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TORSION THEORY FOR LATTICE-ORDERED GROUPS PART II: HOMOGENEOUS l-GROUPS

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Introduction. This note pursues in greater detail some of the discussion initiated in [6]; we use the same terminology and notation of [6]. We shall assume that the reader is familiar with CONRAD [3], and FUCHS [4]. and we shall systematically treat such material as standard theory of lattice-ordered groups (henceforth: l-groups). At the risk of sounding pretentious, we will also assume the reader has the good sense to read [6] before taking this on, although it is by no means a prerequisite.

We write all l-groups additively without regard to commutativity or the lack of it. If A and B are subsets of a set X, we denote (proper) containment by $(A \subset B)$ $A \subseteq B$; $A \setminus B$ stands for the complement of B in A.

Let us start by reviewing the notion of a torsion class from [6]. A class $\mathcal{T}$ of l-groups will be called a *torsion class* if it is closed with respect to taking 1) convex l-subgroups, 2) l-homomorphic images and 3) joins of convex l-subgroups in $\mathcal{T}$. With each torsion class we associate a radical (also denoted by $\mathcal{T}$), so that if G is an l-group then $\mathcal{T}(G)$ is the join of all the convex l-subgroups of G belonging to $\mathcal{T}$. If $\mathcal{T}$ is a torsion class then: a) $\mathcal{T}(A) = A \cap \mathcal{T}(G)$ for each convex l-subgroup A of G, b) $[\mathcal{T}(G)]\, \Phi \subseteq \mathcal{T}(H)$ for each l-epimorphism $\Phi : G \to H$ (proposition 1.1 in [6]). Conversely, any radical satisfying a) and b) gives rise to a torsion class of which it is the associated torsion radical (proposition 1.2 in [6]). Finally, we say that a torsion class is *complete* if it closed under extensions.

In § 4 of [6] we introduced the notion of a homogeneous l-group: G is *homogeneous* if for each torsion class $\mathcal{T}$ either $G \in \mathcal{T}$ or else $\mathcal{T}(G) = 0$. Since torsion radicals are characteristic l-ideals it follows that all characteristically simple l-groups are homogeneous. In this category fall the free abelian l-groups, cardinal sums of reals, rationals or integers, periodic real sequences, etc.

We also developed two criteria for telling when an l-group G is homogeneous. Let $\mathcal{X}_G$ and $\mathcal{X}^G$ denote respectively the torsion class generated by G, and the largest torsion class relative to having $\mathcal{T}(G) = 0$. The two criteria are as follows:

Theorem (4.1 in [6]): *If G is homogeneous then $\mathscr{X}^G$ is complete and meet irreducible in the lattice of all torsion classes. Conversely, if $\mathscr{X}^G$ is meet irreducible then G has a nontrivial homogeneous l-ideal. On the other hand, if $\mathscr{X}$ is any complete, meet irreducible torsion class, there is a homogeneous l-group H such that $\mathscr{X} = \mathscr{X}^H$.*

Theorem (4.2 in [6]). *If G is homogeneous $\mathscr{X}_G$ is join irreducible in the lattice of all torsion classes. Conversely, if $\mathscr{U}$ is a join irreducible torsion class and it covers $\mathscr{U}^{\sim}$, and there is an l-group $H \in \mathscr{U}$ so that $\mathscr{U}^{\sim}(H) = 0$, then H is homogeneous, $\mathscr{X}_H = \mathscr{U}$ and $\mathscr{X}^H$ is the largest torsion class satisfying $\mathscr{T} \cap \mathscr{U} = \mathscr{U}^{\sim}$.*

Note. We use meet and join irreducibility relative to arbitrary meets and joins respectively.

We conclude this introduction with some additional basic facts about homogeneous l-groups:

1. If G is homogeneous so is every convex l-subgroup.
2. If $G = \boxplus\{G_\gamma \mid \gamma \in \Gamma\}$, a cardinal sum of l-groups G_γ, and $G_\gamma \simeq G_\delta$ for $\gamma, \delta \in \Gamma$ and each G_γ is homogeneous, then G is homogeneous.
3. If G is homogeneous and A is a non-trivial convex l-subgroup of G then $\mathscr{X}_G = \mathscr{X}_A$.

The above are easy to prove and the reader is invited to try them.

1. FINITE VALUED, HOMOGENEOUS l-GROUPS

Before launching ahead we need a general lemma concerning the principal torsion classes $\mathscr{X}_G$.

1.1 Lemma. *Let G and H be two l-groups; $H \in \mathscr{X}_G$ if and only if $H = \bigvee_{i\in I} H_i$, where each H_i is a convex l-subgroup of H, and for each $i \in I$ there exist convex l-subgroups $N_i \subseteq D_i$ of G with N_i normal in D_i, such that $D_i/N_i \simeq H_i$.*

Proof. It is clear that if H is a join of convex l-subgroups as described in the lemma then $H \in \mathscr{X}_G$. So all that's needed is to show that the class $\mathscr{T} = \{H \mid H = \bigvee_{i\in I} H_i,\ H_i$ a convex l-subgroup of H and a quotient of a convex l-subgroup of $G\}$ is a torsion class. Obviously, $\mathscr{T}$ is closed under joins of convex l-subgroups in $\mathscr{T}$. Next, suppose $H \in \mathscr{T}$ and K is a convex l-subgroup of H, write $H = \bigvee_{i\in I} H_i$, each H_i a convex l-subgroup of H isomorphic to D_i/N_i, where $N_i \subseteq D_i$ are convex l-subgroups of G and N_i is normal in D_i. Now $K = \bigvee_{i\in I} K \cap H_i$ and $K \cap H_i \simeq D_i^*/N_i$, where $D_i^* \subseteq D_i$ is a convex l-subgroup of G; thus $K \in \mathscr{T}$.

Finally, suppose $\phi : H \to L$ is an l-homomorphism of $H \in \mathscr{T}$ onto the l-group L. If $H = \bigvee_{i\in I} H_i$ as before, then $L = \bigvee_{i\in I} H_i\phi$ and $H_i\phi$ is a quotient of a quotient of a convex l-subgroup of G. Thus $H_i\phi$ is itself a quotient of a convex l-subgroup of G, and we are able to conclude that $L \in \mathcal{T}$. Hence $\mathcal{T}$ is a torsion class, $\mathcal{T} = \mathcal{X}_G$ and the lemma is proved.

It will be useful to make the following definition now: if G is an l-group, $N \subseteq D$ are convex l-subgroups of G with N normal in D, we call D/N a *subquotient* of G. If T is a well ordered set and $\{G_t \mid t \in T\}$ is a family of convex l-subgroups of G, so that $G_s \subseteq G_t$ if $s < t$, and $\bigcup G_t = G$, we call G an *inductive limit* of the G_t.

1.2 Lemma. *Let H be a finite valued l-group; $H \in \mathcal{X}_G$ if and only if H is a cardinal sum of inductive limits of subquotients of G.*

(Note. Here and for the remainder of the section we assume G is a finite valued l-group.)

Proof. Every finite valued l-group H is a cardinal sum of cardinally indecomposable l-groups. (This can be seen as follows: the root system of regular subgroups is a disjoint union of indecomposable – and hence directed – root systems. The indecomposable summands of H are then obtained by considering the l-ideals generated by the special elements "living on" a fixed root system component. It should be clear of course, that if the root system of regular subgroups of H is indecomposable to start with, then H is cardinally indecomposable.)

So we will lose no generality if we prove that whenever $H \in \mathcal{X}_G$ and cardinally indecomposable, then H is isomorphic to an inductive limit of subquotients of G.

Assuming this, the proof from here on breaks down into two parts: a) if $0 < x \in H$ is special then $H(x)$ is isomorphic to a subquotient of G; b) $H = \bigcup\{H(x_j) \mid j \in J\}$ where J is a well ordered set, and each $0 < x_j \in H$ is special, so that $H(x_j) \subset H(x_k)$ if $j < k$.

a) If $0 < x \in H$ is special then $H(x)$ is a lexicographic extension of an l-ideal M of $H(x)$ so that $H(x)/M$ is a subgroup of $\mathbf{R}$, the additive real with the usual order. $H(x) \in \mathcal{X}_G$ and join irreducible in the lattice of convex l-subgroups of H, and must therefore be a subquotient of G.

b) We select a root (root $\equiv$ maximal chain) out of the root system of regular subgroups of H; since H is indecomposable we can state that if we pick for each regular subgroup M on that root, a special element $x_M > 0$ having its value at M, then $H = \bigcup_{x_M > 0} H(x_M)$. By choosing a suitable cofinal, well ordered subset of the root, we obtain the desired well ordered chain of principal convex l-subgroups.

The referee of this note has provided the author with an example to show that the words "inductive limit of" cannot be deleted in lemma 1.2. Consider the o-group A_n of the lexicographic product of n copies of the additive reals, ordered from left to right. Let G be the cardinal sum of the A_n ($n = 1, 2, \ldots$). Let H be the direct sum of copies of the additive reals R_n ($n = 1, 2, \ldots$), lexicographically ordered from left to right. Then H is the inductive limit of convex subgroups which are quotients of G, and hence $H \in \mathcal{X}_G$. But H has infinite sequences $a_1 \gg a_2 \gg a_3 \gg \ldots$ (where $a \gg b$

means that $a \geqq kb$ for each positive integer k), and no totally ordered subquotient of G has this property. Thus H is not a subquotient of G. (Note also that H is indecomposable, as it is an o-group.)

We are now ready for the main theorem of this section; we give it in two stages.

1.3 Theorem. *Suppose G is an indecomposable l-group; G is homogeneous if and only if for each $0 < a \in G$, G is isomorphic to an inductive limit of subquotients of $G(a)$.*

Proof. If G is isomorphic to an inductive limit of subquotients of $G(a)$ for each $0 < a \in G$, then G is clearly homogeneous.

Conversely, suppose $0 < x \in G$ and G is homogeneous. $G(x)$ and G generate the same torsion class $\mathscr{X}$. As $G \in \mathscr{X}$, by lemma 1.2, G is isomorphic to an inductive limit of subquotients of $G(x)$, and we're done.

1.4 Proposition. *G is homogeneous if and only if it is a cardinal sum of indecomposable, homogeneous l-groups, each one being an inductive limit of subquotients of any other in the decomposition.*

Proof. Sufficiency. Suppose $G = \boxplus\{G_\gamma \mid \gamma \in \Gamma\}$ where each G_γ is indecomposable and homogeneous, and further G_γ is an inductive limit of subquotients of G_δ, for $\gamma, \delta \in \Gamma$. If $\mathscr{T}$ is a torsion class and $\mathscr{T}(G) \neq 0$, then $\mathscr{T}(G_\gamma) \neq 0$ for some $\gamma \in \Gamma$, since $\mathscr{T}(G) = \boxplus\mathscr{T}(G_\gamma)$; (proposition 1.3 in [6]). Thus $G_\gamma \in \mathscr{T}$ and hence each $G_\delta \in \mathscr{T}$ since G_δ is an inductive limit of subquotients of G_γ. It follows that $G \in \mathscr{T}$ and G is homogeneous.

Necessity. If G is homogeneous, write $G = \boxplus_{\gamma\in\Gamma}G_\gamma$, where each G_γ is indecomposable. Each G_γ is homogeneous, and $\mathscr{X}_G = \mathscr{X}_{G_\gamma}$ for each $\gamma \in \Gamma$. By lemma 1.2 G_γ is an inductive limit of subquotients of G_δ for all $\gamma, \delta \in \Gamma$.

Recall that an l-group G has *property* (F) if each $0 < g \in G$ exceeds at most finitely many pairwise disjoint elements.

1.4.1 Corollary. *Suppose G has property* (F)*; G is homogeneous if and only if G is a cardinal sum of homogeneous o-groups, any two of which are inductive limits of subquotients of eachother.*

Before going on to the non-finite valued case we should point out the following offshoot of lemma 1.2.

1.5 Proposition. *Let $\mathscr{T}$ be a class of finite valued l-groups closed under taking convex l-subgroups and quotients. Then $\mathscr{T}$ is a torsion class if and only if $\mathscr{T}$ is closed under cardinal sums and unions of chains of convex l-subgroups in $\mathscr{T}$.*

Proof. Necessity is obvious, so we move right on to the sufficiency. Let $H = \bigvee_{i\in I}H_i$, where each H_i is a convex l-subgroup of H belonging to $\mathscr{T}$. In the spirit of lemma 1.2 we may assume without loss of generality that H is indecomposable, and write $H = \bigcup\{H(x_j) \mid j \in J\}$, where J is well ordered, each $x_j > 0$ is special, and $H(x_j) \subset H(x_k)$ for $j < k$. Clearly, each $H(x_j) \subseteq H_{i(j)}$, for a suitable $i(j) \in I$. So each $H(x_j) \in \mathscr{T}$ and hence $H \in \mathscr{T}$, which shows that $\mathscr{T}$ is a torsion class.

2. NOW INFINITELY MANY VALUES

Before saying anything really intelligent about homogeneous l-groups which are not finite valued, we must present some new torsion classes; let α be an infinite cardinal number, and V_α be the class of all l-groups in which all non-zero elements have at most α values.

2.1 Lemma. *V_α is a torsion class for each infinite cardinal number α.*

Proof. Suppose $G \in V_\alpha$ and C is a convex l-subgroup of G; if $0 < c \in C$ and N is a value of c in C then there is a regular subgroup M of G so that $M \cap C = N$, and of course M is a value of c in G. This suffices to show that c has at most α values in C, V_α is therefore closed relative to convex l-subgroups.

Next, suppose $G \in V_\alpha$ and K is an l-ideal of G. If $0 \neq g + K$ its values in G/X are in one to one correspondence with the values of g in G that exceed K. Hence $g + K$ has at most α values, and $G/K \in V_\alpha$.

Finally, suppose $G = \bigvee_{i \in I} G_i$, where each G_i is a convex l-subgroup of G belonging to V_α. G is the subgroup generated by the G_i, and if $0 < g \in G$ then $g = g_{i_1} + \dots + g_{i_n}$, where $0 < g_{i_\lambda}$ and $g_{i_\lambda} \in G_{i_\lambda}$ $(\lambda = 1, 2, \dots, n)$. If M is a value of g in G then M is a value of at least one of the g_{i_λ}. The values of g_{i_λ} in G are in one to one correspondence with its values in G_{i_λ}, whence g has at most α values in G. (Note: it is here that we need the assumption that α be an infinite cardinal number.)

This completes the proof of the lemma.

2.2 Theorem. *Suppose G is a homogeneous, hyper-archimedean l-group. Then either G is finite valued, in which case it is a cardinal sum of copies of a fixed subgroup of $\mathbf{R}$, or else there is an infinite cardinal number α so that each non-zero element of G has precisely α values.*

Proof. Since G is hyper-archimedean all regular subgroups are maximal, and so it follows that if $0 < g \in G$ has at most β values, then any positive element below g has the same property. We ignore the finite valued case, the conclusion being obvious by now. Thus we assume there is an infinite cardinal number α, and an element $0 < a \in G$ having α values. By our remark here, and because G is homogeneous $G \in V_\alpha$.

If some $h \in G$ has fewer values, say γ of them, with $\gamma < \alpha$, then $V_\gamma(G) \neq 0$, which implies that $G \in V_\gamma$, a contradiction. We must conclude then that each non-zero element of G has precisely α values.

For each infinite cardinal number α we can construct a homogeneous, hyper-archimedean l-groups B_α in which each non-zero element has precisely α values. Let I be a set of cardinality α, and let B be the l-group of bounded, integer valued functions on I. Let J be the l-ideal of functions whose supports have cardinality less than α; finally, set $B_\alpha = B/J$.

Since B is hyper-archimedean so is B_α; in fact B is an S-group; that is, it is generated by its singular elements. (Recall that an element $0 < s$ is *singular* if $0 \leqq g \leqq s$ implies that $g \wedge (s - g) = 0$.) The S-groups form a torsion class, so B_α is also an S-group. Define $u \in B$ to be the constant function 1; u is a strong order unit for B, so that $u + J$ is a strong order unit for B_α.

Suppose $0 < g \in B \setminus J$ and let I_g be the support of g. Since I and I_g have the same cardinality there is a bijection between them; it extends to an isomorphism ϕ from B onto $B(g)$ so that $u\phi = g$ and J is mapped onto the l-ideal J_g of all functions in $B(g)$ whose supports have cardinality less than α. Noting that $J_g = J \cap B(g)$ we have, $B_\alpha = B/J \simeq B(g)/J_g = B(g)/B(g) \cap J \simeq B(g) + J/J$, and the last one is the l-ideal of B_α generated by $g + J$.

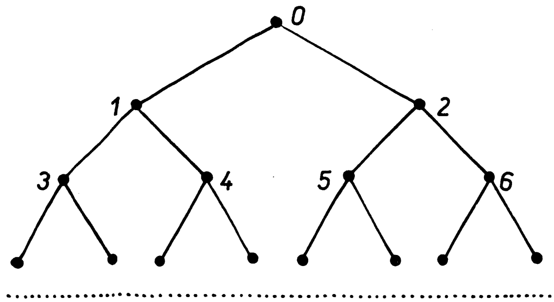
We've shown then that every principal convex l-subgroup of B_α is isomorphic to B_α. This clearly suffices to establish that B_α is homogeneous. Next, partition I into α subsets of cardinality α each. This gives rise to α pairwise disjoint singular elements whose supports all have cardinality α, by passing to the appropriate characteristic functions. This should convince the reader that $u + J$ has α values, and hence that all non-zero elements of B_α have α values.

We give two examples in closing this section:

1. First, theorem 2.2 only gives a necessary condition for homogeneity. For example let G be the l-group of periodic real sequences with integers in all odd components; G is not homogeneous, it is hyper-archimedean, and each non-zero element of G has countably many values.

2. An l-group may be homogeneous, not finite valued, yet contain some finite valued elements. Consider the v-group $V = V(\Lambda, \mathbf{R})$ over the root system Λ pictured below. V is not finite valued; for example there is the function $f \in V$ defined by

$$f(n) = \begin{cases} 1 & \text{if } n = 2^k \\ 0 & \text{otherwise}. \end{cases}$$



Yet V is homogeneous, because each $0 < x \in V$ exceeds a special element $0 < y \in V$ such that $V(y) \simeq V$.

3. VOIDS AND CLOSING COMMENTS

Obviously this note fails to answer certain basic questions about homogeneous l-groups. Unquestionably, a defining condition is needed that circumvents torsion classes; we have it only in the finite valued case.

One interesting question involves homogeneous o-groups: we know now that they are characterized by saying that each positive element "dominates" a subquotient isomorphic to the given group. The question is whether *subquotient* may be replaced by *convex subgroup*. The author has tried in vain to concoct a Hahn group whose skeletal chain Λ is such that for each $\lambda \in \Lambda$ there is a convex subset Γ_λ of Λ, satisfying $\mu \leqq \lambda$ for each $\mu \in \Gamma_\lambda$, whose corresponding Hahn group is isomorphic to the original o-group, yet Γ_λ *cannot always* be chosen to be an order ideal of Λ. The author blithely believes that such examples exist.

We conclude with two examples; the first answers a conjecture of [6], the second raises some new intriguing questions.

a) Let $G = C([0, 1])$; it is shown in [1], corollary 6.11 that $K = \{f \in G \mid f$ vanishes off an open interval in $[0, 1]\}$, is the smallest non-trivial characteristic l-ideal of G. We proved in [6], § 4, that K could not be a torsion radical, and wondered whether G is not actually homogeneous.

Let us see that it is homogeneous: suppose $0 < f \in G$; then there exist $a, b \in (0, 1)$, $a < b$, so that $f(t) > 0$ for each $t \in (a, b)$. Pick two points c and d so that $a < c < d < b$, and let L be the l-ideal of functions that vanish off (a, b). We can define an l-homomorphism ϕ from L onto $C([c, d])$ by restriction; (it is reasonably obvious that ϕ is indeed onto.) What we've proved is that for each $0 < f \in G$ there is a subquotient of $G(f)$ isomorphic to G; it follows that G is homogeneous.

b) Let $G = \boldsymbol{Z} \coprod \boldsymbol{Z}$, the free product as abelian l-groups of the additive integers with themselves; it can be seen from the discussion of [5] that G is the l-subgroup of $C([0, 1])$ generated by $f(t) = t$ and $g(t) = 1 - t$. This says that G is the *full* group of piecewise linear and continuous functions on $[0, 1]$, (finitely many pieces), with integral slopes everywhere. From here one can develop an argument very similar to – but more delicate than – the one in a) to show that G is homogeneous.

The free abelian l-group on any number of generators is a free product of copies of $\boldsymbol{Z} \boxplus \boldsymbol{Z}$, and is homogeneous. In view of this and b) above one ought to wonder whether the free (abelian) product of any number of copies of a (homogeneous) abelian l-group is homogeneous.

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