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THE UNIFORM EXPONENTIAL STABILITY AND THE UNIFORM STABILITY AT CONSTANTLY ACTING DISTURBANCES OF A PERIODIC SOLUTION OF A WAVE EQUATION

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INTRODUCTION

The purpose of this paper is to investigate the uniform exponential stability and the uniform stability at constantly acting disturbances of an ω-periodic solution u_0 of the problem $(\mathscr{P})$, given by the equation

$$u_{tt} - u_{xx} = F(t, x, u, u_t, u_x, \varepsilon)\,, \quad t \geqq 0\,, \quad x \in \langle 0, \pi \rangle \tag{0,1}$$

(ε being a small parameter), by one of the boundary conditions

$$u(t, 0) = u(t, \pi) = 0\,, \tag{0,2}$$

$$u_x(t, 0) + \alpha_0\, u(t, 0) = u(t, \pi) = 0\,, \tag{0,3}$$

$$u_x(t, 0) + \alpha_0\, u(t, 0) = u_x(t, \pi) + \alpha_\pi\, u(t, \pi) = 0 \tag{0,4}$$

(where $\alpha_0, \alpha_\pi \in E_1$ and $t \geqq 0$) and by the condition of periodicity

$$u(t + \omega, x) = u(t, x)\,, \quad t \geqq 0\,, \quad x \in \langle 0, \pi \rangle\,. \tag{0,5}$$

In § 1 the problem of the uniform exponential stability and the uniform stability at constantly acting disturbances of the solution u_0 is transformed to the problem of the uniform exponential stability of the zero solution of an equation of the type

$$u_{tt} - u_{xx} = a(t, x, \varepsilon)\, u + b(t, x, \varepsilon)\, u_t + c(t, x, \varepsilon)\, u_x$$

with the boundary conditions (0,2),

$$u_x(t, 0) = u(t, \pi) = 0\,, \quad t \geqq 0 \tag{0,6}$$

or

$$(0,7) \qquad u_x(t, 0) = u_x(t, \pi) = 0\,, \quad t \geqq 0\,.$$

Then the second Ljapunov method is used. It enables us to obtain sufficient conditions for the uniform exponential stability and the uniform stability at constantly acting disturbances of the solution u_0 in § 2. However, we deal with somewhat more special forms of the right-hand side of the equation (0,1) in § 2.

1. FUNDAMENTAL DEFINITIONS AND THEOREMS

Let the following conditions be fulfilled:

($\mathscr{A}_1$) The function $F(t, x, u, u_t, u_x, \varepsilon)$ together with its partial derivatives

$$\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial u_i}, \frac{\partial F}{\partial \varepsilon}, \frac{\partial^2 F}{\partial u_i \partial u_j}, \frac{\partial^2 F}{\partial u_i \partial t}, \frac{\partial^2 F}{\partial u_i \partial x}, \frac{\partial^2 F}{\partial u_i \partial \varepsilon},$$

$$\frac{\partial^3 F}{\partial u_i \partial u_j \partial x}, \frac{\partial^3 F}{\partial u_i \partial u_j \partial u_k} \quad (i, j, k = 1, 2, 3)$$

is defined and continuous in all variables for $t \geqq 0$, $x \in \langle 0, \pi \rangle$, $(u, u_t, u_x) \in E_3$, $\varepsilon \in \langle 0, \varepsilon_1 \rangle$ $(\varepsilon_1 > 0)$. (We denote $(u, u_t, u_x) = (u_1, u_2, u_3)$. We shall use this notation thoughout the paper if it is convenient.)

$$(\mathscr{A}_2) \qquad F(t + \omega, x, u, u_t, u_x, \varepsilon) = F(t, x, u, u_t, u_x, \varepsilon)$$

everywhere in its domain of definition.

$$(\mathscr{A}_3) \qquad F(t, 0, 0, 0, u_x, \varepsilon) = F(t, \pi, 0, 0, u_x, \varepsilon) = 0$$

for $t \geqq 0$, $u_x \in E_1$, $\varepsilon \in \langle 0, \varepsilon_1 \rangle$ in the case of the boundary conditions (0,2), $F(t, \pi, 0, 0, u_x, \varepsilon) = 0$ for $t \geqq 0$, $u_x \in E_1$, $\varepsilon \in \langle 0, \varepsilon_1 \rangle$ in the case of the boundary conditions (0,3).

We shall sometimes use also the notation ($\mathscr{P}[(*); (0,2)]$) (and similar) instead of ($\mathscr{P}$), if it is important to point out that we deal with an equation $(*)$ that is a special case of (0,1) and with the boundary conditions (0,2). If only the boundary conditions or only the equation are important, we shall denote the problem respectively by ($\mathscr{P}[\,\cdot\,; (0,2)]$) or ($\mathscr{P}[(*); \cdot\,]$); it is then understood that the equation or, respectively, the boundary conditions remain the same in the whole consideration, as long as the notation is used.

We shall denote by ($\mathscr{M}$) the problem arising from ($\mathscr{P}$) by omitting the condition of periodicity (0,5). We shall write ($\mathscr{M}[(*); (0,2)]$), ($\mathscr{M}[(*); \cdot\,]$). etc. sometimes, as in the case of ($\mathscr{P}$).

In the sequel we consider only those solutions of $(\mathscr{P})$ or $(\mathscr{M})$ which are continuous representants of elements of the space

$$C(\langle 0, \infty); W_2^3((0, \pi))) \cap C^1(\langle 0, \infty); W_2^2((0, \pi))) \cap C^2(\langle 0, \infty); W_2^1((0, \pi))) .$$

Let u be a function of variables $t \geqq 0$, $x \in \langle 0, \pi \rangle$. Then we denote by $u(t, \cdot)$ the function resulting from u if t is fixed.

Put

$$(1,1) \qquad \|u(t, \cdot)\|_1 = \left\{ \int_0^\pi [u^2(t, x) + u_x^2(t, x)] \, dx \right\}^{1/2},$$

$$(1,2) \qquad \|u(t, \cdot)\|_2 = \left\{ \int_0^\pi [u^2(t, x) + u_t^2(t, x) + u_x^2(t, x) + u_{tx}^2(t, x) + u_{xx}^2(t, x)] \, dx \right\}^{1/2}.$$

Definition 1,1. We say that a solution u_0 of the problem $(\mathscr{P})$ is *uniformly exponentially stable with respect to the norm* $\|\cdot\|_2$, if there exist $\delta > 0$, $K_1 > 0$, $K_2 > 0$ such that

$$(1,3) \qquad \|u(\tau, \cdot) - u_0(\tau, \cdot)\|_2 < \delta \Rightarrow$$
$$\Rightarrow \|u(t, \cdot) - u_0(t, \cdot)\|_2 \leqq K_1 \|u(\tau, \cdot) - u_0(\tau, \cdot)\|_2 \, e^{-K_2(t-\tau)}, \quad t \geqq \tau$$

for every solution u of the problem $(\mathscr{M})$ and for all $\tau \geqq 0$.

Let us denote

$$(1,4) \qquad u_{tt} - u_{xx} = F(t, x, u, u_t, u_x, \varepsilon) + G(t, x, u, u_t, u_x, \varepsilon) .$$

Definition 1,2. We say that a solution u_0 of the problem $(\mathscr{P})$ is *uniformly stable at constantly acting disturbances with respect to the norm* $\|\cdot\|_2$, if to an arbitrarily chosen $\eta > 0$ there exist $\delta_1 > 0$, $\delta_2 > 0$ such that

$$(1,5) \qquad \{\|u(\tau, \cdot) - u_0(\tau, \cdot)\|_2 < \delta_1, \ \|G(t, \cdot, u, u_t, u_x, \varepsilon)\|_1 < \delta_2$$
$$\text{for all } t \geqq \tau, \text{ satisfying the inequality}$$
$$\|u(t, \cdot) - u_0(t, \cdot)\|_2 < \eta\} \Rightarrow \|u(t, \cdot) - u_0(t, \cdot)\|_2 < \eta \quad \text{for all} \quad t \geqq \tau$$

for each function G fulfilling $(\mathscr{A}_1)$ and $(\mathscr{A}_3)$, for every solution u of $(\mathscr{M}[(1,4); \cdot])$ and for all $\tau \geqq 0$. ($\|G(t, \cdot, u, u_t, u_x, \varepsilon)\|_1$ means in fact $\|G(t, \cdot, u(t, \cdot), u_t(t, \cdot), u_x(t, \cdot), \varepsilon)\|_1$, but we shall use the shorter notation $\|G(t, \cdot, u, u_t, u_x, \varepsilon)\|_1$ in the sequel.)

Throughout the paper we shall deal with the stability with respect to the norm $\|\cdot\|_2$ only, that is why we shall no more repeat it.

The following lemma enables us to transform the problem of stability of a solution u_0 of $(\mathscr{P})$ to the problem of stability of a certain solution v_0 of a problem given by an equation that differs from (0,1) and by the boundary conditions (0,2), (0,6) or (0,7).

Lemma 1,1. *Let u be a solution of $(\mathcal{M})$, $\chi(x)$ a function of the class C^3 for $x \in \in \langle 0, \pi \rangle$ and let*

$$(1,6) \qquad v(t, x) = u(t, x) \, . \, e^{\chi(x)} \quad \textit{for} \quad t \geqq 0 \, , \quad x \in \langle 0, \pi \rangle \, .$$

Then it holds:

(i) *The function v satisfies the equation*

$$(1,7) \qquad v_{tt} - v_{xx} = [(\chi')^2 - \chi''] \, v - 2\chi' v_x + \\ + e^{\chi} F(t, x, e^{-\chi}v, e^{-\chi}v_t, -\chi' e^{-\chi}v + e^{-\chi}v_x, \varepsilon) \, ;$$

(ii) *u satisfies* (0,2) *if and only if v satisfies* (0,2).

(iii) *If $\chi'(0) = \alpha_0$, then u satisfies* (0,3) *if and only if v satisfies* (0,6).

(iv) *If $\chi'(0) = \alpha_0$, $\chi'(\pi) = \alpha_\pi$, then u satisfies* (0,4) *if and only if v satisfies* (0,7).

(v) *The solution u_0 of* ($\mathscr{P}$[(0,1); (0,2)]) (*respectively* ($\mathscr{P}$[(0,1); (0,3)]), *respectively* ($\mathscr{P}$[(0,1); (0,4)])) *is uniformly exponentially stable if and only if the solution $v_0 = u_0 e^{\chi(x)}$ of* ($\mathscr{P}$[(1,7); (0,2)]) (*respectively* ($\mathscr{P}$[(1,7); (0,6)]), *respectively* ($\mathscr{P}$[(1,7); (0,7)])) *is uniformly exponentially stable.*

(vi) *The same proposition as* (v) *holds for the uniform stability at constantly acting disturbances.*

Thus we can deal only with the solution v_0 of ($\mathscr{P}$[(1,7); $\cdot$]) with the boundary conditions (0,2), (0,6) or (0,7).

Let us denote by $F'(t, x, v, v_t, v_x, \varepsilon)$ the right-hand side of the equation (1,7),

$$a(t, x, \varepsilon) = \frac{\partial F'}{\partial v}(t, x, v_0, v_{0t}, v_{0x}, \varepsilon) \, , \quad b(t, x, \varepsilon) = \frac{\partial F'}{\partial v_t}(t, x, v_0, v_{0t}, v_{0x}, \varepsilon) \, ,$$

$$c(t, x, \varepsilon) = \frac{\partial F'}{\partial v_x}(t, x, v_0, v_{0t}, v_{0x}, \varepsilon) \, , \quad d_{ij}(t, x, v, v_t, v_x, \varepsilon) =$$

$$= (2 - \delta_{ij}) \int_0^1 \int_0^1 \frac{\partial^2 F'}{\partial v_i \, \partial v_j}(t, x, v_0 + \alpha\beta v, v_{0t} + \alpha\beta v_t, v_{0x} + \alpha\beta v_x, \varepsilon) \, \beta \, d\alpha \, d\beta \, .$$

$$(i, j = 1, 2, 3 \text{ and } \delta_{ij} \text{ is the Kronecker symbol}) \, ,$$

$$(1,8) \qquad u_{tt} - u_{xx} = au + bu_t + cu_x \, .$$

We shall suppose that

$$(1,9) \qquad \begin{cases} \chi'(0) = \chi'(\pi) = 0 & \text{if we deal with (0,2)} \, , \\ \chi'(\pi) = 0 & \text{if we deal with (0,6)} \, . \end{cases}$$

Then F' fulfils $(\mathscr{A}_3)$ (with (0,6) considered instead of (0,3)).

Theorem 1,1. *Let the zero solution of* $(\mathscr{M}[(1,8); \cdot])$ *be uniformly exponentially stable. Then the solution* v_0 *of* $(\mathscr{P}[(1,7); \cdot])$ *is uniformly stable at constantly acting disturbances.*

Proof. I. First we shall prove that the solution v_0 of $(\mathscr{P}[(1,10); (0,2)])$, where

$$v_{tt} - v_{xx} = F'(t, x, v_0, v_{0t}, v_{0x}, \varepsilon) + a(t, x, \varepsilon)(v - v_0) + \\ + b(t, x, \varepsilon)(v_t - v_{0t}) + c(t, x, \varepsilon)(v_x - v_{0x}), \tag{1,10}$$

is uniformly stable at constantly acting disturbances.

Let $\eta > 0$ be given. We shall look for $\delta_1, \delta_2 > 0$ satisfying all conditions in Definition 1,2, but with the equation

$$u_{tt} - u_{xx} = F'(t, x, v_0, v_{0t}, v_{0x}, \varepsilon) + a(t, x, \varepsilon)(u - v_0) + \\ + b(t, x, \varepsilon)(u_t - v_{0t}) + c(t, x, \varepsilon)(u_x - v_{0x}) + G(t, x, u, u_t, u_x, \varepsilon) \tag{1,10'}$$

instead of (1,4).

It follows immediately from the uniform exponential stability of the zero solution of $(\mathscr{M}[(1,8); (0,2)])$ that the solution v_0 of $(\mathscr{P}[(1,10); (0,2)])$ is also uniformly exponentially stable, i.e. there exist $\delta > 0$, $K_1 > 0$ and $K_2 > 0$ such that (1,3) (with v_0 and $(\mathscr{M}[(1,10); (0,2)])$ instead of u_0 and $(\mathscr{M})$) holds.

We can suppose that $\eta \leqq K_1\delta$. Let us choose $\delta_1 = \eta/2K_1$. Let u be a solution of $(\mathscr{M}[(1,10)'; (0,2)])$ such that $\|u(\tau, \cdot) - v_0(\tau, \cdot)\|_2 < \delta_1$, where $\tau \geqq 0$. If the inequality $\|u(t, \cdot) - v_0(t, \cdot)\|_2 < \delta_1$ does not hold for all $t \geqq \tau$, there exists $T > 0$ such that $\|u(T, \cdot) - v_0(T, \cdot)\|_2 = \delta_1$, $\|u(t, \cdot) - v_0(t, \cdot)\|_2 < \delta_1$ for all $t \in \langle\tau, T)$.

Let v be such a solution of $(\mathscr{M}[(1,10); (0,2)])$ that $v(T, x) = u(T, x)$, $v_t(T, x) = u_t(T, x)$ for all $x \in \langle 0, \pi\rangle$. Let us continue u, v and all members on the right-hand sides of the equations (1,10)' and (1,10) on the whole x-axis as odd and 2π-periodic functions in the variable x. Then

$$u(t, x) - v(t, x) = \frac{1}{2}\int_T^t \int_{x-t+\vartheta}^{x+t-\vartheta} \{a(\vartheta, \sigma, \varepsilon)[u(\vartheta, \sigma) - v(\vartheta, \sigma)] + \\ + b(\vartheta, \sigma, \varepsilon)[u_t(\vartheta, \sigma) - v_t(\vartheta, \sigma)] + c(\vartheta, \sigma, \varepsilon)[u_x(\vartheta, \sigma) - v_x(\vartheta, \sigma)] + \\ + G(\vartheta, \sigma, u(\vartheta, \sigma), u_t(\vartheta, \sigma), u_x(\vartheta, \sigma), \varepsilon)\} \, d\sigma \, d\vartheta$$

for all $t \geqq T$. By standard but laborious calculations it may be proved that there exists $K > 0$ so that

$$\|u(t, \cdot) - v(t, \cdot)\|_2^2 \leqq K(t - T)\int_T^t \{\|u(\vartheta, \cdot) - v(\vartheta, \cdot)\|_2^2 + \\ + \|G(\vartheta, \cdot, u, u_t, u_x, \varepsilon)\|_1^2\} \, d\vartheta \quad \text{for} \quad t \geqq T. \tag{1,11}$$

Lemma 1,2. (FILATOV [2], p. 78.) *Let φ, ψ, f be real continuous functions defined on $\langle a, b\rangle$, $f(t) > 0$ on $\langle a, b\rangle$ and*

$$\varphi(t) \leqq \psi(t) + \int_a^t f(s)\,\varphi(s)\,\mathrm{d}s \quad \text{for} \quad t \in \langle a, b\rangle\,.$$

Then

$$\varphi(t) \leqq \psi(t) + \int_a^t f(\tau)\,\psi(\tau) \exp\left[\int_a^\tau f(s)\,\mathrm{d}s\right] \mathrm{d}\tau\,.$$

Using this lemma, we can derive from (1,11) by easy calculations the inequality

$$\|u(t,\cdot) - v(t,\cdot)\|_2^2 \leqq [K(t-T) + e^{K(t-T)^2}] \int_T^t \|G(\vartheta, \cdot, u, u_t, u_x, \varepsilon)\|_1^2\,\mathrm{d}\vartheta$$

and this yields

$$(1,12) \qquad \|u(t,\cdot) - v_0(t,\cdot)\|_2^2 \leqq 2\|v(t,\cdot) - v_0(t,\cdot)\|_2^2 + \\ + 2[K(t-T) + e^{K(t-T)^2}] \int_T^t \|G(\vartheta, \cdot, u, u_t, u_x, \varepsilon)\|_1^2\,\mathrm{d}\vartheta\,, \quad t \geqq T\,.$$

Suppose that $\|G(\vartheta, \cdot, u, u_t, u_x, \varepsilon)\|_1 < \delta_2$ for all $\vartheta \geqq T$ such that $\|u(\vartheta,\cdot) - v_0(\vartheta,\cdot)\|_2 < \eta$ and let the last inequality hold for all $\vartheta \in \langle T, t\rangle$. Then we have

$$(1,13) \qquad \|u(t,\cdot) - v_0(t,\cdot)\|_2^2 \leqq 2K_1^2\delta_1^2 e^{-2K_2(t-T)} + \\ + 2(t-T)[K(t-T) + e^{K(t-T)^2}]\,\delta_2^2\,.$$

It may be proved that there exist $t_0 > 0$ and $C > 0$ (independent of η) so that

$$(1,14) \qquad \delta_2 = C\delta_1 = (C/2K_1)\,\eta\,,$$

$$2K_1^2 e^{-2K_2t_0} + 2t_0[Kt_0 + e^{Kt_0^2}]\,C^2 < 1\,,$$

$$2K_1^2 e^{-2K_2s} + 2s[Ks + e^{Ks^2}]\,C^2 < 4K_1^2 \quad \text{for} \quad s \in \langle 0, t_0\rangle\,.$$

Thus the estimate $\|u(\vartheta,\cdot) - v_0(\vartheta,\cdot)\|_2 < \eta$ must hold for all $\vartheta \in \langle T, T + t_0\rangle$. But it follows from (1,13) and (1,14) that $\|u(T + t_0,\cdot) - v_0(T + t_0,\cdot)\|_2 < \delta_1$, hence we can prove that $\|u(\vartheta,\cdot) - v_0(\vartheta,\cdot)\|_2 < \eta$ for $\vartheta \in \langle T + t_0, T + 2t_0\rangle$ in the same way. Successively we find that the inequality $\|u(\vartheta,\cdot) - v_0(\vartheta,\cdot)\|_2 < \eta$ holds for all $\vartheta \geqq T$.

II. We can prove that the solution v_0 has the same property as in part I similarly if we deal with the boundary conditions (0,6) (or (0,7)). The only essential differences are:

a) We continue u, v and all members on the right-hand sides of the equations $(1,10)'$ and $(1,10)$ as even, 4π-periodic functions in the variable x for $x \in E_1$ such that $u(t, x) = -u(t, 2\pi - x)$; the function v and all members on the right-hand sides of $(1,10)'$ and $(1,10)$ satisfy the same equality (in the case of the boundary conditions (0,7) we continue all functions mentioned above as even, 2π-periodic functions in the variable x for $x \in E_1$).

b) We can derive the inequality

$$\|u(t, \cdot) - v(t, \cdot)\|_2^2 \leqq K(t - T)^3 \int_T^t \{u(\vartheta, \cdot) - v(\vartheta, \cdot)\|_2^2 + \\ + \|G(\vartheta, \cdot, u, u_t, u_x, \varepsilon)\|_1^2 \, d\vartheta$$

instead of (1,11) in the case of the boundary conditions (0,7).

III. We shall prove the uniform stability at constantly acting disturbances of the solution v_0 of $(\mathscr{P}[1,7); \cdot])$ now.

Let us denote

$$(1,7)' \qquad u_{tt} - u_{xx} = F'(t, x, u, u_t, u_x, \varepsilon) + G'(t, x, u, u_t, u_x, \varepsilon)$$

where G' satisfies $(\mathscr{A}_1)$ and $(\mathscr{A}_3)$.

The right-hand side of $(1,7)'$ can be written in the form

$$F'(t, x, u, u_t, u_x, \varepsilon) + G'(t, x, u, u_t, u_x, \varepsilon) = \\ = F'(t, x, v_0, v_{0t}, v_{0x}, \varepsilon) + a(t, x, \varepsilon)(u - v_0) + b(t, x, \varepsilon)(u_t - v_{0t}) + \\ + c(t, x, \varepsilon)(u_x - v_{0x}) + \sum_{i,j=1}^{3} d_{ij}(t, x, u, u_t, u_x, \varepsilon)(u_i - v_{0i})(u_j - v_{0j}) + \\ + G'(t, x, u, u_t, u_x, \varepsilon).$$

Put

$$G(t, x, u, u_t, u_x, \varepsilon) = \sum_{i,j=1}^{3} d_{ij}(t, x, u, u_t, u_x, \varepsilon)(u_i - v_{0i})(u_j - v_{0j}) + \\ + G'(t, x, u, u_t, u_x, \varepsilon).$$

It may be proved that if u is a solution of $(\mathscr{M}[(1,7)'; \cdot])$ such that $\|u(t, \cdot) - v_0(t, \cdot)\|_2 < \eta'$, then there exists $K' > 0$ (depending on η') so that

$$(1,15) \quad \|\sum_{i,j=1}^{3} d_{ij}(t, \cdot, u, u_t, u_x, \varepsilon)(u_i - v_{0i})(u_j - v_{0j})\|_1 \leqq K'\|u(t, \cdot) - v_0(t, \cdot)\|_2^2 .$$

We can find $\eta > 0$, $\eta \leqq \eta'$ and $\delta_2' > 0$ so that

$$K'\eta^2 + \delta_2' < (C/2K_1)\,\eta$$

where C and K_1 are the constants from (1,14). There exists $\delta_1 > 0$ corresponding to η as in part I of this proof.

Let u solve $(\mathcal{M}[(1,7)'; \cdot])$ and let $\|u(\tau, \cdot) - v_0(\tau, \cdot)\|_2 < \delta_1$ for any $\tau \geqq 0$ and $\|G'(t, \cdot, u, u_t, u_x, \varepsilon)\|_1 < \delta'_2$ for $t \geqq \tau$ such that $\|u(t, \cdot) - v_0(t, \cdot)\|_2 < \eta'$. Because $\eta \leqq \eta'$, it is $\|G'(t, \cdot, u, u_t, u_x, \varepsilon)\|_1 < \delta'_2$ also for all $t \geqq \tau$ such that $\|u(t, \cdot) - v_0(t, \cdot)\|_2 < \eta$. Then these $t \geqq \tau$ satisfy the inequality

$$\|G(t, \cdot, u, u_t, u_x, \varepsilon)\|_1 < K'\eta^2 + \delta'_2 < (C/2K_1)\,\eta = \delta_2 .$$

Using the results of parts I and II, we have the estimate $\|u(t, \cdot) - v_0(t, \cdot)\|_2 < \eta \leqq \eta'$ for all $t \geqq \tau$. QED.

Theorem 1,2. *Let the zero solution of $(\mathcal{M}[(1,8); \cdot])$ be uniformly exponentially stable. Then the solution v_0 of $(\mathcal{P}[(1,7); \cdot])$ is uniformly exponentially stable, too.*

Proof. Writing the solutions v of $(\mathcal{M}[(1,7); \cdot])$ in the form $v = v_0 + u$ with u being a solution of $(\mathcal{M}[(1,16); \cdot])$, where

$$(1,16) \qquad u_{tt} - u_{xx} = a(t, x, \varepsilon)\, u + b(t, x, \varepsilon)\, u_t + c(t, x, \varepsilon)\, u_x + $$
$$+ d_{11}(t, x, u, u_t, u_x, \varepsilon)\, u^2 + d_{12}(t, x, u, u_t, u_x, \varepsilon)\, uu_t + \ldots + d_{33}(t, x, u, u_t, u_x, \varepsilon)\, u_x^2 ,$$

it can be shown that the uniform exponential stability of v_0 is equivalent to the same property of the zero solution of $(\mathcal{M}[(1,16); \cdot])$.

Hence it suffices to prove that the zero solution of $(\mathcal{M}[(1,16); \cdot])$ is uniformly exponentially stable.

Let us continue all solutions of $(\mathcal{M}[1,8); \cdot])$, $(\mathcal{M}[(1,16); \cdot])$ and the right-hand sides of (1,8) and (1,16) on the whole x-axis in accordance with the boundary conditions in the same way as in the proof of Theorem 1,1 in the case of $(\mathcal{M}[(1,10); \cdot])$, $(\mathcal{M}[(1,10)'; \cdot])$, (1,10) and (1,10)'.

The zero solution of $(\mathcal{M}[(1,8); \cdot])$ is uniformly exponentially stable, i.e. there exist $\delta > 0$, $K_1 > 0$, $K_2 > 0$ so that

$$(1,17) \qquad \|v(\tau, \cdot)\|_2 < \delta \Rightarrow \|v(t, \cdot)\|_2 \leqq K_1 \|v(\tau, \cdot)\|_2\, e^{-K_2(t-\tau)}, \quad t \geqq \tau$$

for every solution v of $(\mathcal{M}[(1,8); \cdot])$ and all $\tau \geqq 0$. (In fact, it follows from the linearity of the equation (1,8) that δ may be any positive number here.)

Let us choose $\lambda \in (0, K_2)$ and put

$$(1,18) \qquad \bar{v}(t, x) = e^{\lambda t}\, v(t, x) .$$

We can easily find that v is a solution of (1,8) if and only if $\bar{v}$ solves the equation

$$(1,19) \qquad \bar{v}_{tt} - \bar{v}_{xx} = [a(t, x, \varepsilon) - \lambda b(t, x, \varepsilon) - \lambda^2]\, \bar{v} + $$
$$+ [b(t, x, \varepsilon) + 2\lambda]\, \bar{v}_t + c(t, x, \varepsilon)\, \bar{v}_x .$$

There exist constants $C_1, C_2 > 0$ such that all functions $v, \bar{v}$ fulfilling (1,18) satisfy the inequality

$$(1,20) \qquad \|v(t, \cdot)\|_2 \leqq C_1 e^{-\lambda t} \|\bar{v}(t, \cdot)\|_2 \leqq C_2 \|v(t, \cdot)\|_2 , \quad t \geqq 0 .$$

Hence if $\bar{v}$ is a solution of $(\mathcal{M}[(1,19); \cdot])$, it holds

$$(1,21) \qquad \|\bar{v}(\tau, \cdot)\|_2 < \frac{\delta}{C_1} e^{\lambda\tau} \Rightarrow \|\bar{v}(t, \cdot)\|_2 \leqq$$

$$\leqq K_1 C_2 \|\bar{v}(\tau, \cdot)\|_2 \exp\left[-(K_2 - \lambda)(t - \tau)\right], \quad \tau \geqq 0 , \quad t \geqq \tau .$$

Similarly if $\bar{u}(t, x) = e^{\lambda t} u(t, x)$, then u is a solution of (1,16) if and only if $\bar{u}$ is a solution of the equation

$$(1,22) \qquad \bar{u}_{tt} - \bar{u}_{xx} = \left[a(t, x, \varepsilon) - \lambda b(t, x, \varepsilon) - \lambda^2\right] \bar{u} +$$

$$+ \left[b(t, x, \varepsilon) + 2\lambda\right] \bar{u}_t + c(t, x, \varepsilon) \bar{u}_x +$$

$$+ e^{-\lambda t}\{\left[d_{11} - \lambda d_{12} + \lambda^2 d_{22}\right] \bar{u}^2 + \left[d_{12} - 2\lambda d_{22}\right] \bar{u}\bar{u}_t +$$

$$+ \left[d_{13} - \lambda d_{23}\right] \bar{u}\bar{u}_x + d_{22}\bar{u}_t^2 + d_{23}\bar{u}_t\bar{u}_x + d_{33}\bar{u}_x^2\} ,$$

where $d_{ij} = d_{ij}(t, x, e^{-\lambda t}\bar{u}, e^{-\lambda t}(\bar{u}_t - \lambda\bar{u}), e^{-\lambda t}\bar{u}_x, \varepsilon)$ for $i, j = 1, 2, 3$.

Let us denote $\bar{a} = a - \lambda b - \lambda^2$, $\bar{b} = b + 2\lambda$, $\bar{c} = c$. Let T be an arbitrary non-negative number and let $\tilde{v}, \tilde{u}$ be solutions of $(\mathcal{M}[(1,19); \cdot])$, $(\mathcal{M}[(1,22); \cdot])$ such that

$$(1,23) \qquad \tilde{v}(T, x) = \tilde{u}(T, x) , \quad \tilde{v}_t(T, x) = \tilde{u}_t(T, x) , \quad x \in \langle 0, \pi \rangle .$$

Then $\tilde{v}$ and $\tilde{u}$ satisfy the integro-differential equation

$$\tilde{u}(t, x) - \tilde{v}(t, x) = \frac{1}{2} \int_T^t \int_{x-t+\vartheta}^{x+t-\vartheta} \{\bar{a}(\vartheta, \sigma) \left[\tilde{u}(\vartheta, \sigma) - \tilde{v}(\vartheta, \sigma)\right] +$$

$$+ \bar{b}(\vartheta, \sigma) \left[\tilde{u}_t(\vartheta, \sigma) - \tilde{v}_t(\vartheta, \sigma)\right] + \bar{c}(\vartheta, \sigma) \left[\tilde{u}_x(\vartheta, \sigma) - \tilde{v}_x(\vartheta, \sigma)\right] +$$

$$+ e^{-\lambda\vartheta}\left[d_{11} - \lambda d_{12} + \lambda^2 d_{22}\right] \tilde{u}^2(\vartheta, \sigma) + \left[d_{12} - 2\lambda d_{22}\right] \tilde{u}(\vartheta, \sigma) \tilde{u}(\vartheta, \sigma) +$$

$$+ \left[d_{13} - \lambda d_{23}\right] \tilde{u}(\vartheta, \sigma) \tilde{u}_x(\vartheta, \sigma) + d_{22} \tilde{u}_t^2(\vartheta, \sigma) +$$

$$+ d_{23} \tilde{u}_t(\vartheta, \sigma) \tilde{u}_x(\vartheta, \sigma) + d_{33} \tilde{u}_x^2(\vartheta, \sigma)]\} \, d\sigma \, d\vartheta ,$$

where $d_{ij} = d_{ij}(\vartheta, \sigma, e^{-\lambda\vartheta}\tilde{u}, e^{-\lambda\vartheta}(\tilde{u}_t - \lambda\tilde{u}), e^{-\lambda\vartheta}\tilde{u}_x, \varepsilon)$ for $i, j = 1, 2, 3$ here.

It can be proved by similar calculations as in the case of the inequality (1,11) that if we deal with the boundary conditions (0,2) or (0,6), there exists a constant $k > 0$

so that the following estimate holds:

$$(1,24)\qquad \|\tilde{u}(t,\cdot)-\tilde{v}(t,\cdot)\|_2^2 \leqq k(t-T)\int_T^t \{\|\tilde{u}(\vartheta,\cdot)-\tilde{v}(\vartheta,\cdot)\|_2^2 +$$

$$+ e^{-2\lambda\vartheta}\|[d_{11}(\vartheta,\cdot,e^{-\lambda\vartheta}\tilde{u},e^{-\lambda\vartheta}(\tilde{u}_t-\lambda\tilde{u}),e^{-\lambda\vartheta}\tilde{u}_x,\varepsilon) -$$

$$-\lambda d_{12}(\vartheta,\cdot,e^{-\lambda\vartheta}\tilde{u},e^{-\lambda\vartheta}(\tilde{u}_t-\lambda\tilde{u}),e^{-\lambda\vartheta}\tilde{u}_x,\varepsilon) +$$

$$+\lambda^2 d_{22}(\vartheta,\cdot,e^{-\lambda\vartheta}\tilde{u},e^{-\lambda\vartheta}(\tilde{u}_t-\lambda\tilde{u}),e^{-\lambda\vartheta}\tilde{u}_x,\varepsilon)]\,\tilde{u}^2(\vartheta,\cdot) + \ldots$$

$$\ldots + d_{33}(\vartheta,\cdot,e^{-\lambda\vartheta}\tilde{u},e^{-\lambda\vartheta}(\tilde{u}_t-\lambda\tilde{u}),e^{-\lambda\vartheta}\tilde{u}_x,\varepsilon)\,\tilde{u}_x^2(\vartheta,\cdot)\|_1^2\}\,d\vartheta\,.$$

A similar inequality may be derived also in the case of the boundary conditions (0,7). Nonetheless, in the sequel we shall use (1,24) only, because the rest of the proof is almost the same for the boundary conditions (0,7).

If we restrict ourselves to such $t \geqq T$ that for all $\vartheta \in \langle T, t)$ it holds $\|\tilde{u}(\vartheta,\cdot)\|_2 < Re^{\lambda\vartheta}$ (where R is a positive constant, large enough) and if we use (1,24), we get

$$\|\tilde{u}(t,\cdot)-\tilde{v}(t,\cdot)\|_2^2 \leqq K(t-T)\int_T^t \{\|\tilde{u}(\vartheta,\cdot)-\tilde{v}(\vartheta,\cdot)\|_2^2 + e^{-2\lambda\vartheta}\|\tilde{u}(\vartheta,\cdot)\|_2^4\}\,d\vartheta\,,$$

where K is a positive constant depending on R. Using Lemma 1,2 we can derive

$$\|\tilde{u}(t,\cdot)-\tilde{v}(t,\cdot)\|_2^2 \leqq K(t-T)\,(1+\exp[K(t-T)^2])\int_T^t e^{-2\lambda\vartheta}\|\tilde{u}(\vartheta,\cdot)\|_2^4\,d\vartheta\,,$$

$$(1,25)\qquad \|\tilde{u}(t,\cdot)\|_2^2 \leqq 2\|\tilde{v}(t,\cdot)\|_2^2 +$$

$$+ 2K(t-T)\,(1+\exp[K(t-T)^2])\int_T^t e^{-2\lambda\vartheta}\|\tilde{u}(\vartheta,\cdot)\|_2^4\,d\vartheta\,.$$

Further, let τ be an arbitrary nonnegative number and let us deal with the solutions $\bar{v}$ of $(\mathscr{M}[(1,19);\cdot])$ and $\bar{u}$ of $(\mathscr{M}[(1,22);\cdot])$ satisfying

$$(1,26)\qquad \bar{u}(\tau,x)=\bar{v}(\tau,x)\,,\quad \bar{u}_t(\tau,x)=\bar{v}_t(\tau,x)\,,\quad x\in\langle 0,\pi\rangle\,,$$

$$(1,27)\qquad 0 \neq \|\bar{v}(\tau,\cdot)\|_2 = \|\bar{u}(\tau,\cdot)\|_2 < \frac{\delta}{2K_1C_1C_2}\,e^{\lambda\tau}\,.$$

The constants K_1, C_2 can be surely chosen so that

$$(1,28)\qquad 2K_1C_2 > 1\,.$$

We want to prove that if $\delta > 0$ is sufficiently small, then

$$(1,29)\qquad \|\bar{u}(t,\cdot)\|_2 < 2K_1C_2\|\bar{u}(\tau,\cdot)\|_2\,,\quad t \geqq \tau\,.$$

Let us suppose that this is not true, i.e. that there exists $t_0 > \tau$ such that

$$(1{,}30) \qquad \|\bar{u}(t_0, \cdot)\|_2 = 2K_1C_2\|\bar{u}(\tau, \cdot)\|_2\,,$$

$$(1{,}31) \qquad \|\bar{u}(t, \cdot)\|_2 < \|\bar{u}(t_0, \cdot)\|_2\,, \quad t \in \langle \tau, t_0)\,.$$

There exists $r > 0$ such that

$$(1{,}32) \qquad (\sqrt{2})\, K_1 C_2 \exp\left[-(K_2 - \lambda)\, r\right] < 1\,.$$

Firstly, let us suppose that $t_0 \in (\tau, \tau + 2r)$. Using the relations (1,21), (1,25), (1,27), (1,30) and (1,31) we obtain

$$\|\bar{u}(t_0, \cdot)\|_2^2 = (2K_1C_2\|\bar{u}(\tau, \cdot)\|_2)^2 \leqq 2\|\bar{v}(t_0, \cdot)\|_2^2 +$$

$$+ 2K(t_0 - \tau)\left(1 + \exp\left[K(t_0 - \tau)^2\right]\right) \int_\tau^{t_0} e^{-2\lambda\vartheta} \|\bar{u}(\vartheta, \cdot)\|_2^4 \,\mathrm{d}\vartheta \leqq$$

$$\leqq 2K_1^2C_2^2\|\bar{u}(\tau, \cdot)\|_2^2 + 4rK(1 + \exp(4r^2K))\,(2K_1C_2\|\bar{u}(\tau, \cdot)\|_2)^4 \int_\tau^{+\infty} e^{-2\lambda\vartheta}\,\mathrm{d}\vartheta\,.$$

Thus, we have

$$1 \leqq 16rK(1 + \exp(4r^2K))\,(C_2K_1)^2\, \|\bar{u}(\tau, \cdot)\|_2^2\, \frac{1}{\lambda}\, e^{-2\lambda\tau}\,,$$

$$(1{,}33) \quad 1 \leqq 4rK(1 + \exp(4r^2K))\, \frac{\delta^2 e^{2\lambda\tau}}{C_1^2}\, \frac{1}{\lambda}\, e^{-2\lambda\tau} = 4rK(1 + \exp(4r^2K))\, \frac{\delta^2}{C_1^2\lambda}\,.$$

Secondly, let us suppose that $t_0 \in \langle \tau + 2r, \infty)$. Then t_0 can be expressed in the form $t_0 = \tau + rn + p$, where $p \in \langle r, 2r)$ and n is a natural number.

Let $\bar{v}_n$ be the function of variables t, x defined on $\langle \tau + rn, \infty) \times \langle 0, \pi \rangle$ which is a solution of $(\mathscr{M}[(1{,}19); \cdot])$ on its domain of definition and satisfies

$$(1{,}34) \qquad \bar{v}_n(\tau + rn, x) = \bar{u}(\tau + rn, x)\,,$$

$$\bar{v}_{nt}(\tau + rn, x) = \bar{u}_t(\tau + rn, x)\,, \quad x \in \langle 0, \pi \rangle\,.$$

Using the inequality

$$\|\bar{u}(\tau + rn, \cdot)\|_2 < 2K_1C_2\|\bar{u}(\tau, \cdot)\|_2 < \frac{\delta}{C_1}\, e^{\lambda\tau}\,,$$

we have with help of (1,21) and (1,34)

$$\|\bar{v}_n(t, \cdot)\|_2 \leqq K_1C_2\|\bar{v}_n(\tau + rn, \cdot)\|_2 \exp\left[-(K_2 - \lambda)(t - \tau - rn)\right],\ t \geqq \tau + rn\,,$$

hence

$$\|\bar{v}_n(t, \cdot)\|_2 \leqq K_1 C_2 \|\bar{u}(\tau + rn, \cdot)\|_2 \exp\left[-(K_2 - \lambda)(t - \tau - rn)\right], \quad t \geqq \tau + rn . \tag{1,35}$$

In virtue of (1,25) we have

$$\|\bar{u}(t_0, \cdot)\|_2^2 \leqq 2\|\bar{v}_n(t_0, \cdot)\|_2^2 + 2Kp(1 + \exp(Kp^2)) \int_{\tau+rn}^{t_0} e^{-2\lambda\vartheta} \|\bar{u}(\vartheta, \cdot)\|_2^4 \, d\vartheta$$

and using (1,35) we get

$$\|\bar{u}(t_0, \cdot)\|_2^2 \leqq 2K_1^2 C_2^2 \|\bar{u}(\tau + rn, \cdot)\|_2^2 \exp\left[-2(K_2 - \lambda) p\right] + $$
$$+ 2Kp(1 + \exp(Kp^2)) \int_{\tau+rn}^{t_0} e^{-2\lambda\vartheta} \|\bar{u}(\vartheta, \cdot)\|_2^4 \, d\vartheta .$$

Since $r \leqq p$, it is in virtue of (1,32):

$$\|\bar{u}(t_0, \cdot)\|_2^2 \leqq \|\bar{u}(\tau + rn, \cdot)\|_2^2 + $$
$$+ 2Kp(1 + \exp(Kp^2)) \int_{\tau+rn}^{t_0} e^{-2\lambda\vartheta} \|\bar{u}(\vartheta, \cdot)\|_2^4 \, d\vartheta ,$$

$$\|\bar{u}(t_0, \cdot)\|_2^2 \leqq \|\bar{u}(\tau + rn, \cdot)\|_2^2 + \tag{1,36}$$
$$+ 4Kr(1 + \exp(4Kr^2)) \int_{\tau+rn}^{t_0} e^{-2\lambda\vartheta} \|\bar{u}(\vartheta, \cdot)\|_2^4 \, d\vartheta .$$

Similarly we prove that

$$\|\bar{u}(\tau + ri, \cdot)\|_2^2 \leqq \|\bar{u}(\tau + r(i - 1), \cdot)\|_2^2 + \tag{1,37}$$
$$+ 4Kr(1 + \exp(4Kr^2)) \int_{\tau+r(i-1)}^{\tau+ri} e^{-2\lambda\vartheta} \|\bar{u}(\vartheta, \cdot)\|_2^4 \, d\vartheta , \quad (i = 1, \ldots, n) .$$

It follows from (1,36) and (1,37) that

$$\|\bar{u}(t_0, \cdot)\|_2^2 \leqq \|\bar{u}(\tau, \cdot)\|_2^2 + 4Kr(1 + \exp(4Kr^2)) \int_{\tau}^{t_0} e^{-2\lambda\vartheta} \|\bar{u}(\vartheta, \cdot)\|_2^4 \, d\vartheta \tag{1,38}$$

and by virtue of (1,27), (1,30) and (1,31) we have

$$4K_1^2 C_2^2 \|\bar{u}(\tau, \cdot)\|_2^2 = \|\bar{u}(t_0, \cdot)\|_2^2 \leqq \|\bar{u}(\tau, \cdot)\|_2^2 + $$
$$+ 4Kr(1 + \exp(4Kr^2)) (2K_1 C_2 \|\bar{v}(\tau, \cdot)\|_2)^4 \int_{\tau}^{+\infty} e^{-2\lambda\vartheta} \, d\vartheta ,$$

$$4K_1^2 C_2^2 - 1 \leqq 16Kr(1 + \exp(4Kr^2)) K_1^2 C_2^2 \left(\frac{\delta}{C_1} e^{\lambda\tau}\right)^2 \frac{1}{2\lambda} e^{-2\lambda\tau} . \tag{1,39}$$

Now we can choose $\delta > 0$ so small that neither (1,33) nor (1,39) holds, which contradicts the assumption of the existence of $t_0 > \tau$ satisfying (1,30) and (1,31). Thus the inequality (1,29) holds.

It follows from (1,20), (1,27) and (1,29) that for every solution u of $(\mathcal{M}[(1,16); \cdot])$ and all $\tau \geqq 0$,

$$\|u(\tau, \cdot)\|_2 < \frac{\delta}{2K_1C_2^2} \Rightarrow \|u(t, \cdot)\|_2 \leqq 2K_1C_2^2\|u(\tau, \cdot)\|_2\, e^{-\lambda(t-\tau)}, \quad t \geqq \tau$$

i.e. the zero solution of $(\mathcal{M}[(1,16); \cdot])$ is uniformly exponentially stable. QED.

Now we shall use the second Ljapunov method, which is based on the following theorem:

Theorem 1,3. *Suppose that for $t \geqq 0$ there exists a functional $V(t)$ on the space of (w_1, w_2) such that $w_1 \in W_2^3(0, \pi)$, $w_2 \in W_2^2(0, \pi)$. Assume that if we write $V(t)(u(t, \cdot), u_t(t, \cdot)) = V(t, u)$ for u fulfilling*

$$(1,40) \qquad u \in C(\langle 0, \infty); W_2^3((0, \pi))) \cap C^1(\langle 0, \infty); W_2^2((0, \pi))) \cap \cap C^2(\langle 0, \infty); W_2^1((0, \pi))),$$

it holds:

(1,41) *there exist $\alpha, \beta > 0$ such that*

$$\alpha\|u(t, \cdot)\|_2^2 \leqq V(t, u) \leqq \beta\|u(t, \cdot)\|_2^2$$

for all $t \geqq 0$ and u satisfying (1,40),

(1,42) *there exists $\gamma > 0$ such that*

$$\mathrm{d}V(t, u)/\mathrm{d}t \leqq -\gamma\|u(t, \cdot)\|_2^2$$

for all $t \geqq 0$ and all solutions u of $(\mathcal{M}[(1,8); \cdot])$.

Then the zero solution of $(\mathcal{M}[(1,8); \cdot])$ is uniformly exponentially stable.

Let us denote $(u, u_t, u_x, u_{tx}, u_{xx}) = (u_1, u_2, u_3, u_4, u_5)$. We shall investigate the functional $V(t, u)$ in the form

$$(1,43) \qquad V(t, u) = \int_0^\pi \sum_{i,j=1}^5 A_{ij}(t, x, \varepsilon)\, u_i u_j \,\mathrm{d}x$$

where A_{ij} are functions continuously differentiable with respect to t and x for $t \geqq 0$, $x \in \langle 0, \pi\rangle$ and ω-periodic in the variable t. Let the matrix of coefficients A_{ij} have in the case of the boundary conditions (0,2) the form

$$(1,44)\qquad \begin{pmatrix} A(t,x,\varepsilon), & C(x,\varepsilon), & D(x,\varepsilon), & 0, & 0 \\ C(x,\varepsilon), & B(x,\varepsilon), & 0, & 0, & 0 \\ D(x,\varepsilon), & 0, & B(x,\varepsilon)+G(x,\varepsilon), & E(x,\varepsilon), & F(x,\varepsilon)\,c(t,x,\varepsilon) \\ 0, & 0, & E(x,\varepsilon), & F(x,\varepsilon), & 0 \\ 0, & 0, & F(x,\varepsilon)\,c(t,x,\varepsilon), & 0, & F(x,\varepsilon) \end{pmatrix}$$

where the functions A, B, C, D, E, F, G have all properties of the coefficients $A_{ij}(t, x, \varepsilon)$ $(i, j = 1, 2, \ldots, 5)$ and c is the function on the right-hand side of the equation (1,8). Using the fact that if u is a solution of $(\mathcal{M}[(1,8); (0,2)])$, it satisfies

$$u_{xx}(t,0) + c(t,0,\varepsilon)\,u_x(t,0) = u_{xx}(t,\pi) + c(t,\pi,\varepsilon)\,u_x(t,\pi) = 0\,, \quad t \geqq 0$$

and integrating by parts we can prove that for every solution u of $(\mathcal{M}[(1,8); (0,2)])$ it holds

$$(1,45)\qquad \mathrm{d}V(t,u)/\mathrm{d}t = \int_0^\pi \sum_{i,j=1}^5 B_{ij}(t,x,\varepsilon)\,u_i u_j\,\mathrm{d}x$$

where the matrix of the coefficients B_{ij} has the form

$$(1,46)\qquad \begin{pmatrix} A_t + 2Ca, & A + Ba + Cb - D_x, & Cc - C_x + Ea_x, & Fa_x, & 0 \\ A + Ba + Cb - D_x, & 2C + 2Bb, & Bc - B_x + Eb_x, & Fb_x, & 0 \\ Cc - C_x + Ea_x, & Bc - B_x + Eb_x, & -2C - 2E_x c + 2Ea, & Eb + Fa - F_x c + G, & -E_x - Ec + Fc_t \\ Fa_x, & Fb_x, & Eb + Fa - F_x c + G, & 2E + 2Fb, & Fc - F_x \\ 0, & 0, & -E_x - Ec + Fc_t, & Fc - F_x, & -2E \end{pmatrix}.$$

Similarly, let us choose the matrix $(A_{ij})_{i,j=1,\ldots,5}$ as in (1,44), only put $D(x, \varepsilon) \equiv 0$ in the case of the boundary conditions (0,6) or (0,7). Then, by means of integration by parts and using the boundary conditions, we can find that if u is a solution of $(\mathcal{M}[(1,8); (0,6)])$ or $(\mathcal{M}[(1,8); (0,7)])$, $\mathrm{d}V(t, x)/\mathrm{d}x$ has the form (1,45), where the matrix $(B_{ij})_{i,j=1,\ldots,5}$ is the same as (1,46), only with $D_x(x, \varepsilon) \equiv 0$.

It is quite clear that if it exist a positive definite matrix $(A^*_{ij}(t, x, \varepsilon))_{i,j=1,\ldots,5}$ and a negative definite matrix $(B^*_{ij}(t, x, \varepsilon))_{i,j=1,\ldots,5}$ so that all coefficients A^*_{ij}, B^*_{ij} $(i, j =$

$= 1, \dots, 5)$ are continuous in all variables and ω-periodic in the variable t and if

$$(1,47) \qquad \int_0^\pi \sum_{i,j=1}^{5} A_{ij}(t, x, \varepsilon)\, u_i u_j \,\mathrm{d}x \geqq \int_0^\pi \sum_{i,j=1}^{5} A_{ij}^*(t, x, \varepsilon)\, u_i u_j \,\mathrm{d}x\,,$$

$$(1,48) \qquad \int_0^\pi \sum_{i,j=1}^{5} B_{ij}(t, x, \varepsilon)\, u_i u_j \,\mathrm{d}x \leqq \int_0^\pi \sum_{i,j=1}^{5} B_{ij}^*(t, x, \varepsilon)\, u_i u_j \,\mathrm{d}x$$

for every solution u of $(\mathscr{M}[(1,8); \cdot])$, then the functional $V(t, u)$ satisfies the conditions (1,41) and (1,42), i.e. the zero solution of $(\mathscr{M}[(1,8); \cdot])$ is uniformly exponentially stable.

Thus we shall investigate the conditions under which there exist a positive definite matrix $(A_{ij}^*)_{i,j=1,\dots,5}$ and a negative definite matrix $(B_{ij}^*)_{i,j=1,\dots,5}$ satisfying (1,47) and (1,48). But we shall deal with some rather more special forms of the right-hand side of the equation (1,8) and making use of Theorems 1,3, 1,2, 1,1 and Lemmas 1,3, 1,1, we shall derive sufficient conditions for the uniform exponential stability and the uniform stability at constantly acting disturbances of the periodic solution u_0 of the problem $(\mathscr{P})$ with rather more special forms of the right-hand side of (0,1), corresponding to the right-hand sides of (1,8).

Remark 1,1. We have supposed that (1,9) holds because we used the fact that the right-hand sides of the equations (1,8), (1,10), (1,10)' and (1,16) satisfied $(\mathscr{A}_3)$. This assumption guaranteed the continuity of functions which had resulted from these right-hand sides by their continuation on the whole x-axis in the variable x in the way described in the proofs of Theorems 1,1 and 1,2. Then the solutions of the equations (1,8), (1,10), (1,10)' and (1,16) satisfied the integro-differential relations, used in these proofs.

Now we shall show that the condition (1,9) is no more necessary. Assume that the zero solution of (1,8) is uniformly exponentially stable and that (1,9) does not hold. Then we can put $\chi(x) = \chi_1(x) + \chi_2(x)$, where χ_1, χ_2 are of the class C^3 on $\langle 0, \pi\rangle$, χ_1 satisfies (1,9) and $\chi_1'(0) = \alpha_0$ in the case of the boundary conditions (0,6). If we put $w(t, x) = u(t, x) \,.\, e^{-x_2(x)}$, we can easily find that w solves the equation

$$(1,49) \qquad w_{tt} - w_{xx} = [(\chi_2')^2 + \chi_2'' + a + c\chi_2']\, w + bw_t + [2\chi_2' + c]\, w_x$$

if and only if u solves (1,8) and that the zero solution of (1,49) is uniformly exponentially stable if and only if the zero solution of (1,8) is uniformly exponentially stable. If we recall the definition of the coefficients a, b, c, we see that the equation (1,49) may be written in the form

$$(1,49\text{a}) \quad w_{tt} - w_{xx} = \left[(\chi_1')^2 - \chi_1'' + \frac{\partial F}{\partial u}(t, x, u_0, u_{0t}, u_{0x}, \varepsilon) - \chi_1' \frac{\partial F}{\partial u_x}(t, x, u_0, u_{0t}, u_{0x}, \varepsilon)\right] w +$$
$$+ \frac{\partial F}{\partial u_t}(t, x, u_0, u_{0t}, u_{0x}, \varepsilon)\, w_t + \left[-2\chi_1' + \frac{\partial F}{\partial u_x}(t, x, u_0, u_{0t}, u_{0x}, \varepsilon)\right] w_x \,.$$

Now it is seen that we can write (1,49) also in the form (1,8) but we must consider χ_1 instead of χ in the definition of the function F' and hence also of the coefficients a, b, c in the equation (1,8). But χ_1 satisfies (1,9) and so Theorems 1,1, 1,2 and Lemma 1,1 (again with χ_1 instead of χ) imply that the solution u_0 of $(\mathscr{P})$ is uniformly exponentially stable and uniformly stable at constantly acting disturbances.

2. THE UNIFORM EXPONENTIAL STABILITY THEOREMS

Lemma 2,1. *Let the functions a_1, c_1 be of the class C^1 on $\langle 0, \pi\rangle$, let b_1 by of the class C^1 on $\langle 0, \infty) \times \langle 0, \pi\rangle$ and a_2, b_2 c_2 of the class C^1 on $\langle 0, \infty) \times \langle 0, \pi\rangle \times \times \langle 0, \varepsilon_1\rangle$ and let b_1, a_2, b_2, c_2 be ω-periodic in the variable t.*

Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in \langle 0, \varepsilon_0\rangle$ the zero solution of $(\mathscr{M}[(2,1); (0,2)])$, where

$$u_{tt} - u_{xx} = a_1(x)\, u + b_1(t, x)\, u_t + c_1(x)\, u_x + \varepsilon a_2(t, x, \varepsilon)\, u + \varepsilon b_2(t, x, \varepsilon)\, u_t + \varepsilon c_2(t, x, \varepsilon)\, u_x\,, \tag{2,1}$$

is uniformly exponentially stable, if

$$b_1(t, x) < 0\,, \tag{2,1,1}$$

$$a_1(x) < \left[\min_{x \in \langle 0, \pi\rangle} \exp\left(\int_0^x c_1(\sigma)\, \mathrm{d}\sigma\right)\right] \exp\left(-\int_0^x c_1(\sigma)\, \mathrm{d}\sigma\right) \tag{2,1,2}$$

for $t \in \langle 0, \omega\rangle$, $x \in \langle 0, \pi\rangle$.

Proof. We shall investigate the matrix (1,44) in the form

$$\begin{pmatrix} -B_0(a_1 + \varepsilon a_2) \exp \textstyle\int, & C_0 \exp \textstyle\int, & 0, & 0, & 0 \\ C_0 \exp \textstyle\int, & B_0 \exp \textstyle\int, & 0, & 0, & 0 \\ 0, & 0, & B_0 \exp \textstyle\int, & E_0\eta \exp(-\textstyle\int), & F_0\eta(c_1 + \varepsilon c_2) \exp \textstyle\int \\ 0, & 0, & E_0\eta \exp(-\textstyle\int), & F_0\eta \exp \textstyle\int, & 0 \\ 0, & 0, & F_0\eta(c_1 + \varepsilon c_2) \exp \textstyle\int, & 0, & F_0\eta \exp \textstyle\int \end{pmatrix} \tag{2,1,3}$$

where $\varepsilon \in \langle 0, \varepsilon_1\rangle$, $\eta \in (0, 1)$, $\exp \int \equiv \exp\left(\int_0^x c_1(\sigma)\, \mathrm{d}\sigma\right)$, $\exp(-\int) \equiv \exp\left(-\int_0^x c_1(\sigma)\, \mathrm{d}\sigma\right)$, B_0, C_0, E_0, F_0 are positive constants.

Let us denote $m = \min\limits_{x \in \langle 0, \pi \rangle} \exp\left(\int_0^x c_1(\sigma)\,\mathrm{d}\sigma\right)$ and let $\delta \in \langle 0, 1 \rangle$. Since every solution u of $(\mathcal{M}[(2,1;(0,2)])$ satisfies (0,2), we can use the Rayleigh inequality (see [5]):

$$\int_0^\pi u^2(t, x)\,\mathrm{d}x \leqq \int_0^\pi u_x^2(t, x)\,\mathrm{d}x\,. \tag{2,2}$$

Then we get

$$\begin{aligned}
&\int_0^\pi \{B_0 (\exp \textstyle\int)\, u_x^2 + 2E_0\eta \exp(-\textstyle\int)\, u_x u_{tx} + 2F_0\eta(c_1 + \varepsilon c_2)(\exp \textstyle\int)\, u_x u_{xx}\}\,\mathrm{d}x = \\
&= \int_0^\pi \{[B_0 m(1 - \delta) + B_0 m\delta + B_0(\exp \textstyle\int - m)]\, u_x^2 + \\
&\quad + 2(E_0\eta \exp(-\textstyle\int))^{1/4} (E_0\eta \exp(-\textstyle\int))^{3/4}\, u_x u_{tx} + \\
&\quad + 2(F_0\eta(c_1 + \varepsilon c_2) \exp \textstyle\int)^{1/4} (F_0\eta(c_1 + \varepsilon c_2) \exp \textstyle\int)^{3/4}\, u_x u_{xx}\}\,\mathrm{d}x \geqq \\
&\geqq \int_0^\pi \{B_0 m(1 - \delta)\, u^2 + B_0[m\delta + \exp \textstyle\int - m]\, u_x^2 - |E_0\eta \exp(-\textstyle\int)|^{1/2}\, u_x^2 - \\
&\quad - |E_0\eta \exp(-\textstyle\int)|^{3/2}\, u_{tx}^2 - |F_0\eta(c_1 + \varepsilon c_2) \exp \textstyle\int|^{1/2}\, u_x^2 - \\
&\quad - |F_0\eta(c_1 + \varepsilon c_2) \exp \textstyle\int|^{3/2}\, u_{xx}^2\}\,\mathrm{d}x\,.
\end{aligned}$$

Hence (1,47) holds, where the matrix $(A_{ij}^*)_{i,j=1,\ldots,5}$ has the form

$$\begin{pmatrix}
-B_0(a_1 + \varepsilon a_2) \exp \textstyle\int + B_0 m(1 - \delta), & C_0 \exp \textstyle\int, & 0, & 0, & 0 \\
C_0 \exp \textstyle\int, & B_0 \exp \textstyle\int, & 0, & 0, & 0 \\
0, & 0, & B_0[m\delta + \exp \textstyle\int - m] - |E_0\eta \exp(-\textstyle\int)|^{1/2} - |F_0\eta(c_1 + \varepsilon c_2) \exp \textstyle\int|^{1/2}, & 0, & 0 \\
0, & 0, & 0, & F_0\eta \exp \textstyle\int - |E_0\eta \exp(-\textstyle\int)|^{3/2}, & 0 \\
0, & 0, & 0, & 0, & F_0\eta \exp \textstyle\int - |F_0\eta(c_1 + \varepsilon c_2) \exp \textstyle\int|^{3/2}
\end{pmatrix}$$

We can easily find that the form of the matrix (1,46) corresponding to the special form (2,1,3) of the matrix (1,44) is

(2,1,4)
$$
\begin{aligned}
B_{11} &= -B_0\varepsilon a_{2t}\exp\int + 2C_0(a_1 + \varepsilon a_2)\exp\int\,,\\
B_{12} &= B_{21} = C_0(b_1 + \varepsilon b_2)\exp\int\,,\\
B_{13} &= B_{31} = E_0\eta(a_{1x} + \varepsilon a_{2x})\exp(-\int) + C_0\varepsilon c_2\exp\int\,,\\
B_{14} &= B_{41} = F_0\eta(a_{1x} + \varepsilon a_{2x})\exp\int\,,\quad B_{15} = B_{51} = 0\,,\\
B_{22} &= 2B_0(b_1 + \varepsilon b_2)\exp\int + 2C_0\exp\int\,,\\
B_{23} &= B_{32} = E_0\eta(b_{1x} + \varepsilon b_{2x})\exp(-\int) + B_0\varepsilon c_2\exp\int\,,\\
B_{24} &= B_{42} = F_0\eta(b_{1x} + \varepsilon b_{2x})\exp\int\,,\quad B_{25} = B_{52} = 0\,,\\
B_{33} &= -2C_0\exp\int + 2E_0\eta c_1(c_1 + \varepsilon c_2)\exp(-\int) +\\
&\quad + 2E_0\eta(a_1 + \varepsilon a_2)\exp(-\int)\,,\\
B_{34} &= B_{43} = E_0\eta(b_1 + \varepsilon b_2)\exp(-\int) + F_0\eta(a_1 + \varepsilon a_2)\exp\int -\\
&\quad - F_0\eta c_1(c_1 + \varepsilon c_2)\exp\int\,,\\
B_{35} &= B_{53} = -E_0\eta\varepsilon c_2\exp(-\int) + F_0\eta\varepsilon c_{2t}\exp\int\,,\\
B_{44} &= 2F_0\eta(b_1 + \varepsilon b_2)\exp\int + 2E_0\eta\exp(-\int)\,,\\
B_{45} &= B_{54} = F_0\eta c_2\varepsilon\exp\int\,,\quad B_{55} = -2E_0\eta\exp(-\int)\,.
\end{aligned}
$$

It can be found that if u is a solution of $(\mathscr{M}[(2,1);(0,2)])$, then (1,48) holds with

$$
\begin{aligned}
B^*_{11} &= 2C_0(a_1 + \varepsilon a_2)\exp\int - B_0\varepsilon a_{2t}\exp\int - 2C_0 m(1-\delta) +\\
&\quad + \left|F_0\eta(a_{1x} + \varepsilon a_{2x})\exp\int\right|^{1/2} + \left|E_0\eta(a_{1x} + \varepsilon a_{2x})\exp(-\int)\right| + \left|C_0\varepsilon c_2\exp\int\right|,\\
B^*_{12} &= B^*_{21} = C_0(b_1 + \varepsilon b_2)\exp\int\,,\quad B^*_{13} = B^*_{31} = B^*_{14} = B^*_{41} = B^*_{15} = B^*_{51} = 0\,,\\
B^*_{22} &= 2B_0(b_1 + \varepsilon b_2)\exp\int + 2C_0\exp\int + \left|F_0\eta(b_{1x} + \varepsilon b_{2x})\exp\int\right|^{1/2} +\\
&\quad + \left|E_0\eta(b_{1x} + \varepsilon b_{2x})\exp(-\int)\right| + \left|B_0\varepsilon c_2\exp\int\right|,\\
B^*_{23} &= B^*_{32} = B^*_{24} = B^*_{42} = B^*_{25} = B^*_{52} = 0\,,\\
B^*_{33} &= -2C_0(\exp\int - m + \delta m) + \left|E_0\eta(b_1 + \varepsilon b_2)\exp(-\int) +\right.\\
&\quad \left.+ F_0\eta(a_1 + \varepsilon a_2)\exp\int - F_0\eta c_2(c_1 + \varepsilon c_2)\exp\int\right|^{1/2} +\\
&\quad + \left|E_0\eta(a_{1x} + \varepsilon a_{2x})\exp(-\int)\right| + \left|E_0\eta(b_{1x} + \varepsilon b_{2x})\exp(-\int)\right| +\\
&\quad + \left|C_0\varepsilon c_2\exp\int\right| + \left|B_0\varepsilon c_2\exp\int\right| + \left|E_0\eta\varepsilon c_2\exp(-\int)\right|,\\
B^*_{34} &= B^*_{43} = B^*_{35} = B^*_{53} = 0\,,
\end{aligned}
$$

$$B^*_{44} = 2F_0\eta(b_1 + \varepsilon b_2)\exp\int + 2E_0\eta\exp\left(-\int\right) + \left|F_0\eta(a_{1x} + \varepsilon a_{2x})\exp\int\right|^{3/2} +$$
$$+ \left|F_0\eta(c_{1x} + \varepsilon c_{2x})\exp\int\right|^{3/2} + \left|E_0\eta(b_1 + \varepsilon b_2)\exp\left(-\int\right) +\right.$$
$$\left. + F_0\eta(a_1 + \varepsilon a_2)\exp\int - F_0\eta c_1(c_1 + \varepsilon c_2)\exp\int\right|^{3/2} + \left|F_0\eta\varepsilon c_2\exp\int\right|,$$
$$B^*_{45} = B^*_{54} = 0, \quad B^*_{55} = -2E_0\eta\exp\left(-\int\right) + \left|E_0\eta\varepsilon c_2\exp\left(-\int\right)\right| + \left|F_0\eta\varepsilon c_2\exp\int\right|.$$

The matrix $(A^*_{ij})_{i,j=1,\dots,5}$ is positive definite and the matrix $(B^*_{ij})_{i,j=1,\dots,5}$ is negative definite provided the following inequalities are fulfilled:

$$A^*_{11}A^*_{22} - A^{*2}_{12} > 0, \quad B^*_{11}B^*_{22} - B^{*2}_{12} > 0, \quad A^*_{ii} > 0, \quad B^*_{ii} < 0 \quad (i = 2, 3, 4, 5).$$

It follows from (2,1,2) that there exists $\delta \in (0, 1)$ such that $a_1(x) < (1 - \delta) \cdot \exp\left(-\int\right) m$. Hence if $\varepsilon \geqq 0$, $\eta > 0$ and $C_0/B_0 > 0$ are sufficiently small, then $A^*_{11}A^*_{22} - A^{*2}_{12} > 0$ and $B^*_{11}B^*_{22} - B^{*2}_{12} > 0$. All other required inequalities are fulfilled for sufficiently small $\varepsilon \geqq 0$, $\eta > 0$ and $E_0/F_0 > 0$.

Hence there exists $\varepsilon_0 > 0$ such that if $\varepsilon \in \langle 0, \varepsilon_0 \rangle$, then the zero solution of $(\mathcal{M}[(2,1); (0,2)])$ is uniformly exponentially stable. QED.

We can easily prove the following theorem with help of Lemma 2,1 and the results of § 1.

Theorem 2,1. *Let the right-hand side of the equation*

$$(2,3) \qquad u_{tt} - u_{xx} = g(t, x, u_t) + a(x)\,u + c(x)\,u_x + \varepsilon f(t, x, u, u_t, u_x, \varepsilon)$$

satisfy the conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3)$ *and let* $\chi(x)$ *be any function of the class* C^3 *on* $\langle 0, \pi \rangle$ *such that*

$$(2,1,5) \qquad \frac{\partial}{\partial u_t} g(t, x, u_{0t}) < 0,$$

$$(2,1,6) \qquad a(x) - c(x)\,\chi'(x) - \chi''(x) + \chi'^2(x) <$$
$$< \exp\left(-\int_0^x c(\sigma)\,d\sigma + 2\,\chi(x)\right) \cdot \min_{x \in \langle 0,\pi \rangle} \exp\left(\int_0^x c(\sigma)\,d\sigma - 2\,\chi(x)\right)$$

for $t \in \langle 0, \omega \rangle$ *and* $x \in \langle 0, \pi \rangle$.

Then there exists $\varepsilon_0 > 0$ *such that for* $\varepsilon \in \langle 0, \varepsilon_0 \rangle$ *the solution* u_0 *of* $(\mathcal{P}[(2,3); (0,2)])$ *is uniformly exponentially stable.*

Example. The existence of a 2π-periodic solution of the telegraph equation

$$u_{tt} - u_{xx} = au + bu_t + cu_x + \varepsilon f(t, x, u, u_t, u_x, \varepsilon)$$

(where a, b, c are constants) with the boundary conditions (0,2) is investigated in [1].

Using Theorem 2,1 we can find that if the right-hand side of this equation satisfies $(\mathscr{A}_1)$, $(\mathscr{A}_2)$ and $(\mathscr{A}_3)$, then the 2π-periodic solution is uniformly exponentially stable if $b < 0$, $a - \frac{1}{2}c^2 < 1$. It suffices only to choose $\chi(x) = \frac{1}{2}cx$ in Theorem 2,1.

Lemma 2,2. *Let the functions a_1, c_1 be of the class C^1 on $\langle 0, \pi\rangle$, let c_2 be of the class C^1 on $\langle 0, \pi\rangle \times \langle 0, \varepsilon_1\rangle$ and a_2, b of the class C^1 on $\langle 0, \infty) \times \langle 0, \pi\rangle \times \langle 0, \varepsilon_1\rangle$. Let a_2, b be ω-periodic in the variable t.*

Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0\rangle$ the zero solution of $(\mathscr{M}[(2,4); (0,2)])$, where

$$(2,4) \qquad u_{tt} - u_{xx} = [a_1(x) + \varepsilon a_2(t, x, \varepsilon)]\, u + \varepsilon b(t, x, \varepsilon)\, u_t + [c_1(x) + \varepsilon c_2(x, \varepsilon)]\, u_x\,,$$

is uniformly exponentially stable, if

$$(2,2,1) \qquad b(t, x, 0) < 0\,,$$

$$(2,2,2) \qquad \exp\left(-\int_0^x c_1(\sigma)\,d\sigma\right) . \min_{x\in\langle 0,\pi\rangle} \exp\left(\int_0^x c_1(\sigma)\,d\sigma\right) - a_1(x) > 0\,,$$

$$(2,2,3) \quad a_{2t}(t, x, 0) > 2\left[\exp\left(-\int_0^x c_1(\sigma)\,d\sigma\right) . \min_{x\in\langle 0,\pi\rangle} \exp\left(\int_0^x c_1(\sigma)\,d\sigma\right) - a_1(x)\right] . \max_{\substack{x\in\langle 0,\pi\rangle \\ t\in\langle 0,\omega\rangle}} b(t, x, 0)$$

for $t \in \langle 0, \omega\rangle$, $x \in \langle 0, \pi\rangle$.

This lemma can be proved in the same way as Lemma 2,1. That is why we do not preset all its proof here but only indicate the form of the matrix (1,44).

Let us denote $\exp \int = \exp [\int_0^x (c_1(\sigma) + \varepsilon c_2(\sigma, \varepsilon))\,d\sigma]$, $\exp(-\int) = \exp[-\int_0^x (c_1(\sigma) + \varepsilon c_2(\sigma, \varepsilon))\,d\sigma]$ and let B_0, C_0, E_0, F_0 be positive constants. Then the matrix (1,44) can be chosen in the special form

$$\begin{pmatrix} -B_0(a_1 + \varepsilon a_2)\exp\int, & C_0\varepsilon\exp\int, & 0, & 0, & 0 \\ C_0\varepsilon\exp\int, & B_0\exp\int, & 0, & 0, & 0 \\ 0, & 0, & B_0\exp\int - F_0\varepsilon^2 a_1\exp\int + F_0\varepsilon^2 c_1^2\exp\int, & E_0\varepsilon^3\exp(-\int), & F_0\varepsilon^2(c_1 + \varepsilon c_2)\exp\int \\ 0, & 0, & E_0\varepsilon^3\exp(-\int), & F_0\varepsilon^2\exp\int, & 0 \\ 0, & 0, & F_0\varepsilon^2(c_1 + \varepsilon c_2)\exp\int, & 0, & F_0\varepsilon^2\exp\int \end{pmatrix}.$$

Lemma 2.2. enables us to derive:

Theorem 2,2. *Let the right-hand side of the equation*

$$(2,5) \qquad u_{tt} - u_{xx} = \varepsilon f(t, x, u, u_t, \varepsilon) + \varepsilon c(x, \varepsilon)\, u_x$$

satisfy the conditions $(\mathscr{A}_1)$, $(\mathscr{A}_2)$, $(\mathscr{A}_3)$ *and let* $\chi(x)$ *be any function of the class* C^3 *on* $\langle 0, \pi\rangle$ *such that*

$$(2,2,4) \qquad \frac{\partial}{\partial u_t} f(t, x, u_0, u_{0t}, 0) < 0\,,$$

$$(2,2,5) \qquad \exp(2\,\chi(x))\,.\,\min_{x\in\langle 0,\pi\rangle} \exp(-2\,\chi(x)) + \chi''(x) - \chi'^2(x) > 0\,,$$

$$(2,2,6) \qquad \frac{\partial^2}{\partial u\,\partial t} f(t, x, u_0, u_{0t}, 0) > 2[\exp(2\,\chi(x))\,.\,\min_{x\in\langle 0,\pi\rangle} \exp(-2\,\chi(x)) + \chi''(x) - \chi'^2(x)]\,.$$

$$.\,\max_{\substack{t\in\langle 0,\omega\rangle \\ x\in\langle 0,\pi\rangle}} \frac{\partial}{\partial u_t} f(t, x, u_0, u_{0t}, 0)$$

for $t \in \langle 0, \omega\rangle$, $x \in \langle 0, \pi\rangle$.

Then there exists $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0\rangle$ *the solution* u_0 *of* $(\mathscr{P}[(2,5);(0,2)])$ *is uniformly exponentially stable.*

Remark 2,1. It is seen that the assertion of Theorem 2,2 holds if

$$\frac{\partial}{\partial u_t} f(t, x, u_0, u_{0t}, 0) < 0\,,$$

$$\frac{\partial^2}{\partial u\,\partial t} f(t, x, u_0, u_{0t}, 0) > 2 \max_{\substack{t\in\langle 0,\omega\rangle \\ x\in\langle 0,\pi\rangle}} \frac{\partial}{\partial u_t} f(t, x, u_0, u_{0t}, 0)$$

for $t \in \langle 0, \omega\rangle$, $x \in \langle 0, \pi\rangle$. It suffices to choose $\chi(x) = 0$ in Theorem 2,2.

In the conclusion we shall write theorems analogous to Theorems 2,1 and 2,2, but dealing with the boundary conditions (0,3) and (0,4) instead of (0,2). These theorems could be proved similarly as Theorems 2,1 and 2,2, only without the use of the Rayleigh inequality $\int_0^\pi u^2(t, x)\,dx \leqq \int_0^\pi u_x^2(t, x)\,dx$. Nevertheless, we can use instead the inequality $\frac{1}{4}\int_0^x u^2(t, x)\,dx \leqq \int_0^\pi u_x^2(t, x)\,dx$ (see [5]) for solutions u of the equations corresponding to (2,1) and (2,4) in the case of the boundary conditions (0,3).

Theorem 2,3. *Let the right-hand side of* (2,3) *satisfy* $(\mathscr{A}_1)$, $(\mathscr{A}_2)$, $(\mathscr{A}_3)$ *and let* $\chi(x)$ *be any function of the class* C^3 *on* $\langle 0, \pi\rangle$ *such that* $\chi'(0) = \alpha_0$, (2,1,5) *holds and*

$$a(x) - c(x)\,\chi'(x) - \chi''(x) + \chi'^2(x) <$$

$$< \tfrac{1}{4} \exp\left(-\int_0^x c(\sigma)\,d\sigma + 2\,\chi(x)\right).\,\min_{x\in\langle 0,\pi\rangle} \exp\left(\int_0^x c(\sigma)\,d\sigma - 2\,\chi(x)\right)$$

for $t \in \langle 0, \omega\rangle$, $x \in \langle 0, \pi\rangle$.

Then there exists $\varepsilon_0 > 0$ *such that for* $\varepsilon \in \langle 0, \varepsilon_0 \rangle$ *the solution* u_0 *of* $(\mathscr{P}[(2,3);(0,3)])$ *is uniformly exponentially stable.*

Theorem 2,4. *Let the right-hand side of the equation* (2,5) *satisfy the conditions* $(\mathscr{A}_1), (\mathscr{A}_2), (\mathscr{A}_3)$ *and let* $\chi(x)$ *be any function of the class* C^3 *on* $\langle 0, \pi \rangle$ *such that* $\chi'(0) = \alpha_0$. *Let* (2,2,4) *and* (2,2,5) *hold and let*

$$\frac{\partial^2}{\partial u\, \partial t} f(t, x, u_0, u_{0t}, 0) > 2[\tfrac{1}{4} \exp(2\chi(x)) \, . \min_{x \in \langle 0, \pi \rangle} \exp(-2\chi(x)) + \chi''(x) - \chi'^2(x)] \, . \max_{\substack{t \in \langle 0, \omega \rangle \\ x \in \langle 0, \pi \rangle}} \frac{\partial}{\partial u_t} f(t, x, u_0, u_{0t}, 0)$$

for $t \in \langle 0, \omega \rangle$, $x \in \langle 0, \pi \rangle$.

Then there exists $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0 \rangle$ *the solution* u_0 *of* $(\mathscr{P}[(2,5);(0,3)])$ *is uniformly exponentially stable.*

Theorem 2,5. *Let the right-hand side of* (2,3) *satisfy* $(\mathscr{A}_1), (\mathscr{A}_2), (\mathscr{A}_3)$ *and let* $\chi(x)$ *be any function of the class* C^3 *on* $\langle 0, \pi \rangle$ *such that* $\chi'(0) = \alpha_0$, $\chi'(\pi) = \alpha_\pi$, (2,1,5) *holds and*

$$a(x) - c(x)\, \chi'(x) - \chi''(x) + \chi'^2(x) < 0$$

for $t \in \langle 0, \omega \rangle$, $x \in \langle 0, \pi \rangle$.

Then there exists $\varepsilon_0 > 0$ *such that for* $\varepsilon \in \langle 0, \varepsilon_0 \rangle$ *the solution* u_0 *of* $(\mathscr{P}[(2,3);(0,4)])$ *is uniformly exponentially stable.*

Theorem 2,6. *Let the right-hand side of the equation* (2,5) *satisfy the conditions* $(\mathscr{A}_1), (\mathscr{A}_2), (\mathscr{A}_3)$ *and let* $\chi(x)$ *be any function of the class* C^3 *on* $\langle 0, \pi \rangle$ *such that* $\chi'(0) = \alpha_0$, $\chi'(\pi) = \alpha_\pi$. *Let* (2,2,4) *hold and*

$$\chi''(x) - \chi'^2(x) > 0\,,$$

$$\frac{\partial^2}{\partial u\, \partial t} f(t, x, u_0, u_{0t}, 0) > 2[\chi''(x) - \chi'^2(x)] \, . \max_{\substack{t \in \langle 0, \omega \rangle \\ x \in \langle 0, \pi \rangle}} \frac{\partial}{\partial u_t} f(t, x, u_0, u_{0t}, 0)$$

for $t \in \langle 0, \omega \rangle$, $x \in \langle 0, \pi \rangle$.

Then there exists $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0 \rangle$ *the solution* u_0 *of* $(\mathscr{P}[(2,5);(0,4)])$ *is uniformly exponentially stable.*

Remark 2,2. If we use Theorem 1,1, we immediately conclude that under the assumptions of Theorems 2,1, ..., 2,6 the solution u_0 of the corresponding special form of $(\mathscr{P})$ is uniformly stable at constantly acting disturbances, too.

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