Herbert I. Freedman; John J. Mallet-Paret

Almost Floquet and generalized almost Floquet systems

Czechoslovak Mathematical Journal, Vol. 26 (1976), No. 1, 13–19

Persistent URL: http://dml.cz/dmlcz/101368

Czechoslovak Mathematical Journal, 26 (101) 1976, Praha

ALMOST FLOQUET AND GENERALIZED ALMOST FLOQUET SYSTEMS*)

H. I. FREEDMAN, Edmonton and J. J. MALLET-PARET, Providence

(Received October 3, 1972)

I. INTRODUCTION

By means of the Floquet theorem (see [1]), stability criteria can be obtained for the linear system of ordinary differential equations

$$x' = A(t)\,x \left(' = \frac{\mathrm{d}}{\mathrm{d}t}\right) \tag{1}$$

in the case where $A(t)$ is periodic. Recently, BURTON and MULDOWNEY [4] extended these results to include systems of the type where $A(t)$ is f-periodic, whereas FREEDMAN [5] extended these results to systems where $A(t + \tau) - A(t)$ have certain properties.

It is the prupose of the present paper to combine the results of [4] and [5] in the next section, as well as to further examine criteria for such systems in section III. In the last section, we look at several examples to illustrate the results, including an example where $A(t)$ is a very simple quasi-periodic matrix.

II. GENERALIZED ALMOST FLOQUET THEORY

Definition 1. Let $f(t)$ be an absolutely continuous function on (t_0, ∞), $t_0 \geqq -\infty$, such that

$$f(t) > t \tag{2}$$

for all $t > t_0$. Let

$$B(t) \equiv f'(t)\,A(f(t)) - A(t) \tag{3}$$

for almost all $t > t_0$, and further assume that

$$[B(t), \Phi(t)] = 0\,, \tag{4}$$

*) The research for this paper was partially supported by the Defense Research Board of Canada Grant No. 9540-22.

where $\Phi(t)$ is that fundamental solution of system (1) such that $\Phi(0) = I$, and $[U, V] \equiv UV - VU$. If the above hold, we say that system (1) is a generalized almost Floquet system with respect to f (GAFS $- f$).

Remark 1. If $B(t) = 0$, system (1) reduces to the GFS $- f$ of Burton and Muldowney [4], whereas if $f(t) \equiv t + \tau$, $\tau > 0$, system (1) reduces to the AFS of Freedman [5].

Theorem 1. *Let system* (1) *be a* GAFS $- f$. *Let* $\Psi(t)$ *be that fundamental matrix of*

$$y' = B(t)\, y\,, \tag{5}$$

where $B(t)$ *is defined by* (3) *such that* $\Psi(0) = I$. *Then*

$$\Phi(f(t)) = \Phi(t)\, \Psi(t)\, \Phi(f(0))\,. \tag{6}$$

Proof. $X = \Phi(f(t))$ and $X = \Phi(t)\, \Psi(t)\, \Phi(f(0))$ both satisfy the matrix initial value problem $x' = (A(t) + B(t))\, x$, $x(0) = \Phi(f(0))$. Since this problem has only one solution, Equation (6) follows.

Remark 2. If the appropriate properties of $\Psi(t)$ are known, the results of [4] can be used to obtain further stability criteria for (1).

III. SUFFICIENCY CONDITIONS

In [5], three theorems were proved giving sufficiency conditions for system (1) to be an AFS. With slight modifications, these theorems also go over to the present paper. We now show that Theorem 5 of [5] is a corollary of the Theorem 3 of [5].

Theorem 2. *Let* $B(t)$ *be holomorphic for all* t *and let* $\mathrm{d}^k B(t)/\mathrm{d}t^k$, $A(t) = 0$, $k = 0, 1, 2$. *Then if* $A(t)$ *is continuous, then* $[B(t), A(s)] \equiv 0$ *for all* t, s.

Proof. Define $G_s(t) = [B(t), A(s)]$. Then $\mathrm{d}^k G_s(t)/\mathrm{d}t^k = [B^{(k)}(t), A(s)]$. Since $G_s(t)$ is holomorphic,

$$G_s(t) = \sum_{k=0}^{\infty} \frac{G_s^{(k)}(s)}{k!} (t - s)^k\,.$$

But $G_s^{(k)}(s) = [B^{(k)}(s), A(s)] = 0$ which implies that $G_s(t) \equiv 0$ for all t, s, which proves the theorem.

The fact that the second of the above mentioned theorems of [5] does not imply the first will be shown in the first example of the next section.

The condition $[B(t), A(s)] = 0$ is very strong. The second example of the next section shows that $[B(t), A(t)] = 0$ does not imply AFS in general. However, the following intermediate result does hold.

Theorem 3. *Let $B(t)$ have the same minimal and characteristic polynomials and be such that $[B(t), B'(t)] = 0$ for all t. Then system (1) is a* GAFS $- f$ *if and only if $[B(t), A(t)] = 0$.*

Proof. Since $[B(t), B'(t)] = 0$ and $B(t)$ has the same minimal and characteristic polynomial, there exist continuous functions $b_k(t)$ such that

(7) $$B'(t) = \sum_{k=0}^{n-1} b_k(t)\, B^k(t)\,.$$

This follows immediately from the discussion in [7, Chapter 10]. Also $B^m(t)$ is a polynomial in $B(t)$ of degree less than n since $B(t)$ satisfies its own characteristic polynomial.

Suppose now that $[\Phi(t), B(t)] = 0$. Then also $\Phi(t) = \sum_{k=0}^{n-1} \Phi_k(t)\, B^k(t)$ for $\Phi_k(t) \in C^1[0, \infty)$. Hence $\Phi'(t) = \sum_{k=0}^{n=1} (\Phi_k'(t)\, B^k(t) + k\, \Phi_k(t)\, B^{k-1}(t)\, B'(t))$ since $[B(t), B'(t)] = 0$, and so $[\Phi'(t), B(t)] = 0$. Thus $[A(t), B(t)] = [\Phi'(t)\, \Phi^{-1}(t), B(t)] = 0$, proving the necessity.

Let now $[A(t), B(t)] = 0$. Then $A(t) = \sum_{k=0}^{n-1} a_k(t)\, B^k(t)$ for $a_k(t) \in C[0, \infty)$. $[B(t), \Phi(t)] = 0$ if and only if there exists $u_0(t), u_1(t), \ldots, u_{n-1}(t)$, $u_0(0) = 1$, $u_1(0) = \ldots = u_{n-1}(0) = 0$, $u_k(t) \in C^1[0, \infty)$ such that $\Phi(t) = \sum_{k=0}^{n-1} u_k(t)\, B^k(t)$, i.e. if and only if $\Phi'(t) = \sum_{k=0}^{n-1} u_k'(t)\, B^k(t) + \sum_{k=0}^{n-1} k\, u_k(t)\, B^{k-1}(t)\, B'(t) = A(t)\, \Phi(t) = \sum_{k=0}^{n-1} a_k(t)\,.$ $.\, B^k(t) \sum_{l=0}^{n-1} u_l(t)\, B^l(t)$. Using the expansion of $B'(t)$ in terms of $B(t)$ and writing $B^m(t)$ in terms of $B(t)$ for $n \leqq m \leqq 2n - 2$, this last reduces to an equation of the type

(8) $$\sum_{k=0}^{n-1} u_k'(t)\, B^k(t) = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} f_{kj}(t)\, u_j(t)\, B^k(t)\,,$$

where the $f_{kj}(t)$ are continuous functions of t. Since $I, B(t), \ldots, B^{n-1}(t)$ are linearly independent, equation (8) is equivalent to the system

(9) $$u'(t) = F(t)\, u(t)\,, \quad u(0) = e_1$$

where $u(t) = [u_0, \ldots, u_{n-1}]^T$, $F(t) = (f_{kj}(t))$, and $e_1 = [1, 0, \ldots, 0]^T$. Since this system indeed has a unique continuable solution for all t, the theorem is proved.

In [5], several stability criteria were developed in the case where $B(t)$ is a polynomial. This case is just a special case of the following.

Theorem 4. *Let $A(t) = \Pi(t) + \tilde{A}(t)$, where $\Pi(t+\tau) = \Pi(t)$ and $\tilde{A}(t)$ is functionally commutative. Let the A_i of equation* (11) *below satisfy*

$$(10) \qquad [\Pi(t), A_i] = 0\,.$$

Then system (1) *is an* AFS.

Proof. $[\tilde{A}(s), \tilde{A}(t)] = 0$ for all s, t by functionally commutativity (defined in [2] and [5]). Hence by [2] and [6], $\tilde{A}(t)$ can be written

$$(11) \qquad \tilde{A}(t) = \sum_{i=1}^{N} f_i(t)\, A_i\,,$$

where $1 \leqq N \leqq n^2$, $\{f_i(t)\}_1^N$ is a linearly independent set of scalar functions, and the constant matrices A_i satisfy

$$(12) \qquad [A_i, A_j] = 0\,, \quad i, j = 1, \ldots, N\,.$$

Then $B(t) = A(t + \tau) - A(t) = \sum_{i=1}^{N} (f_i(t + \tau) - f_i(t))\, A_i$. By (10) and (12), $[A(t), B(s)] = 0$ and hence by Theorem 3 of [5] system (1) is almost Floquet.

Remark 3. The above theorem can be modified so as to include the generalized almost Floquet case, if $\Pi(t)$ is chosen as a solution of equation (3) with $B(t) \equiv 0$, and $\tilde{A}(t)$ is such that the $B(t)$ of equation (3) using $\tilde{A}(t)$ is functionally commutative.

Remark 4. For the system of Theorem 4, $\Phi(t + \tau) = \Phi(t)\, \Psi(t)\, \Phi(\tau)$, where $\Psi(t)$ is that fundamental matrix of $y' = B(t)\, y$ such that $\Psi(0) = I$, and $B(t) = \sum_{i=1}^{N} (f_i(t + \tau) - f_i(t))\, A_i$. If for all i, $1 \leqq i \leqq N$, $f_i(t + \tau) - f_i(t) = 0$, then the Floquet theorem holds. Suppose that $f_i(t + \tau) - f_i(t) \neq 0$ for some i's. Then $B(t)$ may be written

$$(13) \qquad B(t) = \sum_{j=1}^{K} g_j(t)\, B_j\,,$$

where $\{g_j(t)\}_1^K$ is a maximal linearly independent subset of $\{f_i(t + \tau) - f_i(t)\}$ and the B_j's are linear combinations of the A_i's, $1 \leqq K \leqq N$. Hence $B(t)$ is functionally commutative since $[B_j, B_k] = 0$. Hence

$$\Psi(t) = \exp\left(\int_0^t B(s)\, ds\right) = \exp\left(\sum_{j=1}^{K} B_j \int_0^t g_i(s)\, ds\right) = \prod_{j=1}^{K} \exp\left(B_j \int_0^t g_j(s)\, ds\right).$$

Hence

$$(14) \qquad \Phi(t + \tau) = \Phi(t) \left(\prod_{j=1}^{K} \exp\left(B_j \int_0^t g_j(s)\, ds\right)\right) \Phi(\tau)\,,$$

and by Corollary 2 of [5]

$$\Phi(t + m\tau) = \Phi(t)\left(\prod_{j=1}^{K} \exp\left(B_j \sum_{l=0}^{m-1} \int_0^{t+l\tau} g_j(s)\,\mathrm{d}s\right)\right)\Phi(\tau)^m . \tag{15}$$

IV. EXAMPLES

Example 1. We here show that Theorem 4 of [5] is not a corollary of Theorem 3 of [5]. Let

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$\alpha(t), \beta(t) \in C^1[0, \infty)$, $\alpha(1) = \alpha'(1) = \beta(2) = \beta'(2) = 0$, $\alpha(\frac{1}{2}) \neq 0$, $\beta(\frac{5}{2}) \neq 0$, $\beta(t) \neq -1$, $\beta(t + 1) - \beta(t) = -1$ for $t > 1$, and $\int_0^1 \alpha(s)\,\mathrm{d}s = 0$. Let

$$A(t) = \begin{cases} tI + \alpha(t) M_2 , & 0 \leqq t \leqq 1 \\ tI , & 1 < t \leqq 2 \\ tI + \beta(t) M_1 , & 2 < t . \end{cases}$$

Then $A(t) \in C^1[0, \infty)$ and

$$B(t) \equiv A(t + 1) - A(t) = \begin{cases} I - \alpha(t) M_2, & 0 \leqq t \leqq 1 \\ I + \beta(t + 1) M_1 , & 1 < t \leqq 2 \\ I + (\beta(t + 1) - \beta(t)) M_1 , & 2 < t . \end{cases}$$

If $\Phi'(t) = A(t)\,\Phi(t)$, $\Phi(0) = I$, then

$$\Phi(t) = \begin{cases} \left[I + \left(\int_0^t \alpha(s)\,\mathrm{d}s\right) M_2\right] \exp\dfrac{t^2}{2} & , \quad 0 \leqq t \leqq 1 \\ I \exp\dfrac{t^2}{2} & , \quad 1 < t \leqq 2 \\ \left[I + \left(\exp\int_2^t \beta(s)\,\mathrm{d}s - 1\right) M_1\right] \exp\dfrac{t^2}{2} , & \quad 2 < t \end{cases}$$

$B(t) \in C^1[0, \infty)$ and $B^{-1}(t)$ exists. Clearly $[A(t), B(t)] = [A(t), B'(t)] = [\Phi(t), B(t)] = [\Phi(t), B'(t)] = 0$. Define $G(t) = B'(t)\,B^{-1}(t)$. Then $[\Phi(t), G(t)] = 0$ and $B(t)$ satisfies the hypotheses of Theorem 4 of [5]. Yet $[A(\frac{1}{2}), B(\frac{3}{2})] = \alpha(\frac{1}{2})\,\beta(\frac{5}{2})\,[M_2, M_1] = -\alpha(\frac{1}{2})\,\beta(\frac{5}{2})\,M_2 \neq 0$. and hence $[A(t), B(s)] \not\equiv 0$.

Example 2. This example shows that it is not true in general that $[B(t), A(t)] = 0$ implies AFS. Let

$$A(t) = \sin \pi t \begin{bmatrix} 0 & 1 \\ t - [t] & 0 \end{bmatrix},$$

where $[t]$ is the greatest integer function.

Let

$$B(t) = A(t+1) - A(t) = -2\sin \pi t \begin{bmatrix} 0 & 1 \\ t-[t] & 0 \end{bmatrix}.$$

Then $[A(t), B(t)] \equiv 0$. Let

$$\Phi(t) = \begin{bmatrix} p(t) & q(t) \\ r(t) & s(t) \end{bmatrix}.$$

$$\begin{bmatrix} p'(t) & q'(t) \\ r'(t) & s'(t) \end{bmatrix} = \Phi'(t) = A(t)\,\Phi(t) = \sin \pi t \begin{bmatrix} r(t) & s(t) \\ (t-[t])\,p(t) & (t-[t])\,q(t) \end{bmatrix}.$$

Further

$$[B(t), \Phi(t)] = -2\sin \pi t \begin{bmatrix} r(t) - (t-[t])\,q(t) & s(t) - p(t) \\ (t-[t])\,(p(t) - s(t)) & (t-[t])\,q(t) - r(t) \end{bmatrix}.$$

If $[B(t), \Phi(t)] = 0$, then the following must hold; $p(t) = s(t)$ and $r(t) = (t-[t])\,q(t)$. Hence the equation $\Phi' = A\Phi$ gives $p = q = r = s \equiv 0$ and $\Phi(t) \equiv 0$, which is a contradiction.

The rest of the examples illustrate Theorem 4 and the remarks following it.

Example 3. Let

$$A(t) = \begin{bmatrix} 0 & p(t) \\ k\,p(t) & 0 \end{bmatrix} + q(t) \begin{bmatrix} a & b \\ kb & a \end{bmatrix},$$

where $p(t+\tau) = p(t)$.

Then

$$B(t) = A(t+\tau) - A(t) = (q(t+\tau) - q(t)) \begin{bmatrix} a & b \\ kb & a \end{bmatrix} = (q(t+\tau) - q(t))\,W.$$

Then $\Psi(t) = \exp\left(W \int_0^t (q(s+\tau) - q(s))\,ds\right)$, and $\Phi(t + m\tau) =$
$= \Phi(t) \exp\left(W \sum_{j=0}^{m-1} \int_0^{t+j\tau} (q(s+\tau) - q(s))\,ds + Rm\tau\right)$, where $R \equiv (1/\tau) \log \Phi(\tau)$.

Example 4. Let $A(t) = \Pi(t) + q(t)\,I$, where $\Pi(t+\tau) = \Pi(t)$. Then $B(t) = A(t+\tau) - A(t) = (q(t+\tau) - q(t))\,I$ and $\Psi(t) = \left(\exp \int_0^t (q(s+\tau) - q(s))\,ds\right) I$. $\Phi(t + m\tau)$ is as in the above example with W replaced by I. An example of a theorem giving stability criteria for this example would be as follows:

Theorem 5. *Let Re (eigenvalues of R) < 0. Let there exist t_0, m_0 such that for $t \geqq t_0$, $m \geqq m_0$, $\sum_{l=0}^{m-1} \int_0^{t+l\tau} [q(s+\tau) - q(s)]\,ds \leqq 0$. Then system (1) is asymptotically stable.*

Example 5. Let $A(t) = \Pi(t) + P(t)$, where $\Pi(t + \tau) = \Pi(t)$, $P(t + \omega) = P(t)$, $[\Pi(t), P(s)] = 0$. Then $B(t) = P(t + \tau) - P(t)$ is periodic of period ω, and $\Psi(t + \omega) = \Psi(t)\,\Psi(\omega)$, $\Phi(t + \tau) = \Phi(t)\,\Psi(t)\,\Phi(\tau)$. Define $g(k)$ and $h(k)$ by $k\tau = g\omega + h$, $0 \leqq h < \omega$. Then $\Psi(t + k\tau) = \Psi(t + h)\,\Psi(\omega)^g$ and $\Phi(t + m\tau) = \Phi(t)\,(\sum_{k=0}^{m-1} \Psi(t + h(k))\,\Psi(\omega)^{g(k)}]\,\Phi(\tau)^m$, and since $0 \leqq h \leqq \omega$, $\Psi(t + h(k))$ is bounded in norm, and hence the stability depends on the eigenvalues of $\Psi(\omega)$ and $\Phi(\tau)$.

Remark 5. The above example could easily be extended to include the case where $A(t)$ is a finite sum of appropriately commuting matrices of incommensurable periods.

ACKNOWLEDGMENT

The authors are indebted to Professor G. J. BUTLER for assistance with Example 2.

References

[1] *R. Bellman:* Stability Theory of Differential Equations, McGraw-Hill (1953).

[2] *J. S. Bogdanov* and *G. N. Chebotarev:* On matrices commuting with their derivatives, Izv. Vyss. Ucebn, Zaved. Matematiks, No. *4* (*11*), (1959) pp. 27—37. (Russian).

[3] *L. Brand:* Differential and Difference Equations, John Wiley & Sons (1966).

[4] *T. A. Burton* and *J. S. Muldowney:* A generalized Floquet theory, Arch. Math. (Basel), *19* (1968) pp. 188—194.

[5] *H. I. Freedman:* Almost Floquet systems, J. Differential Equations, *10* (1971) pp. 345—354.

[6] *V. V. Morozov:* On commutative matrices. Ucen. Zap. Kerel, Ped. Inst. Ser. Fiz.-Math. Nauk, *9* (1952). (Russian).

[7] *H. W. Turnbull* and *A. C. Aitken:* An Introduction to the Theory of Canonical Matrices, Dover Publications, London (1961).

Authors' address: H. I. FREEDMAN, University of Alberta, Department of Mathematics, Edmonton, Canada; J. J. MALLET-PARET, Brown University, Division of Applied Mathematics, Providence, Rhode Island, U.S.A.