), u'(t))$.

We shall write $F(t, u(t)) \in C(\mathscr{D}(u), \mathscr{D}(A^\alpha))$ if the function $\tilde{F}(t) = F(t, u(t))$ is continuous on $\mathscr{D}(u)$ in the norm $\|\cdot\|_{\mathscr{D}(A^\alpha)}$ for every $u \in \mathscr{U}$.

A function $u \in \mathscr{U}$ is called a solution to the equation (1) if u satisfies the equation (1) on $\mathscr{D}(u)$.

Suppose that $F(t, u(t)) \in C(\mathscr{D}(u), \mathscr{D}(A^{1/2}))$.

We shall use the notation $g(t, x)$ if we want to point out the dependence on t and x ($g(t)$ if the dependence on x is not important) for all functions g dependent on $t \in R^1$ and $x \in \bar{\Omega}$. However $g(x)$ denotes a function independent on t.

Let $v : R^+ \to H$ be a solution of the equation (1) whose stability is to be investigated.

Definition 1. We say that the solution $v : R^+ \to H$ of the equation (1) is *uniformly exponentially stable with respect to the norm* $|||\cdot|||$, if there exist positive constants L, ω, ϱ such that

$$|||u(t_0) - v(t_0)||| \leqq \varrho \Rightarrow |||u(t) - v(t)||| \leqq$$
$$\leqq L\exp(-\omega(t - t_0))\, |||u(t_0) - v(t_0)||| \quad \text{for} \quad t \geqq t_0$$

for all solutions u of the equation (1) and all $t_0 \geqq 0$.

If $\varrho = +\infty$, we speak about the *global uniform exponential stability*.

Let us introduce the equation

$$u''(t) + A\,u(t) = F(t, u(t)) + R(t, u(t)) \tag{2}$$

where $R(t, u(t))$ is defined for $u \in \mathscr{U}$ and such $t \in \mathscr{D}(u)$ that $|||u(t) - v(t)||| \leqq r$ ($r > 0$).

Suppose that $R(t, u(t)) \in C(\mathscr{D}(u), \mathscr{D}(A^{1/2}))$.

The function R is called a constanty acting disturbance.

Definition 2. We say that the solution $v : R^+ \to H$ of the equation (1) is *uniformly stable at constantly acting disturbances with respect to the norm* $|||\cdot|||$ if to arbitrarily chosen $\eta \in (0, r]$ there exist positive numbers η_0 and η_D such that $\{|||u(t_0) - v(t_0)||| \leqq \eta_0,\ \|A^{1/2} R(t, u(t))\| \leqq \eta_D$ for such $t \geqq t_0$ that $|||u(t) - v(t)||| \leqq \eta\} \Rightarrow \{|||u(t) - v(t)||| \leqq \eta$ for all $t \geqq t_0\}$ for all solutions u of the equation (2) and all $t_0 \in R^+$ such that $[t_0, \infty) \equiv \mathscr{D}(u)$.

In Section 1 we investigate the global uniform exponential stability of a solution v of the equation

$$u''(t) + A\,u(t) = \varepsilon[\sum a_i f_i(A)\, u(t) + 2b\, u'(t)]$$

where b, a_i, ($i = 1, \ldots, q$) are real numbers; $f_i : [\delta, \infty) \to R^1$ ($i = 1, \ldots, q$) are continuous functions with the following property: there exists a constant F^* such that

$$|f_i(s)| \leqq F^* s^{1/2} \quad \text{for} \quad s \geqq \delta\,, \quad i = 1, \ldots, q\,,$$

$\varepsilon > 0$ being a small parameter.

The symbol $\sum$ stands for $\sum_{i=1}^{q}$.

The global uniform exponential stability of a solution v of the equation

$$u''(t) + A\,u(t) = \varepsilon[\sum a_i(t) f_i(A)\,u(t) + 2\,b(t)\,u'(t)]$$

is investigated in Section 2.

The uniform exponential stability of a solution v of the equation (1) is investigated in Section 3.

The uniform stability at constantly acting disturbances of a solution v of the equation (1) is investigated in Section 4.

The results obtained in Sections 1–4 are applied to a certain special case in Section 5.

We shall not deal with the problem of existence, but we suppose that the solutions of all equations which are dealt with are determined uniquely by their initial conditions.

Remark 1. A constant equipped with a superscript $*$ (for example C^*) has the same meaning throughout the whole paper. Constants without this superscript can have various values in different considerations.

1. THE EQUATION WITH CONSTANT COEFFICIENTS

In this section we shall investigate the equation

$$(1.1)\qquad u''(t) + A\,u(t) = \varepsilon[\sum a_i f_i(A)\,u(t) + 2b\,u'(t)]\,,$$

b, a_i $(i = 1, \ldots, q)$ being real numbers; $f_i : [\delta, \infty) \to R^1$ being continuous functions with the property:

(1.2) there exists a constant F^* such that

$$|f_i(s)| \leqq F^* s^{1/2} \quad \text{for} \quad s \geqq \delta\,, \quad (i = 1, \ldots, q)\,.$$

Let us introduce the initial conditions

$$(1.3)\qquad u(t_0) = \varphi_0\,, \quad u'(t_0) = \varphi_1\,, \quad \varphi_i \in \mathscr{D}(A^{1-i/2})\,, \quad i = 0, 1\,, \quad t_0 \in R^+\,.$$

The following lemma is a special case of Lemma 2.1 from Section 2.

Lemma 1.1. *Let* $\Delta = \Delta(s) = s - \varepsilon \sum a_i f_i(s) - \varepsilon^2 b^2 > 0$ *for* $s \geqq \delta$. *Then the solution of the equation* (1.1) *satisfying the initial conditions* (1.3) *has the form*

$$u(t) = J(t)\,\varphi_0 + K(t - t_0)\,(-\varepsilon b \varphi_0 + \varphi_1)\,,$$

where

$$J(t) = \int_\delta^\infty e^{\varepsilon b(t-t_0)} \cos\left[(t - t_0)\, \Delta^{1/2}(s)\right] \mathrm{d}E(s)\,,$$

$$K(t) = \int_\delta^\infty e^{\varepsilon bt} \sin\left[t\, \Delta^{1/2}(s)\right] \Delta^{-1/2}(s)\, \mathrm{d}E(s)\,.$$

Theorem 1.1. *Suppose $b < 0$. Then there exists $\varepsilon_0 > 0$ so that the zero solution of the equation*

$$u''(t) + A\, u(t) = \varepsilon\left[\sum a_i f_i(A)\, u(t) + 2b\, u'(t)\right]$$

is globally uniformly exponentially stable with respect to the norm $|||\cdot|||$ for $\varepsilon \in (0, \varepsilon_0)$.

Moreover, it is

$$|||u(t)||| \leqq L e^{\varepsilon b(t-t_0)} |||u(t_0)||| \quad \text{for} \quad t \geqq t_0 \geqq 0 \tag{1.4}$$

and for some positive constant L.

Proof. Let us choose numbers $\varepsilon_0, C_1, C_2, C_3$ such that $\varepsilon_0 > 0$,

$$\Delta > 0 \quad \text{for} \quad \varepsilon \in (0, \varepsilon_0)\,, \quad \left[\varepsilon_0^2 b^2 + \Delta\right] \leqq C_1 s\,,$$

$$s\, \Delta^{-1} \leqq C_2\,, \quad \left[\varepsilon_0^2 b^2\, \Delta^{-1} + 1\right] \leqq C_3 \quad \text{for} \quad s \geqq \delta$$

and put

$$C_4 = 1 + (2C_1)^{1/2}\,, \quad C_5 = C_2^{1/2} + (2C_3)^{1/2}\,, \quad C_6 = \varepsilon_0 |b|\, \delta^{-1/2}\,,$$

$$C_7 = \max\left(C_5, C_4 + C_5 C_6\right), \quad L = 2^{1/2} C_7\,.$$

Then

$$\|A\, J(t)\, \varphi_0\|^2 = \int_\delta^\infty s^2 e^{2\varepsilon b(t-t_0)} \left|\cos\left[(t - t_0)\, \Delta^{1/2}\right]\right|^2 \mathrm{d}\|E(s)\, \varphi_0\|^2 \leqq$$

$$\leqq e^{2\varepsilon b(t-t_0)} \int_\delta^\infty s^2\, \mathrm{d}\|E(s)\, \varphi_0\|^2 = e^{2\varepsilon b(t-t_0)} \|A\varphi_0\|^2\,,$$

$$\|A^{1/2}\, J'(t)\, \varphi_0\|^2 = \int_\delta^\infty s e^{2\varepsilon b(t-t_0)} \{\varepsilon b \cos\left[(t - t_0)\, \Delta^{1/2}\right] -$$

$$- \Delta^{1/2} \sin\left[(t - t_0)\, \Delta^{1/2}\right]\}^2\, \mathrm{d}\|E(s)\, \varphi_0\|^2 \leqq$$

$$\leqq 2e^{2\varepsilon b(t-t_0)} \int_\delta^\infty s(\varepsilon_0^2 b^2 + \Delta)\, \mathrm{d}\|E(s)\, \varphi_0\|^2 \leqq 2C_1 e^{2\varepsilon b(t-t_0)} \|A\varphi_0\|^2\,.$$

Thus

$$|||J(t)\, \varphi_0||| \leqq \|A\, J(t)\, \varphi_0\| + \|A^{1/2}\, J'(t)\, \varphi_0\| \leqq C_4 e^{\varepsilon b(t-t_0)} \|A\varphi_0\|\,. \tag{1.5}$$

Similarly

$$\|AK(t - t_0)\,\psi\|^2 = \int_\delta^\infty s^2 e^{2\varepsilon b(t-t_0)} |\sin [(t - t_0)\,\Delta^{1/2}]|^2\, \Delta^{-1}\, \mathrm{d}\|E(s)\,\psi\|^2 \leqq$$

$$\leqq C_2 e^{2\varepsilon b(t-t_0)} \int_\delta^\infty s\, \mathrm{d}\|E(s)\,\psi\|^2 = C_2 e^{2\varepsilon b(t-t_0)} \|A^{1/2}\psi\|^2,$$

$$\|A^{1/2}K'(t - t_0)\,\psi\|^2 = \int_\delta^\infty s e^{2\varepsilon b(t-t_0)} \{\varepsilon b \sin [(t - t_0)\,\Delta^{1/2}]\,\Delta^{-1/2} +$$

$$+ \cos [(t - t_0)\,\Delta^{1/2}]\}^2\, \mathrm{d}\|E(s)\,\psi\|^2 \leqq$$

$$\leqq 2e^{2\varepsilon b(t-t_0)} \int_\delta^\infty s(\varepsilon_0^2 b^2\, \Delta^{-1} + 1)\, \mathrm{d}\|E(s)\,\psi\|^2 \leqq 2C_3 e^{2\varepsilon b(t-t_0)} \|A^{1/2}\psi\|^2$$

and so

$$\text{(1.6)} \qquad |||K(t - t_0)\,\psi||| \leqq C_5 e^{\varepsilon b(t-t_0)} \|A^{1/2}\psi\|, \quad \text{for} \quad \psi \in \mathscr{D}(A^{1/2}).$$

It is

$$\text{(1.7)} \quad \|A^{1/2}\varphi_0\|^2 = \int_\delta^\infty s\, \mathrm{d}\|E(s)\,\varphi_0\|^2 \leqq \delta^{-1} \int_\delta^\infty s^2\, \mathrm{d}\|E(s)\,\varphi_0\|^2 = \delta^{-1}\|A\varphi_0\|^2.$$

It follows from (1.6), (1.7) that

$$\text{(1.8)} \quad |||K(t - t_0)\,(-\varepsilon b\varphi_0 + \varphi_1)||| \leqq C_5(C_6\|A\varphi_0\| + \|A^{1/2}\varphi_1\|)\, e^{\varepsilon b(t-t_0)}.$$

Now we obtain from (1.5), (1.8)

$$|||u(t)||| \leqq |||J(t)\,\varphi_0||| + |||K(t - t_0)\,(-\varepsilon b\varphi_0 + \varphi_1)||| \leqq$$

$$\leqq e^{\varepsilon b(t-t_0)}[(C_4 + C_5 C_6)\,\|A\varphi_0\| + C_5\|A^{1/2}\varphi_1\|] \leqq$$

$$\leqq e^{\varepsilon b(t-t_0)}\, C_7(\|A\varphi_0\| + \|A^{1/2}\varphi_1\|) \leqq L e^{\varepsilon b(t-t_0)} |||u(t_0)|||.$$

This proves the global uniform exponential stability of the zero solution and the estimate (1.4).

2. THE EQUATION WITH VARIABLE COEFFICIENTS

We shall investigate the equation

$$\text{(2.1)} \qquad u''(t) + A\,u(t) = \varepsilon[\textstyle\sum a_i(t)\, f_i(A)\, u(t) + 2\, b(t)\, u'(t)]$$

in this section ($a_i(t)$, $b(t)$ are functions of the variables t, x).

Let b^* ($b^* \leqq 0$), a_i^* ($i = 1, \ldots, q$) be real numbers. We shall write the equation (2.1) in the form

$$(2.1)' \qquad u''(t) - 2\varepsilon b^* u'(t) + A\, u(t) - \varepsilon \sum a_i^* f_i(A)\, u(t) = f^*(t, u(t))$$

where

$$f^*(t, u) = \varepsilon[2(b(t) - b^*)\, u'(t) + \sum(a_i(t) - a_i^*) f_i(A)\, u(t)]\,.$$

We shall further need the following lemmas:

Lemma 2.1 ([1] p. 649). *Let $\Delta = \Delta(s) = s - \varepsilon \sum a_i^* f_i(s) - \varepsilon^2 b^{*2} > 0$ for $s \geqq \delta$. Then if u is a solution of the equation (2.1)′, satisfying the initial conditions (1.3), then*

$$u(t) = J(t)\, \varphi_0 + K(t - t_0)(-\varepsilon b^* \varphi_0 + \varphi_1) + \int_{t_0}^{t} K(t - \tau) f^*(\tau, u(\tau))\, d\tau\,,$$

where

$$J(t) = \int_{\delta}^{\infty} e^{\varepsilon b^*(t - t_0)} \cos\left[(t - t_0)\, \Delta^{1/2}(s)\right] dE(s)\,,$$

$$K(t) = \int_{\delta}^{\infty} e^{\varepsilon b^* t} \sin\left[t\, \Delta^{1/2}(s)\right] \Delta^{-1/2}(s)\, dE(s)\,.$$

Lemma 2.2 ([2] p. 155). *Suppose*

$$\varphi(t) \leqq \alpha e^{-\nu(t - t_0)} + \beta \int_{t_0}^{t} e^{-\nu(t-\tau)}\, p(\tau)\, \varphi(\tau)\, d\tau \quad \text{for} \quad t \geqq t_0$$

where $p(t)$ is a nonnegative continuous function, α, β, ν are constants. Then

$$\varphi(t) \leqq \alpha \exp\left(-\nu(t - t_0) + \beta \int_{t_0}^{t} p(\tau)\, d\tau\right).$$

Let v_1 be a solution of the equation

$$u''(t) - 2\varepsilon b^* u'(t) + A\, u(t) - \varepsilon \sum a_i^* f_i(A)\, u(t) = 0$$

satisfying the initial conditions (1.3). With respect to Lemma 2.1, we can write any solution u of the equation (2.1)′ fulfilling the initial conditions (1.3), in the form

$$(2.2) \qquad u(t) = v_1(t) + \int_{t_0}^{t} K(t - \tau) f^*(\tau, u(\tau))\, d\tau\,.$$

Lemma 2.3. *Suppose that $u \in \mathscr{U}$ and $[t_0, t] \subset \mathscr{D}(u)$. Then to an arbitrarily chosen $\eta > 0$ there exists $\varepsilon_0 > 0$ such that*

$$(2.3) \qquad \left|\left|\left| \int_{t_0}^{t} K(t-\tau)\, F(\tau, u(\tau))\, d\tau \right|\right|\right| \leqq$$

$$\leqq 2(1+\eta) \int_{t_0}^{t} e^{\varepsilon b^*(t-\tau)} \|A^{1/2} F(\tau, u(\tau))\|\, d\tau\,, \quad \textit{if} \quad F(\tau, u(\tau)) \in C([t_0, t], \mathscr{D}(A^{1/2}))\,,$$
$$\varepsilon \in (0, \varepsilon_0)\,.$$

If moreover, $b^* = 0$, $a_i^* = 0$ $(i = 1, \ldots, q)$, *then* (2.3) *is fulfilled also for* $\eta = 0$.

Proof. Let us find $\varepsilon_0 > 0$ such that

$$\Delta(s) > 0\,, \quad s^{1/2} \Delta^{-1/2}(s) \leqq 1 + \eta\,, \quad \varepsilon|b^*|\, \Delta^{-1/2}(s) + 1 \leqq 1 + \eta$$
$$\text{for} \quad s \geqq \delta \quad \text{and} \quad \varepsilon \in (0, \varepsilon_0)\,.$$

Then

$$(2.4) \qquad \left\| A \int_{\delta}^{\infty} e^{\varepsilon b^*(t-\tau)} \sin[(t-\tau)\, \Delta^{1/2}]\, \Delta^{-1/2}\, dE(s)\, F(\tau, u(\tau)) \right\|^2 =$$
$$= e^{2\varepsilon b^*(t-\tau)} \int_{\delta}^{\infty} s^2 |\sin[(t-\tau)\, \Delta^{1/2}]\, \Delta^{-1/2}|^2\, d\|E(s)\, F(\tau, u(\tau))\|^2 \leqq$$
$$\leqq e^{2\varepsilon b^*(t-\tau)} \int_{\delta}^{\infty} s^2 \Delta^{-1}\, d\|E(s)\, F(\tau, u(\tau))\|^2 \leqq (1+\eta)^2\, e^{2\varepsilon b^*(t-\tau)} \|A^{1/2} F(\tau, u(\tau))\|^2$$

$$(2.5) \qquad \left\| A^{1/2} \int_{\delta}^{\infty} e^{\varepsilon b^*(t-\tau)} \{\varepsilon b^* \sin[(t-\tau)\, \Delta^{1/2}]\, \Delta^{-1/2} + \cos[(t-\tau)\, \Delta^{1/2}]\} \right.$$
$$\left. dE(s)\, F(\tau, u(\tau)) \right\|^2 \leqq e^{2\varepsilon b^*(t-\tau)} \int_{\delta}^{\infty} s(\varepsilon|b^*|\, \Delta^{-1/2} + 1)^2\, d\|E(s)\, F(\tau, u(\tau))\|^2 \leqq$$
$$\leqq (1+\eta)^2\, e^{2\varepsilon b^*(t-\tau)} \|A^{1/2} F(\tau, u(\tau))\|^2\,.$$

Using (2.4), (2.5) and the closedness of the operators A, $A^{1/2}$, we get

$$\left|\left|\left| \int_{t_0}^{t} K(t-\tau)\, F(\tau, u(\tau))\, d\tau \right|\right|\right| \leqq$$
$$\leqq \left\| A \int_{t_0}^{t} \int_{\delta}^{\infty} e^{\varepsilon b^*(t-\tau)} \sin[(t-\tau)\, \Delta^{1/2}]\, \Delta^{-1/2}\, dE(s)\, F(\tau, u(\tau))\, d\tau \right\| +$$
$$+ \left\| A^{1/2} \int_{t_0}^{t} \int_{\delta}^{\infty} e^{\varepsilon b^*(t-\tau)} \{\varepsilon b^* \sin[(t-\tau)\, \Delta^{1/2}]\, \Delta^{-1/2} + \cos[(t-\tau)\, \Delta^{1/2}]\} \right.$$
$$\left. dE(s)\, F(\tau, u(\tau))\, d\tau \right\| \leqq \int_{t_0}^{t} \left\{ \left\| A \int_{\delta}^{\infty} e^{\varepsilon b^*(t-\tau)} \sin[(t-\tau)\, \Delta^{1/2}]\, \Delta^{-1/2}\, dE(s)\, F(\tau, u(\tau)) \right\| + \right.$$
$$+ \left\| A^{1/2} \int_{\delta}^{\infty} e^{\varepsilon b^*(t-\tau)} \{\varepsilon b^* \sin[(t-\tau)\, \Delta^{1/2}]\, \Delta^{-1/2} + \cos[(t-\tau)\, \Delta^{1/2}]\} \right.$$

$$\mathrm{d}E(s)\,F(\tau, u(\tau))\Big\|\Big\}\,\mathrm{d}\tau \leqq \int_{t_0}^{t} 2(1+\eta)\,e^{\varepsilon b^*(t-\tau)}\|A^{1/2}\,F(\tau, u(\tau))\|\,\mathrm{d}\tau\,.$$

The last assertion of Lemma 2.3 is evident.

Let us introduce the following condition:

($\mathscr{L}$) There exists a constant K^* such that

$$\|A^{1/2}[(b(t) - b^*)\,u'(t)]\| \leqq K^*|||u(t)|||\,, \tag{2.6}$$

$$\|A^{1/2}[(a_i(t) - a_i^*)\,f_i(A)\,u(t)]\| \leqq K^*|||u(t)|||\,, \quad (i = 1, \ldots, q) \quad \text{for} \quad u \in \mathscr{U}\,. \tag{2.7}$$

Theorem 2.1. *Suppose that there exist real numbers b^*, a_i^* $(i = 1, \ldots, q)$, K^* such that the condition ($\mathscr{L}$) is fulfilled and that the following inequality holds:*

$$b^* + 2(q + 2)\,K^* < 0\,.$$

Then there exists $\varepsilon_0^* > 0$ such that the zero solution of the equation (2.1) is globally uniformly exponentially stable with respect to the norm $|||\cdot|||$ for $\varepsilon \in (0, \varepsilon_0^*)$.

Proof. It is clear that $b^* < 0$. Let us choose $\eta > 0$ so small that

$$b^* + 2(1 + \eta)\,(q + 2)\,K^* < 0\,. \tag{2.8}$$

Let us find ε_0^* as in Lemma 2.3 to this η.

Then with help of (2.2), Theorem 1.1 and Lemma 2.3 we conclude

$$|||u(t)||| \leqq Le^{\varepsilon b^*(t-t_0)}|||u(t_0)||| + \\ + \varepsilon \int_{t_0}^{t} 2(1 + \eta)\,e^{\varepsilon b^*(t-\tau)}(q + 2)\,K^*|||u(\tau)|||\,\mathrm{d}\tau$$

for a positive constant L. Using Lemma 2.2 we can derive the inequality

$$|||u(t)||| \leqq L|||u(t_0)|||\exp\left[\varepsilon b^*(t - t_0) + 2\varepsilon(1 + \eta)\,(q + 2)\,K^*(t - t_0)\right],$$

which, by virtue of (2.8), proves Theorem 2.1.

3. THE GENERAL NONLINEAR EQUATION

We shall investigate the uniform exponential stability of a solution $v : R^+ \to H$ of the equation

$$u''(t) + A\,u(t) = F(t, u(t))\,. \tag{3.1}$$

We do not write the small parameter ε on the right-hand side, because the results derived are valid for the equation without ε as well.

Let us introduce the following assumption for the function F:

$$F(t, v(t) + u(t)) = F(t, v(t)) + F_L(t, u(t)) + F_N(t, u(t)), \tag{3.2}$$

where

$F_L(t, u)$ is linear in the variable u,

$F(t, v(t)) \in C(R^+, \mathscr{D}(A^{1/2}))$,

$F_L(t, u(t)) \in C(\mathscr{D}(u), \mathscr{D}(A^{1/2}))$,

$F_N(t, u(t)) \in C(\mathscr{D}(u), \mathscr{D}(A^{1/2}))$.

Theorem 3.1. *Suppose that* (3.2) *holds. Then if the zero solution of the equation*

$$u''(t) + A\,u(t) = F_L(t, u(t)) + F_N(t, u(t)) \tag{3.3}$$

is uniformly exponentially stable with respect to the norm $|||\cdot|||$, *the solution* v *of the equation* (3.1) *is also uniformly exponentially stable with respect to the norm* $|||\cdot|||$.

Proof. Let us write a solution u of the equation (3.1) in the form $u = v + w$. Since v is a solution of (3.1), w satisfies the equation (3.3). The proposition of Theorem 3.1 follows immediately from the relation $u - v = w$.

The aim of our further considerations is to show that under certain assumptions the uniform exponential stability of a solution v of the equation (3.1) depends on the linear part F_L only. We shall call the equation

$$u''(t) + A\,u(t) = F_L(t, u(t))$$

the linearized equation.

Assume that the following condition is fulfilled:

($\mathscr{EL}$) If $\varphi_i \in \mathscr{D}(A^{1-i/2})$, $i = 0, 1$, then a solution of the linearized equation satisfying the initial conditions (1.3), exists for arbitrary $t_0 \in R^+$.

Remember that according to Lemma 2.1 it holds

$$u(t) = J(t)\,\varphi_0 + K(t - t_0)\,\varphi_1 + \int_{t_0}^{t} K(t - \tau)\,F(\tau, u(\tau))\,d\tau, \tag{3.4}$$

where

$$J(t) = \int_\delta^\infty \cos\left[(t - t_0)\, s^{1/2}\right] \mathrm{d}E(s)\,, \qquad K(t) = \int_\delta^\infty \sin\left(ts^{1/2}\right) s^{-1/2}\, \mathrm{d}E(s)$$

for the solution u of the equation (3.1) satisfying the initial conditions (1.3).

Let us introduce the following condition:

$(\mathcal{NL})$ There exist constants C^*, $R^* > 0$ such that if $u \in \mathcal{U}$, $t \in \mathcal{D}(u)$, then

$$\|A^{1/2} F_L(t, u(t))\| \leqq C^* |||u(t)||| \,; \tag{3.5}$$

if moreover, $|||u(t)||| \leqq R^*$ then

$$\|A^{1/2} F_N(t, u(t))\| \leqq C^* |||u(t)|||^2 \,. \tag{3.6}$$

Theorem 3.2. *Suppose that (3.2) holds for the function F, that the conditions $(\mathcal{NL})$ and $(\mathcal{EL})$ are fulfilled and the zero solution of the linearized equation*

$$u''(t) + A\, u(t) = F_L(t, u(t))$$

is uniformly exponentially stable with respect to the norm $|||\cdot|||$. Then the solution v of the equation

$$u''(t) + A\, u(t) = F(t, u(t))$$

is also uniformly exponentially stable with respect to the norm $|||\cdot|||$.

Proof. It follows from Theorem 3.1 that it suffices to prove the uniform exponential stability of the zero solution of the equation (3.3). But we shall prove several lemmas before.

Lemma 3.1. *Suppose $\varphi_0 \in \mathcal{D}(A)$, $\varphi_1 \in \mathcal{D}(A^{1/2})$. Then*

$$|||J(t)\, \varphi_0 + K(t - t_0)\, \varphi_1||| \leqq 2(\|A\varphi_0\|^2 + \|A^{1/2}\varphi_1\|^2)^{1/2} \,.$$

Proof. We can easily find that

$$\|A\, J(t)\, \varphi_0\|^2 = \int_\delta^\infty s^2 |\cos\left[(t - t_0)\, s^{1/2}\right]|^2\, \mathrm{d}\|E(s)\, \varphi_0\|^2 \leqq \int_\delta^\infty s^2\, \mathrm{d}\|E(s)\, \varphi_0\|^2 = \|A\varphi_0\|^2\,,$$

$$\|A^{1/2}\, J'(t)\, \varphi_0\|^2 = \int_\delta^\infty s^2 |\sin\left[(t - t_0)\, s^{1/2}\right]|^2\, \mathrm{d}\|E(s)\, \varphi_0\|^2 \leqq \int_\delta^\infty s^2\, \mathrm{d}\|E(s)\, \varphi_0\|^2 = \|A\varphi_0\|^2.$$

Hence

$$|||J(t)\, \varphi_0|||^2 = \|A\, J(t)\, \varphi_0\|^2 + \|A^{1/2}\, J'(t)\, \varphi_0\|^2 \leqq 2\|A\varphi_0\|^2 \,. \tag{3.7}$$

Similarly

$$\|A\,K(t - t_0)\,\varphi_1\|^2 = \int_\delta^\infty s|\sin[(t - t_0)\,s^{1/2}]|^2\,\mathrm{d}\|E(s)\,\varphi_1\|^2 \leqq \|A^{1/2}\varphi_1\|^2\,,$$

$$\|A^{1/2}\,K'(t - t_0)\,\varphi_1\|^2 = \int_\delta^\infty s|\cos[(t - t_0)\,s^{1/2}]|^2\,\mathrm{d}\|E(s)\,\varphi_1\|^2 \leqq \|A^{1/2}\varphi_1\|^2\,.$$

Hence

$$(3.8)\qquad |||K(t - t_0)\,\varphi_1|||^2 =$$

$$= \|A\,K(t - t_0)\,\varphi_1\|^2 + \|A^{1/2}\,K'(t - t_0)\,\varphi_1\|^2 \leqq 2\|A^{1/2}\varphi_1\|^2\,.$$

Now (3.7) and (3.8) imply

$$|||J(t)\,\varphi_0 + K(t - t_0)\,\varphi_1|||^2 \leqq 2(|||J(t)\,\varphi_0|||^2 + |||K(t - t_0)\,\varphi_1|||^2) \leqq$$

$$\leqq 4(\|A\varphi_0\|^2 + \|A^{1/2}\varphi_1\|^2)$$

which proves the lemma.

The following three lemmas are easy consequences of Lemma 2.3 and the condition $(\mathscr{N}\mathscr{L})$.

Lemma 3.2. *Let* (3.2) *hold for a function F and let the condition $(\mathscr{N}\mathscr{L})$ be fulfilled. Then*

$$\left|\left|\left|\int_{t_0}^t K(t - \tau)\,[F_L(\tau, u_1(\tau)) - F_L(\tau, u_2(\tau))]\,\mathrm{d}\tau\right|\right|\right| \leqq 2C^* \int_{t_0}^t |||u_1(\tau) - u_2(\tau)|||\,\mathrm{d}\tau$$

for $u_i \in \mathscr{U}$, $t_0 \in R^+$, $t \in R^+$ such that $[t_0, t] \subset \mathscr{D}(u_i)$, $i = 1, 2$.

Lemma 3.3. *Let the assumptions of Lemma* 3.2 *be fulfilled. Then*

$$\left|\left|\left|\int_{t_0}^t K(t - \tau)\,F_L(\tau, u(\tau))\,\mathrm{d}\tau\right|\right|\right| \leqq 2C^* \int_{t_0}^t |||u(\tau)|||\,\mathrm{d}\tau$$

for $u \in \mathscr{U}$, $t_0 \in R^+$, $t \in R^+$ so that $[t_0, t] \subset \mathscr{D}(u)$.

Lemma 3.4. *Let* (3.2) *hold for a function F and let the condition $(\mathscr{N}\mathscr{L})$ be fulfilled. Then*

$$\left|\left|\left|\int_{t_0}^t K(t - \tau)\,F_N(\tau, u(\tau))\,\mathrm{d}\tau\right|\right|\right| \leqq 2C^* \int_{t_0}^t |||u(\tau)|||^2\,\mathrm{d}\tau$$

for $u \in \mathscr{U}$, $t_0 \in R^+$, $t \in R^+$ such that $[t_0, t] \subset \mathscr{D}(u)$ and $|||u(\tau)||| \leqq R^*$ for $\tau \in [t_0, t]$.

Let us continue the proof of Theorem 3.2.

Let $\varphi_i \in \mathscr{D}(A^{1-i/2})$, $i = 0, 1$. Then there exists a solution u_L of the linearized equation satisfying the initial conditions (1.3).

It follows from (3.4) and the assumptions of the theorem that

(3.9) $$u_L(t) = J(t)\,\varphi_0 + K(t - t_0)\,\varphi_1 + \int_{t_0}^{t} K(t - \tau)\,F_L(\tau, u_L(\tau))\,d\tau\,,$$

(3.10) there exist positive numbers L, ω such that

$$|||u_L(t)||| \leqq Le^{-\omega(t-t_0)}|||u_L(t_0)||| \quad \text{for} \quad t \geqq t_0\,.$$

It holds

(3.11) $$u_N(t) = J(t)\,\varphi_0 + K(t - t_0)\,\varphi_1 + \int_{t_0}^{t} K(t - \tau)\,F_L(\tau, u_N(\tau))\,d\tau + \\ + \int_{t_0}^{t} K(t - \tau)\,F_N(\tau, u_N(\tau))\,d\tau$$

for the solution u_N of the nonlinear equation (3.3) satisfying (1.3).

Let $0 < \omega_1 < \omega$, $L_1 > L$.

Let us choose a number $h > 0$ such that

$$Le^{-(\omega-\omega_1)h} < 1\,.$$

Let us find a number $R_1 \in (0, R^*]$ to this h such that

(3.12) $$L + 4C^*R_1he^{(4C^*+2C^*R_1+\omega_1)h} \leqq L_1\,,$$

(3.13) $$Le^{-(\omega-\omega_1)h} + 4C^*R_1he^{(4C^*+2C^*R_1+\omega_1)h} \leqq 1\,.$$

Let us restrict ourselves to such $\varphi_i \in \mathscr{D}(A^{1-i/2})$, $i = 0, 1$ that

$$|||u_L(t_0)||| = |||u_N(t_0)||| = (\|A\varphi_0\|^2 + \|A^{1/2}\varphi_1\|^2)^{1/2} \leqq r^*\,,$$

where r^* is a fixed number, $r^* \in (0, R_1)$.

Suppose

(3.14) there exists a number $\tilde{h}$ such that $0 < \tilde{h} \leqq h$ and

$$|||u_N(\tau)||| < R_1 \quad \text{for} \quad \tau \in [t_0, t_0 + \tilde{h}),$$

$$|||u_N(t_0 + \tilde{h})||| = R_1.$$

Using Lemmas 3.1, 3.3, 3.4 we get from (3.11)

$$|||u_N(t)||| \leqq 2|||u_N(t_0)||| + 2C^*\left\{\int_{t_0}^{t} |||u_N(\tau)|||\,d\tau + \int_{t_0}^{t} |||u_N(\tau)|||^2\,d\tau\right\} \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}]\,.$$

Hence

$$|||u_N(t)||| \leqq 2|||u_N(t_0)||| + 2C^*(1 + R_1) \int_{t_0}^{t} |||u_N(\tau)||| \, d\tau$$

and further

$$(3.15) \qquad |||u_N(t)||| \leqq 2|||u_N(t_0)||| \, e^{2C^*(1+R_1)\tilde{h}} \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}] .$$

Subtracting (3.9) and (3.11), using Lemmas 3.2, 3.4 and the eastimate (3.15), we get

$$|||u_N(t) - u_L(t)||| \leqq 2C^* \int_{t_0}^{t} |||u_N(\tau) - u_L(\tau)||| \, d\tau + 2C^* \int_{t_0}^{t} |||u_N(\tau)|||^2 \, d\tau \leqq$$

$$\leqq 2C^* \int_{t_0}^{t} |||u_N(\tau) - u_L(\tau)||| \, d\tau + 4C^* R_1 \tilde{h} e^{2C^*(1+R_1)\tilde{h}} |||u_N(t_0)||| \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}] .$$

Thus

$$(3.16) \qquad |||u_N(t) - u_L(t)||| \leqq 4C^* R_1 \tilde{h} e^{2C^*(2+R_1)\tilde{h}} |||u_N(t_0)||| \leqq$$

$$\leqq 4C^* R_1 \tilde{h} e^{(4C^* + 2C^* R_1 + \omega_1)\tilde{h}} e^{-\omega_1 (t - t_0)} |||u_N(t_0)||| \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}] .$$

We obtain from (3.10), (3.16) that

$$(3.17) \qquad |||u_N(t)||| \leqq |||u_L(t)||| + |||u_N(t) - u_L(t)||| \leqq$$

$$\leqq [L + 4C^* R_1 \tilde{h} e^{(4C^* + 2C^* R_1 + \omega_1)\tilde{h}}] \, e^{-\omega_1 (t - t_0)} |||u_N(t_0)||| \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}] .$$

As $\tilde{h} \leqq h$ it follows from (3.17) that

$$|||u_N(t)||| \leqq [L + 4C^* R_1 h e^{(4C^* + 2C^* R_1 + \omega_1)h}] \, r^* \leqq L_1 r^* < R_1 \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}] .$$

This is a contradiction with (3.14). Thus

$$(3.18) \qquad |||u_N(\tau)||| < R_1 \leqq R^* \quad \text{for} \quad \tau \in [t_0, t_0 + h] .$$

Using (3.10), (3.16), (3.18), we conclude

$$(3.19) \quad |||u_N(t_0 + h)||| \leqq [L e^{-(\omega - \omega_1)h} + 4C^* R_1 h e^{(4C^* + 2C^* R_1 + \omega_1)h}] \, e^{-\omega_1 h} |||u_N(t_0)||| .$$

It follows from (3.12), (3.13), (3.17), (3.19) that

$$(3.20) \qquad |||u_N(t)||| \leqq L_1 e^{-\omega_1 (t - t_0)} |||u_N(t_0)||| \quad \text{for} \quad t \in [t_0, t_0 + h] ,$$

$$(3.21) \qquad |||u_N(t_0 + h)||| \leqq e^{-\omega_1 h} |||u_N(t_0)||| .$$

The uniform exponential stability of the zero solution of the equation (3.3) follows easily from the inequalities (3.20), (3.21), because

$$|||u_N(t_0 + nh)||| \leqq e^{-\omega_1 nh}|||u_N(t_0)|||$$

for an arbitrary positive integer n.

We can find a positive integer n such that $t = t_0 + nh + s$, $s \in [0, h)$ for $0 \leqq \leqq t_0 < t$. Then

$$|||u_N(t)||| = |||u_N(t_0 + nh + s)||| \leqq L_1 e^{-\omega_1 s}|||u_N(t_0 + nh)||| \leqq$$
$$\leqq L_1 e^{-\omega_1(s+nh)}|||u_N(t_0)||| = L_1 e^{-\omega_1(t-t_0)}|||u_N(t_0)||| .$$

This proves the theorem.

It is clear that we have proved even

Corollary 3.1. *Suppose that the assumptions of Theorem 3.2 are fulfilled and the inequality*

$$|||u_L(t)||| \leqq L e^{-\omega(t-t_0)}|||u_L(t_0)||| \quad \text{for} \quad t \geqq t_0 \geqq 0$$

holds for the solution u_L of the linearized equation. Then if $L_1 > L$, $0 < \omega_1 < \omega$, it is

$$|||u_N(t)||| \leqq L_1 e^{-\omega_1(t-t_0)}|||u_N(t_0)||| \quad \text{for} \quad t \geqq t_0 \geqq 0\,, \quad |||u_N(t_0)||| \leqq r^*\,,$$

where u_N is the solution of (3.3).

4. THE STABILITY AT CONSTANTLY ACTING DISTURBANCES

We shall now investigate the uniform stability at constantly acting disturbances of a solution $v : R^+ \to H$ of the equation

$$u''(t) + A\,u(t) = F(t, u(t))\,. \tag{4.1}$$

Theorem 4.1. *Let* (3.2) *hold for the function F and let the zero solution of the equation*

$$u''(t) + A\,u(t) = F_L(t, u(t)) + F_N(t, u(t)) \tag{4.2}$$

be uniformly stable at constantly acting disturbances with respect to the norm $|||\cdot|||$. Then the solution v is also uniformly stable at constantly acting disturbances with respect to the norm $|||\cdot|||$.

Proof. Let us write a solution u of the equation (4.1) in the form $u = v + w$. The function w fulfils the equation (4.2), because v is a solution of (4.1). The proposition of the theorem follows immediately from the relation $u - v = w$.

Further we shall prove that under certain conditions the solution v of the equation (4.1) is uniformly stable at constantly acting disturbances if the zero solution of the linearized equation is uniformly exponentially stable.

Theorem 4.2. *Let (3.2) hold for the function F and let the conditions $(\mathscr{E}\mathscr{L})$, $(\mathscr{N}\mathscr{L})$ be fulfilled. Suppose that the zero solution of the linearized equation*

$$u''(t) + A\,u(t) = F_L(t, u(t))$$

is uniformly exponentially stable with respect to the norm $|||\cdot|||$. Then the solution v of the equation

$$u''(t) + A\,u(t) = F(t, u(t))$$

is uniformly stable at constantly acting disturbances with respect to the norm $|||\cdot|||$.

Proof. It follows from Theorem 4.1 that it suffices to prove the uniform stability at constantly acting disturbances of the zero solution of the equation (4.2).

The corresponding equation with the disturbance to the equation (4.2) is

$$u''(t) + A\,u(t) = F_L(t, u(t)) + F_N(t, u(t)) + R(t, u(t))\,. \tag{4.3}$$

Remember that if u_L is a solution of the linearized equation fulfilling the initial conditions (1.3), then

$$u_L(t) = J(t)\,\varphi_0 + K(t - t_0)\,\varphi_1 + \int_{t_0}^{t} K(t - \tau)\,F_L(\tau, u_L(\tau))\,d\tau\,,$$

(4.4) there exist positive numbers L, ω such that

$$|||u_L(t)||| \leqq L e^{-\omega(t-t_0)} |||u_L(t_0)||| \quad \text{for} \quad t \geqq t_0 \geqq 0\,.$$

If u_D is a solution of the equation (4.3) fulfilling the initial conditions (1.3), then

$$u_D(t) = J(t)\,\varphi_0 + K(t - t_0)\,\varphi_1 + \int_{t_0}^{t} K(t - \tau)\,F_L(\tau, u_D(\tau))\,d\tau + \tag{4.5}$$
$$+ \int_{t_0}^{t} K(t - \tau)\,F_N(\tau, u_D(\tau))\,d\tau + \int_{t_0}^{t} K(t - \tau)\,R(\tau, u_D(\tau))\,d\tau\,.$$

It follows from Lemma 2.3 that

$$\|A^{1/2}\,R(t, u(t))\| \leqq \eta_D \Rightarrow \left|\left|\left|\int_{t_0}^{t} K(t - \tau)\,R(\tau, u(\tau))\,d\tau\right|\right|\right| \leqq \tag{4.6}$$
$$\leqq 2\,\eta_D(t - t_0) \quad \text{if} \quad u \in \mathscr{U}\,, \quad [t_0, t] \subset \mathscr{D}(u)\,, \quad |||u(\tau)||| \leqq r \quad \text{for} \quad \tau \in [t_0, t]\,.$$

Let $\eta \in (0, r]$ be given. Without loss of generality we may suppose that $\eta \leqq R^*$.

Let us choose a number $h > 0$ such that

$$Le^{-\omega h} < 1 ,$$

further let us choose a number $R_1 \in (0, \eta]$ such that

$$Le^{-\omega h} + 4C^* R_1 h e^{2C^*(2+\eta)h} < 1 .$$

Let us choose $\eta_0 \in (0, R_1)$ to these fixed h, R_1 such that

$$2\eta_0 e^{2C^*(1+\eta)h} < R_1 . \tag{4.7}$$

Finally, we may choose a number $\eta_D > 0$ such that

$$(2\eta_0 + 2\eta_D h)\, e^{2C^*(1+\eta)h} < R_1 , \tag{4.8}$$

$$\begin{aligned} &[Le^{-\omega h} + 4C^* R_1 h e^{2C^*(2+\eta)h}]\, \eta_0 + \\ &+ [2he^{2C^*h} + 4C^* R_1 h^2 e^{2C^*(2+\eta)h}]\, \eta_D \leqq \eta_0 . \end{aligned} \tag{4.9}$$

The theorem will be proved if we can verify the implication

(4.10) If $\{|||u_D(t_0)||| = (\|A\varphi_0\|^2 + \|A^{1/2}\varphi_1\|^2)^{1/2} \leqq \eta_0$,

$\|A^{1/2} R(t, u_D(t))\| \leqq \eta_D$ for all $t \geqq t_0$ such that $|||u_D(t)||| \leqq \eta\}$,

then $|||u_D(t)||| \leqq \eta$ for $t \geqq t_0$.

Suppose

(4.11) there exists a number $\tilde{h} \in (0, h]$ such that

$|||u_D(\tau)||| < R_1$ for $\tau \in [t_0, t_0 + \tilde{h})$,

$|||u_D(t_0 + \tilde{h})||| = R_1$.

Since $R_1 \leqq \eta \leqq R^*$ we can obtain with help of Lemmas 3.1, 3.3, 3.4 and the relation (4.6) the inequality

$$|||u_D(t)||| \leqq 2|||u_D(t_0)||| + 2C^* \int_{t_0}^{t} |||u_D(\tau)|||\, d\tau + 2C^* \int_{t_0}^{t} |||u_D(\tau)|||^2\, d\tau + 2\eta_D(t - t_0) \leqq$$

$$\leqq 2\eta_0 + 2C^*(1 + \eta) \int_{t_0}^{t} |||u_D(\tau)|||\, d\tau + 2\eta_D \tilde{h} \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}] .$$

Hence using Lemma 2.2 we have

$$|||u_D(t_0 + \tilde{h})||| \leqq (2\eta_0 + 2\eta_D \tilde{h})\, e^{2C^*(1+\eta)\tilde{h}} \leqq (2\eta_0 + 2\eta_D h)\, e^{2C^*(1+\eta)h} < R_1 . \tag{4.12}$$

This is a contradiction with (4.11). Thus we have proved that

$$|||u_D(t)||| < R_1 \leqq \eta \quad \text{for} \quad t \in [t_0, t_0 + h]. \tag{4.13}$$

Using Lemmas 3.2, 3.4, the relation $\eta \leqq R^*$ and (4.12), (4.13) we conclude

$$|||u_L(t) - u_D(t)||| \leqq 2C^* \int_{t_0}^{t} |||u_L(\tau) - u_D(\tau)||| \, d\tau + 2C^* \int_{t_0}^{t} |||u_D(\tau)|||^2 \, d\tau +$$

$$+ 2\eta_D(t - t_0) \leqq 2C^* \int_{t_0}^{t} |||u_L(\tau) - u_D(\tau)||| \, d\tau + 4C^* R_1 h(\eta_0 + \eta_D h) \, e^{2C^*(1+\eta)h} +$$

$$+ 2\eta_D h \quad \text{for} \quad t \in [t_0, t_0 + h].$$

Hence it follows by virtue of Lemma 2.2

$$|||u_L(t) - u_D(t)||| \leqq \{4C^* R_1 h e^{2C^*(1+\eta)h} \eta_0 + \tag{4.14}$$
$$+ [2h + 4C^* R_1 h^2 e^{2C^*(1+\eta)h}] \, \eta_D\} \, e^{2C^* h} \quad \text{for} \quad t \in [t_0, t_0 + h].$$

With respect to (4.14) we have

$$|||u_D(t_0 + h)||| \leqq |||u_L(t_0 + h)||| + |||u_L(t_0 + h) - u_D(t_0 + h)||| \leqq \tag{4.15}$$
$$\leqq [L e^{-\omega h} + 4C^* R_1 h e^{2C^*(2+\eta)h}] \, \eta_0 + [2h e^{2C^* h} + 4C^* R_1 h^2 e^{2C^*(2+\eta)h}] \, \eta_D.$$

According to (4.9), (4.15), it is

$$|||u_D(t_0 + h)||| \leqq \eta_0. \tag{4.16}$$

The implication (4.10) follows from (4.13) and (4.16): Let us find an integer $n \geqq 0$ for $t \geqq t_0 \geqq 0$ such that $t = t_0 + nh + s$, $s \in [0, h)$.

Then if we successively use (4.16), we get $|||u_D(t_0 + nh)||| \leqq \eta_0$ and with respect to (4.13) it is $|||u_D(t)||| = |||u_D(t_0 + nh + s)||| \leqq \eta$, which proves (4.10).

5. AN EXAMPLE

Let Ω be the parallelepiped $\Omega = (0, \pi c_1) \times (0, \pi c_2) \times \ldots \times (0, \pi c_n)$, $c_i > 0$ $(i = 1, \ldots, n)$, $H = L_2(\Omega)$, $p \geqq 1$ integer.

Let the operator A be defined by

$$A \, v(x) = (-1)^p [\sum_{i=1}^{n} D_i^2]^p \, v(x) \quad \text{for} \quad v \in \mathscr{D}(A) = \tag{5.1}$$

$$= \Big\{ u(x) \in L_2(\Omega) \mid u(x) = \sum_{\substack{1 \leqq k_i < \infty \\ i = 1, \ldots, n}} u_k \sin \frac{k_1 x_1}{c_1} \ldots \sin \frac{k_n x_n}{c_n},$$

$$k = [k_1, \ldots, k_n] \sum_{\substack{1 \leqq k_i < \infty \\ i=1,\ldots,n}} \left[\left(\frac{k_1}{c_1}\right)^2 + \ldots + \left(\frac{k_n}{c_n}\right)^2\right]^{2p} u_k^2 < \infty\Bigg\},$$

$$D_i = \frac{\partial}{\partial x_i}, \quad \text{(in the sense of distributions)}.$$

Suppose that

(5.2) $$F(t, u(t)) = f(t, x, u'(t), f_1(A)\, u(t), \ldots, f_q(A)\, u(t)),$$

where $f_i(s)$ are continuous functions for $s \geqq \left[\sum_{i=1}^{n} \left(\frac{1}{c_i}\right)^2\right]^p, \quad i = 1, \ldots, q;$

(5.3) there exists a constant F^* so that

$$|f_i(s)| \leqq F^* s^{1/2} \quad \text{for} \quad s \geqq \left[\sum_{i=1}^{n} \left(\frac{1}{c_i}\right)^2\right]^p, \quad i = 1, \ldots, q,$$

(5.4) $$F(t, u(t)) \in C(\mathscr{D}(u), \mathscr{D}(A^{1/2}));$$

(5.5) $$F(t + T, u(t)) = F(t, u(t)) \quad \text{for} \quad u \in \mathscr{U}, \quad t \in \mathscr{D}(u), \quad T > 0.$$

Suppose that there exists a T-periodic solution $v : R^+ \to H$ of the equation

$$u''(t) + A\, u(t) = \varepsilon F(t, u(t)).$$

We shall investigate the stability of this solution. (The existence is investigated in [1] for $p = 1$.)

The aim of this section is to verify the assumptions on the operator A and to prove the existence of constants K^*, $R^* > 0$, C^* satisfying the conditions $(\mathscr{L})$ and $(\mathscr{N}\mathscr{L})$.

Remark 5.1. The value of R^* and C^* is not important, but the value of K^* is important for the stability. We shall show the way in which K^* may be determined, but we shall not look for its explicit expression in order not to complicate our considerations.

The symbol $\sum_k$ will mean $\sum_{\substack{1 \leqq k_i < \infty \\ i=1,\ldots,n}}$ in several following lemmas.

Lemma 5.1. *The operator A is selfadjoint.*

Proof. According to Neumann theorem ([3] p. 121) it suffices to prove that $\mathscr{R}(A) = H$. Let $g \in H$ and let us write it in the form

$$g(x) = \sum_k g_k \sin \frac{k_1 x_1}{c_1} \ldots \sin \frac{k_n x_n}{c_n}, \quad k = [k_1, \ldots, k_n], \quad x = [x_1, \ldots, x_n] \in \Omega.$$

The relation

$$v(x) = \sum_k \left[\left(\frac{k_1}{c_1}\right)^2 + \ldots + \left(\frac{k_n}{c_n}\right)^2\right]^{-p} g_k \sin\frac{k_1x_1}{c_1} \ldots \sin\frac{k_nx_n}{c_n}, \quad x = [x_1, \ldots, x_n] \in \Omega$$

determines the element $v \in \mathcal{D}(A)$ such that $Av = g$.

Lemma 5.2. *The spectrum $\sigma(A)$ of the operator A consists of the point spectrum*

$$\sigma(A) = \left\{\lambda_k = \left[\sum_{i=1}^n \left(\frac{k_i}{c_i}\right)^2\right]^p \middle| \, k = [k_1, \ldots, k_n], \quad 1 \leqq k_i < \infty, \right.$$
$$\left. k_i \textit{ integers}, \quad (i = 1, \ldots, n)\right\}$$

and the eigenfunctions corresponding to the eigenvalue λ_k is

$$\sin\frac{k_1x_1}{c_1} \ldots \sin\frac{k_nx_n}{c_n}.$$

Further,

$$\delta = \left[\sum_{i=1}^n \left(\frac{1}{c_i}\right)^2\right]^p.$$

Proof. The discreteness of $\sigma(A)$ follows from the completeness of the system

$$\sin\frac{k_1x_1}{c_1} \ldots \sin\frac{k_nx_n}{c_n}, \quad 1 \leqq k_i < \infty, \quad (i = 1, \ldots, n)$$

of eigenfunctions of the operator A in the space H (see [3] p. 136).

Lemma 5.3. *Let $u \in \mathcal{D}(A)$ have the form*

$$u(x) = \sum_k u_k \sin\frac{k_1x_1}{c_1} \ldots \sin\frac{k_nx_n}{c_n}. \tag{5.6}$$

Then

$$A\,u(x) = \sum_k \left[\left(\frac{k_1}{c_1}\right)^2 + \ldots + \left(\frac{k_n}{c_n}\right)^2\right]^p u_k \sin\frac{k_1x_1}{c_1} \ldots \sin\frac{k_nx_n}{c_n}.$$

If

$$v \in \mathcal{D}(A^{1/2}) = \left\{u(x) \in L_2(\Omega) \mid u(x) = \sum_k u_k \sin\frac{k_1x_1}{c_1} \ldots \sin\frac{k_nx_n}{c_n},\right.$$
$$\left. k = [k_1, \ldots, k_n], \quad \sum_k \left[\left(\frac{k_1}{c_1}\right)^2 + \ldots + \left(\frac{k_n}{c_n}\right)^2\right]^p u_k^2 < \infty\right\}$$

has the form (5.6), *where we write* v_k *instead of* u_k, *then*

$$A^{1/2} v(x) = \sum_k \left[\left(\frac{k_1}{c_1}\right)^2 + \ldots + \left(\frac{k_n}{c_n}\right)^2\right]^{p/2} v_k \sin \frac{k_1 x_1}{c_1} \ldots \sin \frac{k_n x_n}{c_n} .$$

Proof. Follows immediately from Lemma 5.2 and from the convergence of the series

$$\sum_k \left[\left(\frac{k_1}{c_1}\right)^2 + \ldots + \left(\frac{k_n}{c_n}\right)^2\right]^{2p} u_k^2 , \quad \sum_k \left[\left(\frac{k_1}{c_1}\right)^2 + \ldots + \left(\frac{k_n}{c_n}\right)^2\right]^{p} v_k^2 .$$

Lemma 5.4. *There exists a constant* K_0^* (K_0^* *depends on* n, p) *such that if* $u \in \mathscr{D}(A^{1/2})$, *then*

$$n^{-1/2} \left\| \sum_{i=1}^n D_i^p u(x) \right\| \leqq \left\| A^{1/2} u(x) \right\| \leqq K_0^* \left\| \sum_{i=1}^n D_i^p u(x) \right\| .$$

Proof. If $u(x)$ has the form (5.6), then

$$\left\| \sum_{i=1}^n D_i^p u(x) \right\|^2 = \left(\frac{\pi}{2}\right)^n c_1 \ldots c_n \sum_k \left[\left(\frac{k_1}{c_1}\right)^p + \ldots + \left(\frac{k_n}{c_n}\right)^p\right]^2 u_k^2 , \tag{5.7}$$

$$\left\| A^{1/2} u(x) \right\|^2 = \left(\frac{\pi}{2}\right)^n c_1 \ldots c_n \sum_k \left[\left(\frac{k_1}{c_1}\right)^2 + \ldots + \left(\frac{k_n}{c_n}\right)^2\right]^p u_k^2 . \tag{5.8}$$

The lemma follows from (5.7), (5.8) and from the inequalities

$$\left(\frac{k_1}{c_1}\right)^{2p_1} \ldots \left(\frac{k_n}{c_n}\right)^{2p_n} \leqq \left[\left(\frac{k_1}{c_1}\right)^p + \ldots + \left(\frac{k_n}{c_n}\right)^p\right]^2 \quad \text{for} \quad \sum_{i=1}^n p_i = p , \quad p_i \geqq 0 ,$$

$$\left[\left(\frac{k_1}{c_1}\right)^p + \ldots + \left(\frac{k_n}{c_n}\right)^p\right]^2 \leqq n \left[\left(\frac{k_1}{c_1}\right)^{2p} + \ldots + \left(\frac{k_n}{c_n}\right)^{2p}\right] \leqq n \left[\left(\frac{k_1}{c_1}\right)^2 + \ldots + \left(\frac{k_n}{c_n}\right)^2\right]^p .$$

Lemma 5.5. *There exists a constant* K_1^* *so that if* $u \in \mathscr{D}(A^{1/2})$, *then*

$$\|u\|_{W_2^p(\Omega)} \leqq K_1^* \left\| A^{1/2} u \right\| .$$

Proof. Let us write $u(x)$ in the form (5.6). Then if $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| \leqq p$, it follows from (5.7) and Lemma 5.4 that

$$\|D^\alpha u\|^2 = \left(\frac{\pi}{2}\right)^n c_1 \ldots c_n \sum_k \left[\left(\frac{k_1}{c_1}\right)^{\alpha_1} \ldots \left(\frac{k_n}{c_n}\right)^{\alpha_n}\right]^2 u_k^2 \leqq$$

$$\leqq \left(\frac{\pi}{2}\right)^n c_1 \ldots c_n K \sum_k \left[\left(\frac{k_1}{c_1}\right)^p + \ldots + \left(\frac{k_n}{c_n}\right)^p\right]^2 u_k^2 = K \left\| \sum_{i=1}^n D_i^p u \right\|^2 \leqq Kn \left\| A^{1/2} u \right\|^2$$

for a certain constant K. Now the lemma follows immediately from

$$\|u\|^2_{W_2{}^p(\Omega)} = \sum_{|\alpha| \leq p} \|D^\alpha u\|^2 .$$

Put $s^* = \frac{1}{2}(n + 1)$ if n is odd, $s^* = \frac{1}{2}(n + 2)$ if n is even.

Lemma 5.6. (Sobolev embedding theorem) ([4] p. 72). *There exists a constant K_2^* such that to each $u \in W_2^{s^*}(\Omega)$ there exists a continuous representant of this function satisfying*

$$\|u(x)\|_{C(\bar\Omega)} = \max_{x \in \bar\Omega} |u(x)| \leq K_2^* \|u(x)\|_{W_2{}^{s^*}(\Omega)} .$$

From now on we shall understand under solutions only those for which $u'(t, x)$, $f_i(A)\, u(t, x)$ $(i = 1, \ldots, q)$ are continuous for $t \in \mathscr{D}(u)$, $x \in \bar\Omega$, and we shall call them continuous representants (of the solutions).

Suppose that

(5.9) there exist continuous **G**-derivatives of the function f with respect to the variables u', $f_i(A)\, u$ $(i = 1, \ldots, q)$ up to the second order.

Then

$$F(t, v(t) + u(t)) = F(t, v(t)) + 2\, b(t)\, u'(t) + \sum_{i=1}^{q} a_i(t) f_i(A)\, u(t) +$$
$$+ \sum_{i,j=1}^{q} r_{ij}(t, u(t)) f_i(A)\, u(t) f_j(A)\, u(t) + \sum_{i=1}^{q} r_i(t, u(t))\, u'(t) f_i(A)\, u(t) ,$$

where

(5.10) $$b(t) = \frac{1}{2} \frac{\partial f}{\partial u'} (t, x, v'(t), f_1(A)\, v(t), \ldots, f_q(A)\, v(t)) ,$$

$$a_i(t) = \frac{\partial f}{\partial f_i} (t, x, v'(t), f_1(A)\, v(t), \ldots, f_q(A)\, v(t)) , \quad i = 1, \ldots, q ,$$

(5.11) $$r_{ij}(t, u(t)) = \int_0^1 \int_0^1 \tilde r_{ij}(t, v(t) + \vartheta\sigma\, u(t))\, \sigma \, d\vartheta \, d\sigma ,$$

$$r_i(t, u(t)) = \int_0^1 \int_0^1 \tilde r_i(t, v(t) + \vartheta\sigma\, u(t))\, \sigma \, d\vartheta \, d\sigma ,$$

where

$$\tilde r_{ij}(t, u(t)) = \frac{\partial^2 f}{\partial f_i\, \partial f_j} (t, x, u'(t), f_1(A)\, u(t), \ldots, f_q(A)\, u(t)) ,$$

$$\tilde r_i(t, u(t)) = \frac{\partial^2 f}{\partial u'\, \partial f_i} (t, x, u'(t), f_1(A)\, u(t), \ldots, f_q(A)\, u(t)) , \quad i, j = 1, \ldots, q .$$

Let $g = g(t, x, u', f_1, \ldots, f_q)$. Let us denote by $M_k(g)$ the system of all derivatives of the type

$$\frac{\partial^\Gamma g}{\partial^{\gamma_{-1}} x\, \partial^{\gamma_0} u'\, \partial^{\gamma_1} f_1 \ldots \partial^{\gamma_q} f_q}, \quad \Gamma = \sum_{i=-1}^{q} \gamma_i, \quad 0 \leqq \Gamma \leqq k .$$

Further, let us denote

$$\varrho^* = \max\left(\sup_{t \in R^+, x \in \bar\Omega} |v'(t, x)|, \quad \max_{i=1,\ldots,q} \sup_{t \in R^+, x \in \bar\Omega} |f_i(A)\, v(t, x)|\right).$$

Proposition 5.1. *Suppose that* (5.9) *holds, there exist all derivatives of systems* $M_p(b)$, $M_p(a_i)$, $(i = 1, \ldots, q)$ *and they are continuous on* $[0, T] \times \bar\Omega \times K_{\varrho^*}$, *where* $b(t)$, $a_i(t)$ $(i = 1, \ldots, q)$ *are defined by the relation* (5.10), $K_{\varrho^*} = \{y \in E_{q+1} \mid \|y\|_{E_{q+1}} \leqq \varrho^*\}$. *Let* $2s^* \leqq p + 1$. *Then the condition* $(\mathscr{L})$ *is fulfilled.*

Proof. Because $p \geqq 2s^* - 1 \geqq n > \frac{1}{2}n$, it holds by virtue of Lemmas 5.5, 5.6

$$\|v'(t)\|_{C(\bar\Omega)} \leqq K_2^* \|v'(t)\|_{W_2^p(\Omega)} \leqq K_1^* K_2^* \|A^{1/2}\, v'(t)\| \leqq K_1^* K_2^* |||v(t)||| ,$$

$$\|f_i(A)\, v(t)\|_{C(\bar\Omega)} \leqq K_1^* K_2^* \|A^{1/2} f_i(A)\, v(t)\| \leqq K_1^* K_2^* F^* \|A\, v(t)\| \leqq K_1^* K_2^* F^* |||v(t)||| ,$$
$$(i = 1, \ldots, q)$$

and so $\varrho^* < +\infty$ and there exists a continuous representant of the solution.

Now we shall verify the condition (2.6). Using Lemma 5.4 we get

$$\|A^{1/2}[(b(t) - b^*)\, u'(t)]\| \leqq K_0^* \|\sum_{i=1}^{n} D_i^p[(b(t) - b^*)\, u'(t)]\| \leqq$$

$$\leqq K_0^* \sum_{i=1}^{n} \|D_i^p[(b(t) - b^*)\, u'(t)]\| \leqq K_0^* \sum_{i=1}^{n} \sum_{j=0}^{p-s^*} \binom{p}{j} \|D_i^j(b(t) - b^*)\, D_i^{p-j}\, u'(t)\| +$$

$$+ K_0^* \sum_{i=1}^{n} \sum_{j=p-s^*+1}^{p} \binom{p}{j} \|D_i^j(b(t) - b^*)\, D_i^{p-j}\, u'(t)\| .$$

Further in this proof, C_i will mean some constants. Thus the inequality (2.6) will be proved if we prove

$$(5.12) \quad \|D_i^j(b(t) - b^*)\, D_i^{p-j}\, u'(t)\| \leqq C_1 |||u(t)||| \quad \text{for} \quad i = 1, \ldots, n, \quad j = 0, \ldots, p .$$

We can easily find that $D_i^j(b(t) - b^*)$ is a linear combination of members

$$(5.13) \quad m_j(b - b^*)\, (D_i^1 v')^{\beta_{01}} \ldots (D_i^{\delta_0} v')^{\beta_{0\delta_0}}$$
$$(D_i^1 f_1(A)\, v)^{\beta_{11}} \ldots (D_i^{\delta_1} f_1(A)\, v)^{\beta_{1\delta_1}}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(D_i^1 f_q(A)\, v)^{\beta_{q1}} \ldots (D_i^{\delta_q} f_q(A)\, v)^{\beta_{q\delta_q}} ,$$

where

$m_j(b - b^*)$ is a derivative from the system $M_j(b - b^*)$ and

$$\beta_{m\delta_m} = 0 \Rightarrow \beta_{mk} = 0\,, \quad \text{for} \quad k = 1, \dots \quad \text{and} \quad \delta_m = 0\,, \tag{5.14}$$

$$\sum_{m=0}^{q} \sum_{k=1}^{\delta_m} k\beta_{mk} \leqq j\,.$$

As the functions $m_j(b - b^*)$ are continuous on $[0, T] \times \bar{\Omega} \times K_{\varrho^*}$, they are bounded, i.e.

$$\sup_{t \in [0,T]} \|m_j(b(t) - b^*)\|_{C(\bar{\Omega})} \leqq C_2 \quad \text{for} \quad j = 0, \dots, p\,. \tag{5.15}$$

We shall prove (5.12) for $0 \leqq j \leqq p - s^*$ first. All derivatives D_i^ν in (5.13) are of orders $\leqq p - s^*$ in this case, because if we suppose that this is not true, there exists at least one $\delta_m > p - s^*$. But then (5.14) implies

$$\delta_m \leqq \delta_m \beta_{m\delta_m} \leqq \sum_{n=0}^{q} \sum_{k=1}^{\delta_n} k\beta_{nk} \leqq j \leqq p - s^*$$

which is a contradiction. Hence according to the embedding theorem (Lemma 5.6) and Lemma 5.5 it is

$$\|D_i^\nu v'(t)\|_{C(\bar{\Omega})} \leqq K_2^* \|v'(t)\|_{W_2^p(\Omega)} \leqq K_1^* K_2^* \|A^{1/2} v'(t)\| \leqq K_1^* K_2^* |||v(t)||| \leqq C_3\,,$$

$$\|D_i^\nu f_k(A)\, v(t)\|_{C(\bar{\Omega})} \leqq K_2^* \|f_k(A)\, v(t)\|_{W_2^p(\Omega)} \leqq K_1^* K_2^* \|A^{1/2} f_k(A)\, v(t)\| \leqq$$
$$\leqq K_1^* K_2^* F^* \|A\, v(t)\| \leqq K_1^* K_2^* F^* |||v(t)||| \leqq C_3\,, \quad k = 1, \dots, q\,.$$

Thus

$$\|D_i^j(b(t) - b^*)\, D_i^{p-j} u'(t)\| \leqq C_4 \|D_i^{p-j} u'(t)\| \leqq C_4 \|u'(t)\|_{W_2^p(\Omega)} \leqq C_4 K_1^* |||u(t)|||\,.$$

It remains to prove the estimate (5.12) for $p - s^* + 1 \leqq j \leqq p$. Then $p - j + s^* \leqq \leqq 2s^* - 1 \leqq p$ in this case and so

$$\|D_i^{p-j} u'(t)\|_{C(\bar{\Omega})} \leqq K_2^* \|u'(t)\|_{W_2^p(\Omega)} \leqq K_1^* K_2^* \|A^{1/2} u'(t)\| \leqq K_1^* K_2^* |||u(t)|||\,.$$

Hence it suffices to prove

$$\|D_i^j(b(t) - b^*)\| \leqq C_5\,.$$

We shall prove the following proposition first:

There is no more than one derivative D_i^ν of an order higher than $p - s^*$ in each member (5.13).

Suppose that this is not true. Then we can find at least two different expressions of the type

$$(D_i^{\nu_1})^{\beta_{m_1\nu_1}},\ (D_i^{\nu_2})^{\beta_{m_2\nu_2}} \quad \text{and} \quad \nu_i \geqq p - s^* + 1\,, \quad \beta_{m_i\nu_i} \neq 0\,, \quad i = 1, 2$$

in at least one member (5.13). But then

$$\sum_{n=0}^{q} \sum_{k=1}^{\delta_n} k\beta_{nk} \geqq \nu_1 \beta_{m_1 \nu_1} + \nu_2 \beta_{m_2 \nu_2} \geqq 2p - 2s^* + 2 \geqq p + 1 > p$$

and this contradicts (5.14).

All derivatives from (5.13) of orders $\leqq p - s^*$ can be estimated by a constant independent on t, in the same way as above. According to (5.15) it remains to estimate

$$\|D_i^\nu v'(t)\| \quad \text{and} \quad \|D_i^\nu f_k(A)\, v(t)\| \quad \text{in the case} \quad p \geqq \nu \geqq p - s^* + 1\,, \quad k = 1, \ldots, q\,.$$

Using Lemma 5.5 we get

$$\|D_i^\nu v'(t)\| \leqq \|v'(t)\|_{W_2{}^p(\Omega)} \leqq K_1^* \|A^{1/2} v'(t)\| \leqq K_1^* |||v(t)||| \leqq C_6\,,$$

$$\|D_i^\nu f_k(A)\, v(t)\| \leqq \|f_k(A)\, v(t)\|_{W_2{}^p(\Omega)} \leqq K_1^* \|A^{1/2} f_k(A)\, v(t)\| \leqq K_1^* F^* \|A\, v(t)\| \leqq$$
$$\leqq K_1^* F^* |||v(t)||| \leqq C_6\,, \quad k = 1, \ldots, q\,.$$

This completes the proof of the inequality (2.6) from the condition ($\mathscr{L}$).

The proof of the inequality (2.7) is analogous to that of (2.6) and that is why we omit it.

Let us denote

$$F_L(t, u(t)) = 2b(t)\, u'(t) + \sum_{i=1}^{q} a_i(t) f_i(A)\, u(t)\,,$$

$$F_N(t, u(t)) = \sum_{i,j=1}^{q} r_{ij}(t, u(t)) f_i(A)\, u(t) f_j(A)\, u(t) + \sum_{i=1}^{q} r_i(t, u(t))\, u'(t) f_i(A)\, u(t)$$

for $b(t)$, $a_i(t)$, $r_{ij}(t, u(t))$, $r_i(t, u(t))$ determined by the relations (5.10), (5.11).

Proposition 5.2. *Suppose that (5.9) holds, that there exists a closed sphere K_ϱ, $\operatorname{Int} K_\varrho \supset K_{\varrho^*}$ so that all the derivatives of the systems $M_p(\tilde{r}_{ij})$, $M_p(\tilde{r}_i)$, $M_p(b)$, $M_p(a_i)$, $(i, j = 1, \ldots, q)$ exist and are continuous on $[0, T] \times \bar{\Omega} \times K_\varrho$. Let $2s^* \leqq p + 1$. Then the condition ($\mathscr{N}\mathscr{L}$) is fulfilled.*

Proof. The inequality (3.5) can be proved analogously as Proposition 5.1. Therefore we shall deal with the condition (3.6) only.

Let C_i be constants.

Let $R^* > 0$ sufficiently small so that

$$\begin{aligned} (5.16) \qquad & [v'(t, x) + \vartheta\sigma u'(t, x), f_1(A)\,(v(t, x) + \vartheta\sigma u(t, x)), \ldots \\ & \ldots, f_q(A)\,(v(t, x) + \vartheta\sigma u(t, x))] \in K_\varrho \quad \text{for} \quad \vartheta, \sigma \in [0, 1]\,, \quad u \in \mathscr{U}\,, \quad x \in \bar{\Omega} \\ & \text{and such} \quad t \in \mathscr{D}(u) \quad \text{that} \quad |||u(t)||| \leqq R^*\,. \end{aligned}$$

Further, let $V(t)$, $W(t)$ denote some of the functions $u'(t), f_j(A)\, u(t)$, $(j = 1, \ldots, q)$; $r(t, u(t))$ denotes one of the functions $r_{ij}(t, u(t))$, $r_i(t, u(t))$, $(i, j = 1, \ldots, q)$.

Remember that

$$\|D_i^\nu V(t)\|_{C(\bar\Omega)} \leqq K_2^* \|V(t)\|_{W_2^{\nu+s^*}(\Omega)} \leqq K_2^* \|V(t)\|_{W_2^p(\Omega)} \leqq$$
$$\leqq K_1^* K_2^* \|A^{1/2} V(t)\| \leqq K_1^* K_2^* C_0 |||u(t)||| \quad \text{for} \quad \nu \leqq p - s^* .$$

Surely it suffices to prove that

$$\|A^{1/2}[r(t, u(t))\, V(t)\, W(t)]\| \leqq C_1 |||u(t)|||^2 \tag{5.17}$$

for each of the expressions $r(t, u(t))\, V(t)\, W(t)$ from the function F_N.

Because

$$\|A^{1/2}[r(t, u(t))\, V(t)\, W(t)]\| \leqq K_0^* \sum_{i=1}^{n} \|D_i^p[r(t, u(t))\, V(t)\, W(t)]\| =$$
$$= K_0^* \sum_{i=1}^{n} \left\| \sum_{k=0}^{p} \binom{p}{k} D_i^k r(t, u(t)) \sum_{m=0}^{p-k} \binom{p-k}{m} D_i^m V(t)\, D_i^{p-k-m} W(t) \right\| ,$$

it suffices to prove

$$\|D_i^k r(t, u(t))\, D_i^m V(t)\, D_i^{p-k-m} W(t)\| \leqq C_2 |||u(t)|||^2 \quad \text{for} \quad |||u(t)||| \leqq R^* , \tag{5.18}$$
$$i = 1, \ldots, n , \quad k = 0, \ldots, p , \quad m = 0, \ldots, p - k .$$

The index i in the derivative will be ommited in the sequel and so D will mean one of the derivatives D_i $(i = 1, \ldots, n)$.

We can easily find that $D^j\, r(t, u(t))$ is a linear combination of members of the form

$$\int_0^1 \int_0^1 [m_j(\tilde r)\, (D^1(v' + \vartheta\sigma u'))^{\beta_{01}} \ldots (D^{\delta_0}(v' + \vartheta\sigma u'))^{\beta_{0\delta_0}} \cdot \tag{5.19}$$
$$(D^1 f_1(A)\,(v + \vartheta\sigma u))^{\beta_{11}} \ldots (D^{\delta_1} f_1(A)\,(v + \vartheta\sigma u))^{\beta_{1\delta_1}}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(D^1 f_q(A)\,(v + \vartheta\sigma u))^{\beta_{q1}} \ldots (D^{\delta_q} f_q(A)\,(v + \vartheta\sigma u))^{\beta_{q\delta_q}}]\, \sigma\, d\vartheta\, d\sigma ,$$

where $m_j(\tilde r)$ is some derivative from the system $M_j(\tilde r)$,

$$\beta_{m\delta_m} = 0 \Rightarrow \beta_{mk} = 0 \quad \text{for} \quad k = 1, \ldots, \quad \text{and} \quad \delta_m = 0 , \quad \sum_{m=0}^{q} \sum_{k=1}^{\delta_m} k\beta_{mk} \leqq j .$$

The boundedness of the expressions $m_j(\tilde r)$ follows from their continuity on $[0, T] \times \bar\Omega \times K_\varrho$ and (5.16).

We shall prove (5.18) for $0 \leqq k \leqq p - s^*$ first. All derivatives D^ν in (5.19) are of orders $\leqq p - s^*$ in this case.

Hence

$$\|D^\nu(v'(t) + \vartheta\sigma\, u'(t))\|_{C(\bar\Omega)} \leqq K_2^*\|v'(t) + \vartheta\sigma\, u'(t)\|_{W_2^{\nu+s^*}(\Omega)} \leqq$$
$$\leqq K_2^*\|v'(t) + \vartheta\sigma\, u'(t)\|_{W_2^p(\Omega)} \leqq K_1^*K_2^*\|A^{1/2}(v'(t) + \vartheta\sigma\, u'(t))\| \leqq$$
$$\leqq K_1^*K_2^*(|||v(t)||| + R^*) \leqq C_3\,,$$

similarly

$$\|D^\nu f_i(A)\,(v(t) + \vartheta\sigma\, u(t))\|_{C(\bar\Omega)} \leqq C_3\,, \quad (i = 1, \ldots, q)\,.$$

If $0 \leqq m \leqq p - s^*$ then $p - k - m \leqq p$. Hence

$$\|D^m\, V(t)\|_{C(\bar\Omega)} \leqq K_1^*K_2^*C_0|||u(t)|||\,, \quad \|D^{p-k-m}\, W(t)\| \leqq \|W(t)\|_{W_2^p(\Omega)} \leqq$$
$$\leqq K_1^*\|A^{1/2}\, W(t)\| \leqq K_1^*C_0|||u(t)|||\,.$$

If $p - s^* + 1 \leqq m \leqq p$ then $p - k - m \leqq p - s^*$ and so

$$\|D^m\, V(t)\| \leqq K_1^*C_0|||u(t)|||\,, \quad \|D^{p-k-m}\, W(t)\|_{C(\bar\Omega)} \leqq K_1^*K_2^*C_0|||u(t)|||\,.$$

Now the estimate (5.18) is clear.

It remains to prove (5.18) for $p - s^* + 1 \leqq k \leqq p$. Then $m \leqq p - s^*$, $p - k - m \leqq p - s^*$ in this case and so

$$\|D^m\, V(t)\|_{C(\bar\Omega)} \leqq K_1^*K_2^*C_0|||u(t)|||\,, \quad \|D^{p-k-m}\, W(t)\|_{C(\bar\Omega)} \leqq K_1^*K_2^*C_0|||u(t)|||\,.$$

Similarly as in the proof of Proposition 5.1 we can find that at most one derivative D^ν from (5.19) is of an order $> p - s^*$. But this derivative must be of an order $\leqq p$. Hence it holds

$$\|D^\nu[v'(t) + \vartheta\sigma\, u'(t)]\| \leqq K_1^*|||v(t) + \vartheta\sigma\, u(t)||| \leqq C_4$$

or

$$\|D^\nu[f_i(A)\,(v(t) + \vartheta\sigma\, u(t))]\| \leqq K_1^*F^*|||v(t) + \vartheta\sigma\, u(t)||| \leqq C_4\,.$$

All other expressions in (5.19) can be estimated by a constant C_3 as above.

This proves the proposition.

We can summarize our results in the following theorem:

Theorem 5.1. *Suppose that there exists a T-periodic solution $v : R^+ \to H$ of the equation*

$$u''(t) + A\, u(t) = \varepsilon F(t, u(t))$$

where F satisfies the conditions (5.2)–(5.5), (5.9), the operator A is defined by the relation (5.1) and the condition $(\mathscr{E}\mathscr{L})$ is fulfilled. Let K_ϱ be the sphere from Proposi-

tion 5.2. Let there exist all derivatives of the systems $M_p(b), M_p(a_i), M_p(\tilde{r}_{ij}), M_p(\tilde{r}_i)$ $(i, j = 1, \ldots, q)$ *and let them be continuous on* $[0, T] \times \bar{\Omega} \times K_\varrho$. *Further, let* $2s^* \leqq p + 1$.

Then the conditions $(\mathscr{L})$ *and* $(\mathscr{N}\mathscr{L})$ *are fulfilled with some constants* K^*, C^*, $R^* > 0$.

Moreover, if $b^* + 2(q + 2) K^* < 0$, $\varepsilon \in (0, \varepsilon_0^*)$, *then the solution* v *is uniformly exponentially stable with respect to the norm* $|||\cdot|||$ *and uniformly stable at constantly acting disturbances with respect to the norm* $|||\cdot|||$.

Remark 5.3. We can put

$$b^* = \tfrac{1}{2}\Big(\sup_{\substack{t\in[0,T]\\ x\in\bar{\Omega}}} b(t, x) + \inf_{\substack{t\in[0,T]\\ x\in\bar{\Omega}}} b(t, x)\Big),$$

$$a_i^* = \tfrac{1}{2}\Big(\sup_{\substack{t\in[0,T]\\ x\in\bar{\Omega}}} a_i(t, x) + \inf_{\substack{t\in[0,T]\\ x\in\bar{\Omega}}} a_i(t, x)\Big), \quad (i = 1, \ldots, q)$$

for rough estimates.

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