Czechoslovak Mathematical Journal

Alois Švec

On 3-dimensional Lie algebras of vector fields

Czechoslovak Mathematical Journal, Vol. 25 (1975), No. 4, 661–672

Persistent URL: http://dml.cz/dmlcz/101362

Czechoslovak Mathematical Journal, 25 (100) 1975, Praha

ON 3-DIMENSIONAL LIE ALGEBRAS OF VECTOR FIELDS

ALOIS ŠVEC, Praha

(Received November 18, 1974)

In series of papers [1]–[5], I devoted myself to the study of real hypersurfaces of the complex space $\mathscr{C}^2$. The local differential geometry of such hypersurfaces consists (at least partly) in the study of 3-dimensional Lie algebras of vector fields on a 3-manifold. Because of this I present a more systematic study of such algebras.

Let G be a 3-dimensional Lie group, g its Lie algebra; let X_1, X_2, X_3 be independent left invariant fields on G. Let $\varphi : G \to G$ be a (local) diffeomorphism. If $d\varphi(X_i(x)) = X_i(\varphi(x))$ for $i = 1, 2, 3$ and each $x \in \operatorname{Dom} \varphi$, φ is a restriction of a left motion $L_g(x) = gx$ of G; denote by $\mathscr{M}(G)$ the pseudogroup of such diffeomorphisms. Further, denote by $\mathscr{M}^*(G)$ the pseudogroup of (local) diffeomorphisms $\psi : G \to G$ such that $d\psi(X_\alpha(x)) \in \{X_\alpha(\psi(x))\}$ for each $x \in \operatorname{Dom} \psi$ and $\alpha = 1, 2$. I am going to show the infinitesimal version of the fact that generally (the exceptions being singled out) $\mathscr{M}(G) = \mathscr{M}^*(G)$.

1. Let L be a 3-dimensional Lie algebra of vector fields on a 3-dimensional differentiable manifold; everything be of class C^∞. Suppose the existence of two 1-dimensional subspaces t, t' of L such that the plane spanned by them is not a subalgebra of L. A basis (v_1, v_2, v_3) of L is called *canonical* if $v_1 \in t$, $v_2 \in t'$ (or $v_1 \in t'$, $v_2 \in t$ respectively) and $v_3 = [v_1, v_2]$.

Lemma. *The canonical basis may be chosen in such a way that we have one of the following cases (here, $p \in \mathscr{R}$ and $\varepsilon^2 = \varepsilon_1^2 = \varepsilon_2^2 = 1$):*

(L_1^p) $\quad [v_1, v_2] = v_3\,, \quad [v_1, v_3] = [v_2, v_3] = pv_1 - pv_2 + v_3\,;$

(L_2^p) $\quad [v_1, v_2] = v_3\,, \quad [v_1, v_3] = pv_2 + v_3\,, \quad [v_2, v_3] = 0\,;$

(L_3^p) $\quad [v_1, v_2] = v_3\,, \quad [v_1, v_3] = pv_1 + \varepsilon_1 v_2\,, \quad [v_2, v_3] = \varepsilon_2 v_1 - pv_2\,;$

(L_4) $\quad [v_1, v_2] = v_3\,, \quad [v_1, v_3] = v_1 + \varepsilon v_2\,, \quad [v_2, v_3] = -v_2\,;$

(L_5) $\quad [v_1, v_2] = v_3\,, \quad [v_1, v_3] = \varepsilon v_2\,, \quad [v_2, v_3] = 0\,;$

(L_6) $\quad [v_1, v_2] = v_3\,, \quad [v_1, v_3] = v_1\,, \quad [v_2, v_3] = -v_2\,;$

(L_7) $\quad [v_1, v_2] = v_3\,, \quad [v_1, v_3] = [v_2, v_3] = 0\,.$

Proof. Let (v_1, v_2, v_3), (w_1, w_2, w_3) be two canonical bases of L. Then

(1) $$[v_1, v_2] = v_3\,, \quad [v_1, v_3] = a_1 v_1 + a_2 v_2 + a_3 v_3\,,$$
$$[v_2, v_3] = b_1 v_1 + b_2 v_2 + b_3 v_3\,;$$
$$[w_1, w_2] = w_3\,, \quad [w_1, w_3] = A_1 w_1 + A_2 w_2 + A_3 w_3\,,$$
$$[w_2, w_3] = B_1 w_1 + B_2 w_2 + B_3 w_3\,;$$

(2) $$v_1 = \alpha w_1\,, \quad v_2 = \beta w_2\,, \quad v_3 = \alpha\beta w_3\,; \quad \alpha\beta \neq 0\,.$$

From the Jacobi identities,

(3) $$a_1 + b_2 = 0\,, \quad a_1 b_3 - a_3 b_1 = 0\,, \quad a_2 b_3 - a_3 b_2 = 0\,;$$
$$A_1 + B_2 = 0\,, \quad A_1 B_3 - A_3 B_1 = 0\,, \quad A_2 B_3 - A_3 B_2 = 0\,.$$

Further,

(4) $$a_1 = \alpha\beta A_1\,, \quad a_2 = \alpha^2 A_2\,, \quad a_3 = \alpha A_3\,, \quad b_1 = \beta^2 B_1\,,$$
$$b_2 = \alpha\beta B_2\,, \quad b_3 = \beta B_3\,,$$

and the result follows.

Theorem. *Let L be as above. Let $\mathscr{L}(L)$ be the Lie algebra of infinitesimal automorphisms of L, i.e., the Lie algebra of vector fields u on M such that $[v, u] = 0$ for each $v \in L$. Let $\mathscr{L}^*(L)$ be the Lie algebra of vector fields u on M such that $[t, u] \subset t$ and $[t', u] \subset t'$. Then the following conditions are equivalent:* (i) $\mathscr{L}(L) \neq \mathscr{L}^*(L)$, (ii) $\dim \mathscr{L}^*(L) = 8$, (iii) *L is equal to L_1^0 or L_2^p or L_3^0 or L_5 or L_6 or L_7 respectively.*

Proof. (1) Consider the algebra L_1^p, and let

(5) $$u = A v_1 + B v_2 + C v_3$$

be a vector field. Because of

(6) $$[v_1, u] = (v_1 A + pC)\, v_1 + (v_1 B - pC)\, v_2 + (v_1 C + B + C)\, v_3\,,$$
$$[v_2, u] = (v_2 A + pC)\, v_1 + (v_2 B - pC)\, v_2 + (v_2 C - A + C)\, v_3\,,$$
$$[v_3, u] = (v_3 A - pA - pB)\, v_1 + (v_3 B + pA + pB)\, v_2 + (v_3 C - A - B)\, v_3\,,$$

$u \in \mathscr{L}^*(L_1^p)$ if and only if

(7) $$v_2 A = -pC\,; \quad v_1 B = pC\,; \quad v_1 C = -B - C\,, \quad v_2 C = A - C\,.$$

The integrability condition of $(7_{3,4})$ is $v_3C = v_1A + v_2B + A + B$. Set $D := v_1A$, $E := v_2B$, then

(8) $$v_1A = D\,;\quad v_2B = E\,;\quad v_3C = A + B + D + E\,.$$

The integrability conditions of (7) + (8) are

$$v_3A + v_2D = p(B + C)\,,\quad v_3B - v_1E = p(C - A)\,,$$
$$v_3B + v_1D + v_1E = -p(A + B + C) - D\,,$$
$$v_3A - v_2D - v_2E = p(A + B - C) + E\,.$$

Set $F := v_3A$, $G := v_3B$, then

(9) $$v_3A = F\,;\quad v_3B = G\,;$$
$$v_1D = -p(2A + B) - D - 2G\,,\quad v_2D = p(B + C) - F\,;$$
$$v_1E = p(A - C) + G\,,\quad v_2E = -p(A + 2B) - E + 2F\,.$$

The integrability conditions of (7)–(9) are

$$v_1F - v_3D = p^2C + pD + F\,,\qquad v_2F = p^2C - p(A + B + E) + F\,,$$
$$v_1G = p^2C + p(A + B + D) + G\,,\quad v_2G - v_3E = p^2C - pE + G\,,$$
$$v_3D + v_1F - 2v_2G = -p^2C + pE - F\,,$$
$$v_3E - 2v_1F + v_2G = -p^2C - pD - G\,.$$

Set $H := v_1F - \frac{1}{2}pE$, then

(10) $$v_3D = -p^2C - pD + \tfrac{1}{2}pE - F + H\,;$$
$$v_3E = -p^2C - \tfrac{1}{2}pD + pE - G + H\,;$$
$$v_1F = \tfrac{1}{2}pE + H\,,\quad v_2F = p^2C - p(A + B + E) + F\,;$$
$$v_1G = p^2C + p(A + B + D) + G\,,\quad v_2G = -\tfrac{1}{2}pD + H\,.$$

The integrability conditions of (9) + (10) are

$$v_1H + 2v_3G = -\tfrac{9}{2}p^2A - 4p^2B - \tfrac{3}{2}p^2C - 2pD + \tfrac{1}{2}pE - pF - \tfrac{11}{2}pG + H\,,$$
$$v_2H + v_3F = -\tfrac{1}{2}p^2A - p^2C - pD + pE - pF - pG + H\,,$$
$$v_1H - v_3G = \tfrac{1}{2}p^2B - p^2C - pD + pE - pF - pG + H\,,$$
$$v_2H - 2v_3F = 4p^2A + \tfrac{9}{2}p^2B - \tfrac{3}{2}p^2C - \tfrac{1}{2}pD + 2pE - \tfrac{11}{2}pF - pG + H\,,$$

and we get

$$(11)\qquad \begin{aligned} v_3F &= -\tfrac{3}{2}p^2A - \tfrac{3}{2}p^2B + \tfrac{1}{6}p^2C - \tfrac{1}{6}pD - \tfrac{1}{3}pE + \tfrac{3}{2}pF\,; \\ v_3G &= -\tfrac{3}{2}p^2A - \tfrac{3}{2}p^2B - \tfrac{1}{6}p^2C - \tfrac{1}{3}pD - \tfrac{1}{6}pE - \tfrac{3}{2}pG\,; \\ v_1H &= -\tfrac{3}{2}p^2A - p^2B - \tfrac{7}{6}p^2C - \tfrac{4}{3}pD + \tfrac{5}{6}pE - pF - \tfrac{5}{2}pG + H\,, \\ v_2H &= p^2A + \tfrac{3}{2}p^2B - \tfrac{7}{6}p^2C - \tfrac{5}{6}pD + \tfrac{4}{3}pE - \tfrac{5}{2}pF - pG + H\,. \end{aligned}$$

The integrability conditions of $(10_3) + (11_1)$ and $(11_3) + (11_4)$ are

$$(12)\qquad \begin{aligned} v_3H &= \tfrac{1}{2}p^2(A+B) + \tfrac{1}{12}p(4-15p)(D+E) - \tfrac{1}{2}p(F-G)\,, \\ v_3H &= \tfrac{1}{2}p^2(A+B) - \tfrac{1}{12}p(8+3p)(D+E) - \tfrac{1}{2}p(F-G) \end{aligned}$$

respectively. Thus

$$(13)\qquad p(1-p)(D+E) = 0\,.$$

Suppose $p \neq 0, 1$; then

$$(14)\qquad D + E = 0\,.$$

Applying v_1, v_2, v_3 to (14), we get

$$(15)\qquad \begin{aligned} &p(A+B+C) + D + G = 0\,, \quad p(A+B-C) + E - F = 0 \\ &2H = 2p^2C + \tfrac{3}{2}pD - \tfrac{3}{2}pE + F + G\,. \end{aligned}$$

Thus

$$(16)\qquad F - G = 2p(A+B)$$

from (14), $(15_{1,2})$. Applying v_1, we get

$$(17)\qquad D = -pC\,, \quad E = pC$$

because of (14) and (16). Applying v_2 to (17_1), we get

$$(18)\qquad F = p(A+B)\,, \quad G = -p(A+B)$$

because of (16). Finally,

$$(19)\qquad H = -\tfrac{1}{2}p^2C$$

from (15_3). Thus

$$(20)\qquad \begin{aligned} v_1A &= -pC\,, & v_2A &= -pC\,, & v_3A &= p(A+B)\,,\\ v_1B &= pC\,, & v_2B &= pC\,, & v_3B &= -p(A+B)\,,\\ v_1C &= -B-C\,, & v_2C &= A-C\,, & v_3C &= A+B\,, \end{aligned}$$

and $u \in \mathscr{L}(L_1^p)$ for $p \neq 0, 1$, i.e., $\mathscr{L}^*(L_1^p) = \mathscr{L}(L_1^p)$ for $p \neq 0, 1$.
Suppose $p = 1$. Then

$$(21)\qquad \begin{aligned}
&v_1A = D\,,\quad v_2A = -C\,,\quad v_3A = F\,;\\
&v_1B = C\,,\quad v_2B = E\,,\quad v_3B = G\,;\\
&v_1C = -B - C\,,\quad v_2C = A - C\,,\quad v_3C = A + B + D + E\,;\\
&v_1D = -2A - B - D - 2G\,,\quad v_2D = B + C - F\,,\\
&v_3D = -C - D + \tfrac{1}{2}E - F + H\,;\\
&v_1E = A - C + G\,,\quad v_2E = A - 2B - E + 2F\,,\\
&v_3E = -C - \tfrac{1}{2}D + E - G + H\,;\\
&v_1F = \tfrac{1}{2}E + H\,,\quad v_2F = -A - B + C - E + F\,,\\
&v_3F = -\tfrac{3}{2}A - \tfrac{3}{2}B + \tfrac{1}{6}C - \tfrac{1}{6}D - \tfrac{1}{3}E + \tfrac{3}{2}F\,;\\
&v_1G = A + B + C + D + G\,,\quad v_2G = -\tfrac{1}{2}D + H\,,\\
&v_3G = -\tfrac{3}{2}A - \tfrac{3}{2}B - \tfrac{1}{6}C - \tfrac{1}{3}D - \tfrac{1}{6}E - \tfrac{3}{2}G\,;\\
&v_1H = -\tfrac{3}{2}A - B - \tfrac{7}{6}C - \tfrac{4}{3}D + \tfrac{5}{6}E - F - \tfrac{5}{2}G + H\,,\\
&v_2H = A + \tfrac{3}{2}B - \tfrac{7}{6}C - \tfrac{5}{6}D + \tfrac{4}{3}E - \tfrac{5}{2}F - G + H\,.
\end{aligned}$$

The integrability conditions of $(21_{17}) + (21_{18})$ and $(21_{19}) + (21_{21})$ are $2C = = D + 3E$ and $2C = -3D - E$ respectively, i.e.,

$$(22)\qquad D = -C\,,\quad E = C\,.$$

Applying v_1, v_2, v_3 to $C + D = 0$, we get

$$(23)\qquad G = -A - B\,,\quad F = A + B\,,\quad H = -\tfrac{1}{2}E\,,$$

i.e., $\mathscr{L}^*(L_1^1) = \mathscr{L}(L_1^1)$.

Let $p = 0$. Then

(24)
$$v_1A = D\,,\quad v_2A = 0\,,\quad v_3A = F\,;$$
$$v_1B = 0\,,\quad v_2B = E\,,\quad v_3B = G\,;$$
$$v_1C = -B - C\,,\quad v_2C = A - C\,,\quad v_3C = A + B + D + E\,;$$
$$v_1D = -D - 2G\,,\quad v_2D = -F\,,\quad v_3D = -F + H\,;$$
$$v_1E = G\,,\quad v_2E = -E + 2F\,,\quad v_3E = -G + H\,;$$
$$v_1F = H\,,\quad v_2F = F\,,\quad v_3F = 0\,;$$
$$v_1G = G\,,\quad v_2G = H\,,\quad v_3G = 0\,;$$
$$v_1H = H\,,\quad v_2H = H\,.$$

The integrability conditions of this system reduce to

(25)
$$v_3H = 0\,.$$

The system (24) + (25) being completely integrable, we have dim $\mathscr{L}^*(L_1^0) = 8$.

(2) Let $L = L_2^p$. Then

(26)
$$[v_1, u] = v_1A\,.\,v_1 + (v_1B + pC)\,v_2 + (v_1C + B + C)\,v_3\,,$$
$$[v_2, u] = v_2A\,.\,v_1 + v_2B\,.\,v_2 + (v_2C - A)\,v_3\,,$$

i.e., $v_1C = -B - C$, $v_2C = A$ for $u \in \mathscr{L}^*(L_2^p)$. The integrability condition being $v_3C = v_1A + v_2B + A$, our starting point are the equations

(27)
$$v_1A = D\,,\quad v_2A = 0\,;\quad v_1B = -pC\,,\quad v_2B = E\,;$$
$$v_1C = -B - C\,,\quad v_2C = A\,,\quad v_3C = A + D + E\,.$$

The integrability conditions are

$$v_3A + v_2D = 0\,,\quad v_3B - v_1E = pA\,,\quad v_3A - v_2D - v_2E = 0\,,$$
$$v_3B + v_1D + v_1E = pA - D\,.$$

For $F := v_3A$, $G := v_3B$, we get

(28)
$$v_3A = F\,;\quad v_3B = G\,;$$
$$v_1D = 2pA - D - 2G\,,\quad v_2D = -F\,;$$
$$v_1E = -pA + G\,,\quad v_2E = 2F\,.$$

The integrability conditions of (27) + (28) are

$$v_3D - v_1F = -F\,, \quad v_2F = 0\,, \quad v_1G = -p(A + D) + G\,, \quad v_3E - v_2G = 0\,,$$
$$v_3D + v_1F - 2v_2G = -F\,, \quad v_3E - 2v_1F + v_2G = 0\,,$$

Set $H := v_3D$, then

$$v_3D = H\,; \quad v_3E = F + H\,; \quad v_1F = F + H\,, \quad v_2F = 0\,; \tag{29}$$
$$v_1G = -p(A + D) + G\,, \quad v_2G = F + H\,.$$

The integrability conditions of (28) + (29) imply

$$v_3F = 0\,; \quad v_3G = 0\,; \quad v_1H = pF\,, \quad v_2H = 0\,; \tag{30}$$

the integrability of conditions (29), (30) reduce to

$$v_3H = 0\,. \tag{31}$$

The system (27)–(31) being completely integrable, we have $\dim \mathscr{L}^*(L_2^p) = 8$.

(3) Let $L = L_3^p$. Then

$$\begin{aligned} [v_1, u] &= (v_1A + pC)\,v_1 + (v_1B + \varepsilon_1C)\,v_2 + (v_1C + B)\,v_3\,, \\ [v_2, u] &= (v_2A + \varepsilon_2C)\,v_1 + (v_2B - pC)\,v_2 + (v_2C - A)\,v_3\,, \\ [v_3, u] &= (v_3A - pA - \varepsilon_2B)\,v_1 + (v_3B - \varepsilon_1A + pB)\,v_2 + v_3C.\,v_3\,. \end{aligned} \tag{32}$$

Let $u \in \mathscr{L}^*(L_3^p)$, then

$$v_1B + \varepsilon_1C = 0\,, \quad v_2A + \varepsilon_2C = 0\,, \quad v_1C + B = 0\,, \quad v_2C - A = 0\,.$$

From the last two equations, $v_3C = v_1A + v_2B$, and our starting points is the system

$$v_1A = D\,, \quad v_2A = -\varepsilon_2C\,; \quad v_1B = -\varepsilon_1C\,, \quad v_2B = E\,; \tag{33}$$
$$v_1C = -B\,, \quad v_2C = A\,, \quad v_3C = D + E\,.$$

Its integrability conditions are

$$v_3A + v_2D = \varepsilon_2B\,, \quad v_3B - v_1E = \varepsilon_1A\,,$$
$$v_3B + v_1D + v_1E = \varepsilon_1A - pB\,, \quad v_3A - v_2D - v_2E = pA + \varepsilon_2B\,.$$

Set $F := v_3 A$, $G := v_3 B$, then the prolongation of (33) is

$$v_3 A = F\,; \quad v_3 B = G\,; \tag{34}$$

$$v_1 D = 2\varepsilon_1 A - pB - 2G\,, \quad v_2 D = \varepsilon_2 B - F\,;$$

$$v_1 E = -\varepsilon_1 A + G\,, \quad v_2 E = -pA - 2\varepsilon_2 B + 2F\,.$$

Set $H := v_1 F - \frac{1}{2} pD$; the integrability conditions of (33) + (34) imply

$$v_3 D = \varepsilon_1 \varepsilon_2 C - \tfrac{1}{2} pD + H\,; \quad v_3 E = \varepsilon_1 \varepsilon_2 C + \tfrac{1}{2} pE + H\,; \tag{35}$$

$$v_1 F = \tfrac{1}{2} pD + H\,, \qquad v_2 F = \varepsilon_2 pC - \varepsilon_2 E\,;$$

$$v_1 G = -\varepsilon_1 pC - \varepsilon_1 D\,, \quad v_2 G = -\tfrac{1}{2} pE + H\,.$$

The integrability conditions of (34) + (35) are

$$v_3 F = \left(\varepsilon_1 \varepsilon_2 - \tfrac{1}{2} p^2\right) A - \tfrac{3}{2} \varepsilon_2 pB + \tfrac{3}{2} pF\,; \tag{36}$$

$$v_3 G = \tfrac{3}{2} \varepsilon_1 pA + \left(\varepsilon_1 \varepsilon_2 - \tfrac{1}{2} p^2\right) B - \tfrac{3}{2} pG\,;$$

$$v_1 H = -\tfrac{1}{2} p^2 B + \varepsilon_1 F - pG\,, \quad v_2 H = \tfrac{1}{2} p^2 A - pF - \varepsilon_2 G\,.$$

The integrability condition of $(36_3) + (36_4)$ is

$$v_3 H = \varepsilon_1 \varepsilon_2 (D + E)\,; \tag{37}$$

the integrability condition of $(36_3) + (37)$ reduces to

$$p(pA + \varepsilon_2 B - F) = 0\,. \tag{38}$$

Let $p \neq 0$; then

$$F = pA + \varepsilon_2 B\,. \tag{39}$$

Applying v_1 and v_2 to this equation, we get

$$H = -\varepsilon_1 \varepsilon_2 C + \tfrac{1}{2} pD\,, \quad E = pC \tag{40}$$

respectively. Applying v_1 and v_3 to (40_2), we get

$$G = \varepsilon_1 A - pB\,, \quad D = -pC \tag{41}$$

respectively. Thus $u \in \mathscr{L}^*(L_3^p)$, $p \neq 0$, implies $u \in \mathscr{L}(L_3^p)$.

In the case $p = 0$, it is easy to see that the system (33)–(37) is completely integrable. Thus $\dim \mathscr{L}^*(L_3^0) = 8$.

(4) Let $L = L_4$. Then

$$[v_1, u] = (v_1A + C)\,v_1 + (v_1B + \varepsilon C)\,v_2 + (v_1C + B)\,v_3\,, \tag{42}$$
$$[v_2, u] = v_2A\,.\,v_1 + (v_2B - C)\,v_2 + (v_2C - A)\,v_3\,,$$
$$[v_3, u] = (v_3A - A)\,v_1 + (v_3B - \varepsilon A + B)\,v_2 + v_3C\,.\,v_3\,.$$

From $v_1C = -B$, $v_2C = A$, we get $v_3C = v_1A + v_2B$, and we may write

$$v_1A = D\,, \quad v_2A = 0\,; \quad v_1B = -\varepsilon C\,, \quad v_2B = E\,; \tag{43}$$
$$v_1C = -B\,, \quad v_2C = A\,, \quad v_3C = D + E$$

for $u \in \mathscr{L}^*(L_4)$. The integrability conditions of (43) allow us to write

$$v_3A = F\,; \quad v_3B = G\,; \tag{44}$$
$$v_1D = 2\varepsilon A - B - 2G\,, \quad v_2D = -F\,;$$
$$v_1E = -\varepsilon A + G\,, \quad v_2E = -A + 2F\,,$$

and a further differentiation yields

$$v_3D = H - \tfrac{1}{2}D\,; \quad v_3E = H + \tfrac{1}{2}E\,; \tag{45}$$
$$v_1F = H + \tfrac{1}{2}D\,, \quad v_2F = 0\,; \quad v_1G = -\varepsilon(C + D)\,, \quad v_2G = H - \tfrac{1}{2}E\,.$$

Finally,

$$v_3F = -\tfrac{1}{2}A + \tfrac{3}{2}F\,; \quad v_3G = \tfrac{3}{2}\varepsilon A - \tfrac{1}{2}B - \tfrac{3}{2}G\,; \tag{46}$$
$$v_1H = -\tfrac{1}{2}B + \varepsilon F - G\,, \quad v_2H = \tfrac{1}{2}A - F\,.$$

The integrability conditions are

$$v_3H = 0\,, \quad v_3H = -\tfrac{1}{2}(C - E)\,, \quad 3D = -2C - E\,.$$

Thus

$$D = -C\,, \quad E = C\,. \tag{47}$$

Applying v_1, v_2, v_3 to (47_2), we get

$$G = \varepsilon A - B\,, \quad F = A\,, \quad H = -\tfrac{1}{2}C \tag{48}$$

respectively. Thus $\mathscr{L}^*(L_4) = \mathscr{L}(L_4)$.

(5) Let $L = L_5$. We have

(49)
$$[v_1, u] = v_1A \,.\, v_1 + (v_1B + \varepsilon C)\, v_2 + (v_1C + B)\, v_3\,,$$
$$[v_2, u] = v_2A \,.\, v_1 + v_2B \,.\, v_2 + (v_2C - A)\, v_3\,.$$

By the same procedure, we get successively

(50)
$$v_1A = D\,,\quad v_2A = 0\,;\quad v_1B = -\varepsilon C\,,\quad v_2B = E\,;$$
$$v_1C = -B\,,\quad v_2C = A\,,\quad v_3C = D + E\,;$$

(51)
$$v_3A = F\,;\quad v_3B = G\,;$$
$$v_1D = 2\varepsilon A - 2G\,,\quad v_2D = -F\,;\quad v_1E = G - \varepsilon A\,,\quad v_2E = 2F\,;$$

(52)
$$v_3D = H\,;\quad v_3E = H\,;\quad v_1F = H\,,\quad v_2F = 0\,;\quad v_1G = -\varepsilon D\,,\quad v_2G = H\,;$$

(53)
$$v_3F = 0\,;\quad v_3G = 0\,;\quad v_1H = \varepsilon F\,,\quad v_2H = 0\,,$$

(54)
$$v_3H = 0$$

for $u \in \mathscr{L}^*(L_5)$. The system (50)–(54) being completely integrable, dim $\mathscr{L}^*(L_5) = 8$.

(6) Let $L = L_6$. Then

(55)
$$[v_1, u] = (v_1A + C)\, v_1 + v_1B \,.\, v_2 + (v_1C + B)\, v_3\,,$$
$$[v_2, u] = v_2A \,.\, v_1 + (v_2B - C)\, v_2 + (v_2C - A)\, v_3\,,$$

and we get the completely integrable system

(56)
$$v_1A = D\,,\quad v_2A = 0\,;\quad v_1B = 0\,,\quad v_2B = E\,;$$
$$v_1C = -B\,,\quad v_2C = A\,,\quad v_3C = D + E\,;$$

(57)
$$v_3A = F\,;\quad v_3B = G\,;$$
$$v_1D = -B - 2G\,,\quad v_2D = -F\,;\quad v_1E = G\,,\quad v_2E = -A + 2F\,;$$

(58)
$$v_3D = H - \tfrac{1}{2}D\,;\quad v_3E = H + \tfrac{1}{2}E\,;$$
$$v_1F = H + \tfrac{1}{2}D\,,\quad v_2F = 0\,;\quad v_1G = 0\,,\quad v_2G = H - \tfrac{1}{2}E\,;$$

(59)
$$v_3F = -\tfrac{1}{2}A + \tfrac{3}{2}F\,;\quad v_3G = -\tfrac{1}{2}B - \tfrac{3}{2}G\,;$$
$$v_1H = -\tfrac{1}{2}B - G\,,\quad v_2H = \tfrac{1}{2}A - F\,,$$

(60)
$$v_3H = 0$$

for $u \in \mathscr{L}^*(L_6)$. Thus dim $\mathscr{L}^*(L_6) = 8$.

(7) Let $L = L_7$. Then

(61)
$$[v_1, u] = v_1 A . v_1 + v_1 B . v_2 + (v_1 C + B) v_3 ,$$
$$[v_2, u] = v_2 A . v_1 + v_2 B . v_2 + (v_2 C - A) v_3 .$$

The result follows from the complete integrability of the system

(62)
$$v_1 A = D , \quad v_2 A = 0 ; \quad v_1 B = 0 , \quad v_2 B = E ;$$
$$v_1 C = -B , \quad v_2 C = A , \quad v_3 C = D + E ;$$

(63)
$$v_3 A = F ; \quad v_3 B = G ;$$
$$v_1 D = -2G , \quad v_2 D = -F ; \quad v_1 E = G , \quad v_2 E = 2F ;$$

(64)
$$v_3 D = H , \quad v_3 E = H ; \quad v_1 F = H , \quad v_2 F = 0 ; \quad v_1 G = 0 , \quad v_2 G = H ;$$

(65)
$$v_3 F = 0 ; \quad v_3 G = 0 ; \quad v_1 H = 0 , \quad v_2 H = 0 ,$$

(66)
$$v_3 H = 0$$

for $u \in \mathscr{L}^*(L_7)$; namely, $\dim \mathscr{L}^*(L_7) = 8$.

2. Let us add two remarks.

(1) Let G be the group of matrices of the form

(67)
$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ \gamma & \delta & \varphi \end{pmatrix}, \quad \alpha\beta\varphi \neq 0 ,$$

and let L be one of the algebras of the type $L_1^p, \ldots, L_7$. Denote by $B_G(L)$ the G-structure on M^3 generated by the section (v_1, v_2, v_3). Then it is possible to prove the following theorem:

The conditions (i)–(iii) *of our Theorem are equivalent to the following one*: (iv) *the G-structure* $B_G(L)$ *contains a section* (w_1, w_2, w_3) *satisfying* $[w_1, w_2] = w_3$, $[w_1, w_3] = [w_2, w_3] = 0$.

(2) Let $M^{2n-1} \subset \mathscr{C}^n$ be a real hypersurface of the complex space, and let $\Gamma(M^{2n-1})$ be the pseudogroup of (local) biholomorphic mappings of $\mathscr{C}^n$ preserving M^{2n-1}. One of the problems is to determine hypersurfaces which are transitive with respect to $\Gamma(M^{2n-1})$. It turns out that the problem to determine all possible numbers $\dim \Gamma(M^{2n-1})$ is equivalent to the following one:

Let M^{2n-1} be a differentiable manifold, and let L be a Lie algebra of vector fields on M^{2n-1}. Suppose that $\dim L = 2n - 1$ and that there are two subalgebras $K_1, K_2 \subset L$ such that $\dim K_1 = \dim K_2 = n - 1$, $K_1 \cap K_2 = \{0\}$, $[K_1, K_2] = L$. Denote by $\mathscr{L}^*(L; K_1, K_2)$ the Lie algebra of vector fields u on M^{2n-1} satisfying $[K_1, u] \subset K_1$, $[K_2, u] \subset K_2$. We have to determine all possible values of $\dim \mathscr{L}^*(L; K_1, K_2)$.

Bibliography

[1] *A. Švec:* On transitive submanifolds of $\mathscr{C}^2$ and $\mathscr{C}^3$. Czech. Math. J., *23 (98)* 1973, 306—338.

[2] *A. Švec:* On certain groups of holomorphic maps. Acta Univ. Carolinae, Vol. *13*, 3—27.

[3] *A. Švec:* On a partial product structure. Czech. Math. J., *24 (99)* 1974, 107—113.

[4] *A. Švec:* On a group of holomorphic transformations in $\mathscr{C}^2$. Czech. Math. J., *24* (99) 1974, 97—106.

[5] *A. Švec:* On a partial complex structure. Czech. Math. J. *25 (100)* 1975, 653—660.

Author's address: 118 00 Praha 1, Malostranské nám. 25, ČSSR (Matematicko-fyzikální fakulta UK).