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ON THE EXISTENCE OF PARALLEL NORMAL VECTOR FIELDS OF SURFACES IN E^4

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In the last time, there have been achieved many results about submanifolds possessing a parallel normal vector field; see, p.ex., [1] and [2]. In what follows, I am going to show a simple condition which ensures the existence of such a vector field in the case of a surface of a 4-space.

Let $M \subset E^4$ be a surface of class C^∞. Let $\{U_\alpha\}$ be its cover such that in each domain U_α there is a field of orthonormal frames $\{M; v_1, v_2, v_3, v_4\}$ such that $v_1, v_2 \in T(M)$, $v_3, v_4 \in N(M)$; $N(M)$ be the normal bundle of M. The fundamental equations of our surface are

$$
\begin{aligned}
(1) \qquad \mathrm{d}M &= \omega^1 v_1 + \omega^2 v_2\,,\\
\mathrm{d}v_1 &= \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4\,,\\
\mathrm{d}v_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4\,,\\
\mathrm{d}v_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4\,,\\
\mathrm{d}v_4 &= -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_3\,;
\end{aligned}
$$

$$(2) \qquad \mathrm{d}\omega^i = \omega^j \wedge \omega_j^i\,, \quad \mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j\,; \quad \omega_i^j + \omega_j^i = 0\,;$$

$$(3) \qquad \omega^3 = \omega^4 = 0\,.$$

From (3), we get (by means of exterior differentiation and the application of Cartan's lemma) the existence of functions $a_1, \ldots, b_3, \alpha_1, \ldots, \beta_4, A_1, \ldots, B_5$ such that

$$(4) \qquad \omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = 0\,, \quad \omega^1 \wedge \omega_1^4 + \omega^2 \wedge \omega_2^4 = 0\,;$$

$$
\begin{aligned}
(5) \qquad \omega_1^3 &= a_1\omega^1 + a_2\omega^2\,, \quad \omega_2^3 = a_2\omega^1 + a_3\omega^2\,,\\
\omega_1^4 &= b_1\omega^1 + b_2\omega^2\,, \quad \omega_2^4 = b_2\omega^1 + b_3\omega^2\,;
\end{aligned}
$$

(6) $\{da_1 - 2a_2\omega_1^2 - b_1\omega_3^4\} \wedge \omega^1 + \{da_2 + (a_1 - a_3)\omega_1^2 - b_2\omega_3^4\} \wedge \omega^2 = 0$,

$\{da_2 + (a_1 - a_3)\omega_1^2 - b_2\omega_3^4\} \wedge \omega^1 + \{da_3 + 2a_2\omega_1^2 - b_3\omega_3^4\} \wedge \omega^2 = 0$,

$\{db_1 - 2b_2\omega_1^2 + a_1\omega_3^4\} \wedge \omega^1 + \{db_2 + (b_1 - b_3)\omega_1^2 + a_2\omega_3^4\} \wedge \omega^2 = 0$,

$\{db_2 + (b_1 - b_3)\omega_1^2 + a_2\omega_3^4\} \wedge \omega^1 + \{db_3 + 2b_2\omega_1^2 + a_3\omega_3^4\} \wedge \omega^2 = 0$;

(7)
$$\begin{aligned}
da_1 - 2a_2\omega_1^2 - b_1\omega_3^4 &= \alpha_1\omega^1 + \alpha_2\omega^2,\\
da_2 + (a_1 - a_3)\omega_1^2 - b_2\omega_3^4 &= \alpha_2\omega^1 + \alpha_3\omega^2,\\
da_3 + 2a_2\omega_1^2 - b_3\omega_3^4 &= \alpha_3\omega^1 + \alpha_4\omega^2,\\
db_1 - 2b_2\omega_1^2 + a_1\omega_3^4 &= \beta_1\omega^1 + \beta_2\omega^2,\\
db_2 + (b_1 - b_3)\omega_1^2 + a_2\omega_3^4 &= \beta_2\omega^1 + \beta_3\omega^2,\\
db_3 + 2b_2\omega_1^2 + a_3\omega_3^4 &= \beta_3\omega^1 + \beta_4\omega^2;
\end{aligned}$$

(8)

$$\{d\alpha_1 - 3\alpha_2\omega_1^2 - \beta_1\omega_3^4\} \wedge \omega^1 + \{d\alpha_2 + (\alpha_1 - 2\alpha_3)\omega_1^2 - \beta_2\omega_3^4\} \wedge \omega^2 = \\ = (2a_2K + b_1k)\,\omega^1 \wedge \omega^2,$$

$$\{d\alpha_2 + (\alpha_1 - 2\alpha_3)\omega_1^2 - \beta_2\omega_3^4\} \wedge \omega^1 + \{d\alpha_3 + (2\alpha_2 - \alpha_4)\omega_1^2 - \beta_3\omega_3^4\} \wedge \omega^2 = \\ = (a_3K - a_1K + b_2k)\,\omega^1 \wedge \omega^2,$$

$$\{d\alpha_3 + (2\alpha_2 - \alpha_4)\omega_1^2 - \beta_3\omega_3^4\} \wedge \omega^1 + \{d\alpha_4 + 3\alpha_3\omega_1^2 - \beta_4\omega_3^4\} \wedge \omega^2 = \\ = (-2a_2K + b_3k)\,\omega^1 \wedge \omega^2,$$

$$\{d\beta_1 - 3\beta_2\omega_1^2 + \alpha_1\omega_3^4\} \wedge \omega^1 + \{d\beta_2 + (\beta_1 - 2\beta_3)\omega_1^2 + \alpha_2\omega_3^4\} \wedge \omega^2 = \\ = (2b_2K - a_1k)\,\omega^1 \wedge \omega^2,$$

$$\{d\beta_2 + (\beta_1 - 2\beta_3)\omega_1^2 + \alpha_2\omega_3^4\} \wedge \omega^1 + \{d\beta_3 + (2\beta_2 - \beta_4)\omega_1^2 + \alpha_3\omega_3^4\} \wedge \omega^2 = \\ = (b_3K - b_1K - a_2k)\,\omega^1 \wedge \omega^2,$$

$$\{d\beta_3 + (2\beta_2 - \beta_4)\omega_1^2 + \alpha_3\omega_3^4\} \wedge \omega^1 + \{d\beta_4 + 3\beta_3\omega_1^2 + \alpha_4\omega_3^4\} \wedge \omega^2 = \\ = (-2b_2K - a_3k)\,\omega^1 \wedge \omega^2;$$

(9)
$$\begin{aligned}
d\alpha_1 - 3\alpha_2\omega_1^2 - \beta_1\omega_3^4 &= A_1\omega^1 + (A_2 - a_2K - \tfrac{1}{2}b_1k)\,\omega^2,\\
d\alpha_2 + (\alpha_1 - 2\alpha_3)\omega_1^2 - \beta_2\omega_3^4 &= (A_2 + a_2K + \tfrac{1}{2}b_1k)\,\omega^1 + (A_3 + a_1K - \tfrac{1}{2}b_2k)\,\omega^2,\\
d\alpha_3 + (2\alpha_2 - \alpha_4)\omega_1^2 - \beta_3\omega_3^4 &= (A_3 + a_3K + \tfrac{1}{2}b_2k)\,\omega^1 + (A_4 + a_2K - \tfrac{1}{2}b_3k)\,\omega^2,\\
d\alpha_4 + 3\alpha_3\omega_1^2 - \beta_4\omega_3^4 &= (A_4 - a_2K + \tfrac{1}{2}b_3k)\,\omega^1 + A_5\omega^2,\\
d\beta_1 - 3\beta_2\omega_1^2 + \alpha_1\omega_3^4 &= B_1\omega^1 + (B_2 - b_2K + \tfrac{1}{2}a_1k)\,\omega^2,
\end{aligned}$$

$$d\beta_2 + (\beta_1 - 2\beta_3)\,\omega_1^2 + \alpha_2\omega_3^4 = (B_2 + b_2K - \tfrac{1}{2}a_1k)\,\omega^1 + (B_3 + b_1K + \tfrac{1}{2}a_2k)\,\omega^2\,,$$
$$d\beta_3 + (2\beta_2 - \beta_4)\,\omega_1^2 + \alpha_3\omega_3^4 = (B_3 + b_3K - \tfrac{1}{2}a_2k)\,\omega^1 + (B_4 + b_2K + \tfrac{1}{2}a_3k)\,\omega^2\,,$$
$$d\beta_4 + 3\beta_3\omega_1^2 + \alpha_4\omega_3^4 = (B_4 - b_2K - \tfrac{1}{2}a_3k)\,\omega^1 + B_5\omega^2\,,$$

where

$$(10)\qquad K = a_1a_3 - a_2^2 + b_1b_3 - b_2^2\,,\quad k = (a_1 - a_3)\,b_2 - (b_1 - b_3)\,a_2\,,$$

i.e.,

$$(11)\qquad d\omega_1^2 = -K\omega^1 \wedge \omega^2\,,\quad d\omega_3^4 = -k\omega^1 \wedge \omega^2\,.$$

Let $\{M;\, v_1^*, v_2^*, v_3^*, v_4^*\}$ be another field of frames over U_α; let

$$(12)\qquad v_1 = \varepsilon_1 \cos\varrho\,.\,v_1^* - \varepsilon_1 \sin\varrho\,.\,v_2^*\,,\quad v_2 = \sin\varrho\,.\,v_1^* + \cos\varrho\,.\,v_2^*\,,$$
$$v_3 = \varepsilon_2\cos\sigma\,.\,v_3^* - \varepsilon_2\sin\sigma\,.\,v_4^*\,,\quad v_4 = \sin\sigma\,.\,v_3^* + \cos\sigma\,.\,v_4^*\,;$$
$$\varepsilon_1^2 = \varepsilon_2^2 = 1\,.$$

Write $dM = \Omega^i v_i^*$, $dv_i^* = \Omega_i^j v_j^*$; the corresponding functions be denoted by *. From (1) and (1*),

$$(13)\qquad \Omega^1 = \varepsilon_1\cos\varrho\,.\,\omega^1 + \sin\varrho\,.\,\omega^2\,;\quad \Omega^2 = -\varepsilon_1\sin\varrho\,.\,\omega^1 + \cos\varrho\,.\,\omega^2\,;$$

$$(14)\qquad \Omega_1^2 = \varepsilon_1\omega_1^2 + d\varrho\,,\quad \Omega_3^4 = \varepsilon_2\omega_3^4 + d\sigma\,;$$

$$(15)\qquad \Omega_1^3 = \varepsilon_1\varepsilon_2\cos\varrho\cos\sigma\,.\,\omega_1^3 + \varepsilon_2\sin\varrho\cos\sigma\,.\,\omega_2^3 + \varepsilon_1\cos\varrho\sin\sigma\,.\,\omega_1^4 + \sin\varrho\sin\sigma\,.\,\omega_2^4\,,$$
$$\Omega_2^3 = -\varepsilon_1\varepsilon_2\sin\varrho\cos\sigma\,.\,\omega_1^3 + \varepsilon_2\cos\varrho\cos\sigma\,.\,\omega_2^3 - \varepsilon_1\sin\varrho\sin\sigma\,.\,\omega_1^4 + \cos\varrho\sin\sigma\,.\,\omega_2^4\,,$$
$$\Omega_1^4 = -\varepsilon_1\varepsilon_2\cos\varrho\sin\sigma\,.\,\omega_1^3 - \varepsilon_2\sin\varrho\sin\sigma\,.\,\omega_2^3 + \varepsilon_1\cos\varrho\cos\sigma\,.\,\omega_1^4 + \sin\varrho\cos\sigma\,.\,\omega_2^4\,,$$
$$\Omega_2^4 = \varepsilon_1\varepsilon_2\sin\varrho\sin\sigma\,.\,\omega_1^3 - \varepsilon_2\cos\varrho\sin\sigma\,.\,\omega_2^3 - \varepsilon_1\sin\varrho\cos\sigma\,.\,\omega_1^4 + \cos\varrho\cos\sigma\,.\,\omega_2^3$$

and

$$(16)\qquad a_1^* = \varepsilon_2\cos\sigma\,.\,(\cos^2\varrho\,.\,a_1 + \varepsilon_1\sin 2\varrho\,.\,a_2 + \sin^2\varrho\,.\,a_3) + \sin\sigma\,.\,(\cos^2\varrho\,.\,b_1 + \varepsilon_1\sin 2\varrho\,.\,b_2 + \sin^2\varrho\,.\,b_3)\,,$$

$$a_2^* = \varepsilon_2 \cos\sigma \,.\, \left(-\tfrac{1}{2}\sin 2\varrho \,.\, a_1 + \varepsilon_1 \cos 2\varrho \,.\, a_2 + \tfrac{1}{2}\sin 2\varrho \,.\, a_3\right) + \\ + \sin\sigma \,.\, \left(-\tfrac{1}{2}\sin 2\varrho \,.\, b_1 + \varepsilon_1 \cos 2\varrho \,.\, b_2 + \tfrac{1}{2}\sin 2\varrho \,.\, b_3\right),$$

$$a_3^* = \varepsilon_2 \cos\sigma \,.\, \left(\sin^2\varrho \,.\, a_1 - \varepsilon_1 \sin 2\varrho \,.\, a_2 + \cos^2\varrho \,.\, a_3\right) + \\ + \sin\sigma \,.\, \left(\sin^2\varrho \,.\, b_1 - \varepsilon_1 \sin 2\varrho \,.\, b_2 + \cos^2\varrho \,.\, b_3\right),$$

$$b_1^* = -\varepsilon_2 \sin\sigma \,.\, \left(\cos^2\varrho \,.\, a_1 + \varepsilon_1 \sin 2\varrho \,.\, a_2 + \sin^2\varrho \,.\, a_3\right) + \\ + \cos\sigma \,.\, \left(\cos^2\varrho \,.\, b_1 + \varepsilon_1 \sin 2\varrho \,.\, b_2 + \sin^2\varrho \,.\, b_3\right),$$

$$b_2^* = \varepsilon_2 \sin\sigma \,.\, \left(\tfrac{1}{2}\sin 2\varrho \,.\, a_1 - \varepsilon_1 \cos 2\varrho \,.\, a_2 - \tfrac{1}{2}\sin 2\varrho \,.\, a_3\right) - \\ - \cos\sigma \,.\, \left(\tfrac{1}{2}\sin 2\varrho \,.\, b_1 - \varepsilon_1 \cos 2\varrho \,.\, b_2 - \tfrac{1}{2}\sin 2\varrho \,.\, b_3\right),$$

$$b_3^* = -\varepsilon_2 \sin\sigma \,.\, \left(\sin^2\varrho \,.\, a_1 - \varepsilon_1 \sin 2\varrho \,.\, a_2 + \cos^2\varrho \,.\, a_3\right) + \\ + \cos\sigma \,.\, \left(\sin^2\varrho \,.\, b_1 - \varepsilon_1 \sin 2\varrho \,.\, b_2 + \cos^2\varrho \,.\, b_3\right).$$

Thus

(17)
$$a_1^* + a_3^* = \varepsilon_2 \cos\sigma \,.\, (a_1 + a_3) + \sin\sigma \,.\, (b_1 + b_3), \\ b_1^* + b_3^* = -\varepsilon_2 \sin\sigma \,.\, (a_1 + a_3) + \cos\sigma \,.\, (b_1 + b_3),$$

and we get from (7) and (17),

(18)
$$\alpha_1^* + \alpha_3^* = \varepsilon_1\varepsilon_2 \cos\varrho \cos\sigma \,.\, (\alpha_1 + \alpha_3) + \varepsilon_2 \sin\varrho \cos\sigma \,.\, (\alpha_2 + \alpha_4) + \\ + \varepsilon_1 \cos\varrho \sin\sigma \,.\, (\beta_1 + \beta_3) + \sin\varrho \sin\sigma \,.\, (\beta_2 + \beta_4),$$

$$\alpha_2^* + \alpha_4^* = -\varepsilon_1\varepsilon_2 \sin\varrho \cos\sigma \,.\, (\alpha_1 + \alpha_3) + \varepsilon_2 \cos\varrho \cos\sigma \,.\, (\alpha_2 + \alpha_4) - \\ -\varepsilon_1 \sin\varrho \sin\sigma \,.\, (\beta_1 + \beta_3) + \cos\varrho \sin\sigma \,.\, (\beta_2 + \beta_4),$$

$$\beta_1^* + \beta_3^* = -\varepsilon_1\varepsilon_2 \cos\varrho \sin\sigma \,.\, (\alpha_1 + \alpha_3) - \varepsilon_2 \sin\varrho \sin\sigma \,.\, (\alpha_2 + \alpha_4) + \\ + \varepsilon_1 \cos\varrho \cos\sigma \,.\, (\beta_1 + \beta_3) + \sin\varrho \cos\sigma \,.\, (\beta_2 + \beta_4),$$

$$\beta_2^* + \beta_4^* = \varepsilon_1\varepsilon_2 \sin\varrho \sin\sigma \,.\, (\alpha_1 + \alpha_3) - \varepsilon_2 \cos\varrho \sin\sigma \,.\, (\alpha_2 + \alpha_4) - \\ - \varepsilon_1 \sin\varrho \cos\sigma \,.\, (\beta_1 + \beta_3) + \cos\varrho \cos\sigma \,.\, (\beta_2 + \beta_4).$$

From (10) and (11),

(19)
$$K^* = K, \quad k^* = \varepsilon_1\varepsilon_2 k.$$

The *mean curvature vector* being

(20)
$$\xi = (a_1 + a_3)\, v_3 + (b_1 + b_3)\, v_4,$$

we have

$$(21) \qquad \xi^* = \xi ;$$

for the *mean curvature*

$$(22) \qquad H = \|\xi\|^2 = (a_1 + a_3)^2 + (b_1 + b_3)^2 ,$$

we get

$$(23) \qquad H^* = H .$$

Let $m \in M$ be a fixed point. Introduce the linear maps

$$(24) \qquad C^T_m : T_m(M) \to T_m(M) , \quad C^N_m : T_m(M) \to N_m(M)$$

by

$$(25) \qquad C^T_m(v) = \pi^N_m(v\xi) , \quad C^N_m(v) = \pi^N_m(v\xi) \quad \text{for} \quad v \in T_m(M) ,$$

π^T_m or π^N_m being the orthogonal projection onto $T_m(M)$ or $N_m(M)$ resp. Because of

$$(26) \quad \mathrm{d}\xi = - \{(a_1 + a_3)(a_1\omega^1 + a_2\omega^2) + (b_1 + b_3)(b_1\omega^1 + b_2\omega^2)\} v_1 - \\ - \{(a_1 + a_3)(a_2\omega^1 + a_3\omega^2) + (b_1 + b_3)(b_2\omega^1 + b_3\omega^2)\} v_2 + \\ + \{(\alpha_1 + \alpha_3)\omega^1 + (\alpha_2 + \alpha_4)\omega^2\} v_3 + \\ + \{(\beta_1 + \beta_3)\omega^1 + (\beta_2 + \beta_4)\omega^2\} v_4 ,$$

we get

$$(27) \qquad C^T(v_1) = - \{(a_1 + a_3) a_1 + (b_1 + b_3) b_1\} v_1 - \\ - \{(a_1 + a_3) a_2 + (b_1 + b_3) b_2\} v_2 ,$$

$$C^T(v_2) = - \{(a_1 + a_3) a_2 + (b_1 + b_3) b_2\} v_1 - \\ - \{(a_1 + a_3) a_3 + (b_1 + b_3) b_3\} v_2 ;$$

$$(28) \qquad C^N(v_1) = (\alpha_1 + \alpha_3) v_3 + (\beta_1 + \beta_3) v_4 , \\ C^N(v_2) = (\alpha_2 + \alpha_4) v_3 + (\beta_2 + \beta_4) v_4 .$$

From (18),

$$(29) \qquad \begin{vmatrix} \alpha_1^* + \alpha_3^* & \beta_1^* + \beta_3^* \\ \alpha_2^* + \alpha_4^* & \beta_2^* + \beta_4^* \end{vmatrix} = \varepsilon_1\varepsilon_2 \begin{vmatrix} \alpha_1 + \alpha_3 & \beta_1 + \beta_3 \\ \alpha_2 + \alpha_4 & \beta_2 + \beta_4 \end{vmatrix} .$$

Let us prove the following auxiliary results: (i) *If* $C^N(v) = 0$ *for each* $v \in T(M)$ *and* $H \neq 0$ *on* M, *then* $k = 0$. (ii) *Let the normal bundle* $N(M)$ *be trivial. Then* $k = 0$ *if and only if there is, in* $N(M)$, *a non-zero parallel vector field.*

Indeed, suppose $C^N = 0$, i.e.,

$$\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4 = \beta_1 + \beta_3 = \beta_2 + \beta_4 = 0 . \tag{30}$$

Then (9) implies

$$\begin{aligned} &A_1 + A_3 + a_3K + \tfrac{1}{2}b_2k = 0 , \quad A_2 + A_4 - \tfrac{1}{2}(b_1 + b_3)\,k = 0 ,\\ &A_2 + A_4 + \tfrac{1}{2}(b_1 + b_3)\,k = 0 , \quad A_3 + A_5 + a_1K - \tfrac{1}{2}b_2k = 0 ,\\ &B_1 + B_3 + b_3K - \tfrac{1}{2}a_2k = 0 , \quad B_2 + B_4 + \tfrac{1}{2}(a_1 + a_3)\,k = 0 ,\\ &B_2 + B_4 - \tfrac{1}{2}(a_1 + a_3)\,k = 0 , \quad B_3 + B_5 + b_1K + \tfrac{1}{2}a_2k = 0 ; \end{aligned} \tag{31}$$

from ($31_{2,3,6,7}$), we get $(a_1 + a_3)\,k = (b_1 + b_3)\,k = 0$ and $k = 0$ because of $H \neq 0$. Let $v = xv_3 + yv_4$ be a normal vector field. Then

$$\begin{aligned} dv = &- (x\omega_1^3 + y\omega_1^4)\,v_1 - (x\omega_2^3 + y\omega_2^4)\,v_2 + \\ &+ (dx - y\omega_3^4)\,v_3 + (dy + x\omega_3^4)\,v_4 \end{aligned} \tag{32}$$

and v is parallel if and only if

$$dx - y\omega_3^4 = 0 , \quad dy + x\omega_3^4 = 0 . \tag{33}$$

The integrability conditions of (33) are $x\, d\omega_3^4 = y\, d\omega_3^4 = 0$, i.e., $xk = yk = 0$.

Suppose the space E^4 to be oriented and the moving frames to be positively oriented. Then

$$\varepsilon_1\varepsilon_2 = 1 . \tag{34}$$

Further, suppose that the bundle $N(M)$ is trivial and $\xi \neq 0$ on M. The frames may be chosen in such a way that v_3 and ξ are linearly dependent, i.e.,

$$b_1 + b_3 = 0 ; \tag{35}$$

from (17_2),

$$\sin \sigma = 0 . \tag{36}$$

Introduce the form

$$\tau = (a_1 + a_3)^2\, \omega_3^4 ; \tag{37}$$

we get

$$\tau^* = \varepsilon_2 \tau \tag{38}$$

and

$$d\tau = \{2(\alpha_1 + \alpha_3)(\beta_2 + \beta_4) - 2(\alpha_2 + \alpha_4)(\beta_1 + \beta_3) - (a_1 + a_3)^2 k\} \omega^1 \wedge \omega^2 . \tag{39}$$

From (7),

$$\tau = (a_1 + a_3)\{(\beta_1 + \beta_3)\omega^1 + (\beta_2 + \beta_4)\omega^2\} . \tag{40}$$

The integral formula

$$\int_{\partial M} (a_1 + a_3)\{(\beta_1 + \beta_3)\omega^1 + (\beta_2 + \beta_4)\omega^2\} = \int_M (2 \det C^N - Hk)\omega^1 \wedge \omega^2 \tag{41}$$

and (28) imply the following

Theorem. *Let $M \subset E^4$ be a surface of class C^∞, and suppose*: (i) *the normal bundle $N(M)$ is trivial*; (ii) *$H \neq 0$ on M*; (iii) *on M,* $\det C^N \geqq 0$ *and $k \leqq 0$ or* $\det C^N \leqq 0$ *and $k \geqq 0$ resp.*; (iv) *$C^N(T_m(M)) \subset \xi_m$ for each point $m \in \partial M$. Then $k = 0$ on M, i.e., M admits a parallel normal vector field.*

Notice that $H =$ const. implies $\det C^N = 0$. Indeed: from $(7_{1,3})$, we get $\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4 = 0$.

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