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TOLERANCES AND CONGRUENCES ON TREE ALGEBRAS

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The concept of tolerance was introduced by E. C. ZEEMAN [3] and studied on various types of algebras in [4], [5], [6], [7].

A tolerance is a reflexive and symmetric binary relation on a set.

Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra; A is its set of elements, $\mathscr{F}$ is the set of operations on it. Let ξ be a tolerance on A. We say that ξ is compatible with $\mathfrak{A}$, if and only if the following condition is satisfied: If $f \in \mathscr{F}$ is an n-ary operation, where n is a positive integer, and $x_1, \ldots, x_n, y_1, \ldots, y_n$ are elements of A such that $(x_i, y_i) \in \xi$ for $i = 1, \ldots, n$, then $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \xi$.

Tree algebras were introduced by L. NEBESKÝ [1]. A tree algebra (M, P) is an algebra with a non-empty finite set M of elements and with a ternary operation P satisfying the following axioms:

I. $P(u, u, v) = u$;

II. $P(u, v, w) = P(v, u, w) = P(u, w, v)$;

III. $P(P(u, v, w), v, x) = P(u, v, P(w, v, x))$;

IV. if $P(u, v, x) \neq P(v, w, x) \neq P(u, w, x)$, then $P(u, v, x) = P(u, w, x)$.

L. Nebeský has proved that there exists a one-to-one correspondence between tree algebras and trees; to a tree algebra (M, P) a tree T corresponds whose vertex set is M and $x = P(u, v, w)$ if and only if the vertex x of T is the common vertex of the path connecting u and v, the path connecting u and w and the path connecting v and w.

If (M, P) is a tree algebra and ξ a tolerance on M, then ξ is compatible with (M, P) if and only if the following assertion holds: If $x_1, x_2, y_1, y_2, z_1, z_2$ are elements of M, $(x_1, x_2) \in \xi$, $(y_1, y_2) \in \xi$, $(z_1, z_2) \in \xi$, then $(P(x_1, y_1, z_1), P(x_2, y_2, z_2)) \in \xi$.

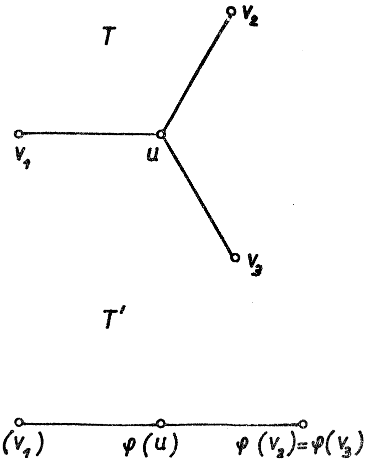
If ξ is a tolerance compatible with (M, P) and moreover ξ is transitive, then ξ is a congruence on (M, P).

We shall prove two theorems concerning tolerances and congruences on tree algebras.

Theorem 1. *Let T be a tree, let (M, P) be the tree algebra corresponding to T. Let ξ be a tolerance on M. Then the following two assertions are equivalent:*

(1) *ξ is compatible with (M, P).*

(2) *If $u \in M$, $v \in M$, $(u, v) \in \xi$, then $(x, y) \in \xi$ for any two vertices x, y of the path connecting u and v in T.*



Proof. (1) $\Rightarrow$ (2). Let x, y be two vertices of the path connecting u and v in T. Without loss of generality let x lie between u and y. Then $P(u, x, y) = x$, $P(v, x, y) = y$. As $(u, v) \in \xi$, $(x, x) \in \xi$, $(y, y) \in \xi$, we must have $(P(u, x, y), P(v, x, y)) = (x, y) \in \xi$.

(2) $\Rightarrow$ (1). Let $x_1, x_2, y_1, y_2, z_1, z_2$ be elements of M (vertices of T), let $(x_1, x_2) \in \xi$, $(y_1, y_2) \in \xi$, $(z_1, z_2) \in \xi$. Let X (or Y, or Z) be the path connecting the vertices x_1, x_2 (or y_1, y_2, or z_1, z_2 respectively) in T. Let $u_1 = P(x_1, y_1, z_1)$, $u_2 = P(x_2, y_2, z_2)$. First suppose that x_1, y_1, z_1 are all distinct from u_1. Let B_x (or B_y, or B_z) be the branch of T outgoing from u_1 and containing x_1 (or y_1, or z_1 respectively). The branches B_x, B_y, B_z are pairwise distinct. If $u_1 = u_2$, then $(u_1, u_2) \in \xi$ and the assertion holds. Thus suppose that $u_1 \neq u_2$. This means that at least two of the vertices x_2, y_2, z_2 lie in the same branch outgoing from u_1. Without loss of generality let x_2, y_2 lie in the same branch B outgoing from u_1; the branch B may coincide with some of the branches B_x, B_y, B_z, but obviously at most with one of them. Without loss of generality let $B \neq B_x$. Then X goes through u_1. If z_2 does not belong to B, then the

path $R(x_2, z_2)$ goes through u_1 and has a common subpath $R'(u_1, x_2)$ with X; this subpath $R'(u_1, x_2)$ connects u_1 and x_2. The path $S(y_2, z_2)$ contains also u_1. Then u_2 is the common vertex of $R(x_2, z_2)$ and $S(y_2, z_2)$ which is in B and whose distance from u_1 is maximal. As u_2 belongs to $R'(u_1, x_2)$ and $R'(u_1, x_2)$ is a subpath of X, the vertices u_1, u_2 belong both to X and $(u_1, u_2) \in \xi$. If z_2 belongs to B, then either $B = B_z$, or Z contains u_1. If $B = B_z$, then Y contains u_1. All vertices x_2, y_2, z_2 are in B. Suppose that Y contains u_1. Let $Q(u_1, x_2)$, $Q'(u_1, y_2)$ be the paths connecting u_1 with x_2 and y_2 respectively. Let w be the common vertex of $Q(u_1, x_2)$, $Q'(u_1, y_2)$ whose distance from u_1 is maximal. The path connecting w and x_2 has only one common vertex w with the path connecting w and y_2; their union is the path connecting x_2 and y_2 (this path is unique, because T is a tree). Thus u_2 lies on this path. But $Q(u_1, x_2)$ is a subpath of X and $Q'(u_1, y_2)$ is a subpath of Y. This means that u_2 belongs either to X, or to Y. As u_1 belongs to both X and Y, we have $(u_1, u_2) \in \xi$. Analogously if Z contains u_1. Thus the proof is complete for the case when x_1, y_1, z_1 are all distinct from u_1. Now let u_1 coincide with one of the vertices x_1, y_1, z_1. If $u_1 = x_1$, then the above proof is adapted so that B_x is not a branch, but the one-vertex subgraph of T consisting of u_1; analogously if $u_1 = y_1$ or $u_1 = z_1$.

Theorem 2. *Let T be a tree, let (M, P) be the tree algebra corresponding to T. Let ξ be an equivalence on M. Then the following two assertions are equivalent:*

(1) *ξ is a congruence on (M, P).*

(2) *Each equivalence class of ξ induces a subtree of T.*

Proof. (1) $\Rightarrow$ (2). As ξ is a congruence on (M, P), it is a tolerance compatible with (M, P). Thus all vertices of a path connecting two vertices of one equivalence class of ξ belong to this equivalence class and the subgraph of T induced by this class is connected. Any connected subgraph of a tree is its subtree.

(2) $\Rightarrow$ (1). The assertion (2) from this theorem implies the assertion (2) from Theorem 1. According to Theorem 1, ξ is then a tolerance compatible with (M, P). As ξ is transitive, it is a congruence on (M, P).

In [2] a connected homomorphism of a graph G onto a graph G' is defined as a homomorphism φ of G onto G' such that for each vertex y of G' the set of all vertices x of G such that $\varphi(x) = y$ induces a connected subgraph of G.

Corollary. *Let T, T' be trees, let (M, P), (M', P') be the tree algebras corresponding to them. Then each connected homomorphism of T onto T' is a homomorphism of (M, P) onto (M', P') and vice versa.*

There exist also homomorphisms of one tree onto another which are not connected. An example is in Fig. 1. This homomorphism φ is not a homomorphism of (M, P) onto (M', P'), because $P(v_1, v_2, v_3) = u$, $P'(\varphi(v_1), \varphi(v_2), \varphi(v_3)) = \varphi(v_2) \neq \varphi(u)$.

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