Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 25 (1975), No. 4, 607–618

Persistent URL: http://dml.cz/dmlcz/101356

Czechoslovak Mathematical Journal, 25 (100) 1975, Praha

EIGENVECTORS OF ACYCLIC MATRICES

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(Received September 2, 1974)

It is shown that for acyclic matrices, i.e. symmetric matrices the graph of which does not contain any circuit, close relations exist between the signs of coordinates of eigenvectors and the position of the corresponding eigenvalues in the natural ordering according to magnitude.

1. Notation and preliminaries. In the whole paper, n will be an integer, $n \geqq 2$. The set $\{1, 2, \ldots, n\}$ will be denoted by N. All numbets, vectors and matrices will be real. We shall be using terminology of the graph theory (see e.g. [1]) and certain elementary facts about trees, such as:

(1,1) *A tree is a connected graph not containing any circuit.*

(1,2) *Every tree with $n \geqq 2$ vertices has at least one end-vertex (i.e. a vertex adjacent to a single edge).*

(1,3) *There is exactly one path between any two vertices of a tree.*

(1,4) *A tree with n vertices has $n - 1$ edges.*

We shall be denoting vertices of a graph by numbers, usually $1, 2, \ldots, n$. We shall also speak about "removing" edges or vertices and the adjacent edges from a graph, in the obvious sense. So the following assertions hold:

(1,5) *If we remove from a tree one edge the resulting graph will have exactly two components (both are trees, maybe with a single vertex).*

(1,6) *If we remove from a tree a vertex of degree s and all adjacent edges, the resulting graph will have s components.*

(1,7) *If we remove from a tree with $n \geqq 2$ vertices, an end-vertex and the adjacent edge, a tree with $n - 1$ vertices will result.*

(1,8) *Let T be a tree with the set of vertices $N = \{1, 2, \ldots, n\}$. Then the $n - 1$ linear forms $x_i - x_k$ where (i, k), $i < k$, are edges of T, are linearly independent.*

Proof. Assume

$$\sum_{\substack{(i,k)\in T \\ i<k}} a_{ik}(x_i - x_k) = 0$$

where not all a_{ik} are equal to zero. Let T_1 be the subgraph of T with the set of vertices N but those edges (i, k) only for which $a_{ik} \neq 0$. Then T_1 has at least one edge and each of its components is a tree by (1, 1); thus its component with at least one edge has at least one end-vertex by (1, 2). If this vertex is n and the single adjacent edge $(n - 1, n)$ then x_n is contained in the sum above in the only term

$$a_{n-1,n}(x_{n-1} - x_n)$$

where $a_{n-1,n} \neq 0$, in contradiction with the assumption that this sum is zero.

Let us turn now to the main definition. An $n \times n$ matrix $A = (a_{ik})$ will be called *acyclic* if it is symmetric and if for any mutually distinct indices $k_1, k_2, \ldots, k_s$ ($s \geqq 3$) in N the equality

$$a_{k_1k_2}a_{k_2k_3} \ldots a_{k_sk_1} = 0$$

is fulfilled.

This has clearly the following graph-theoretical meaning. Let us assign to any symmetric $n \times n$ matrix $A = (a_{ik})$ a (non-directed) graph $G(A)$ with the vertex set N and exactly those edges (i, k), $i \neq k$, for which $a_{ik} \neq 0$. It is well known that A is irreducible iff $G(A)$ is connected. Let us recall that an $n \times n$ matrix $A = (a_{ik})$ is irreducible if no decomposition $N = N_1 \cup N_2$, $N_1 \neq \emptyset \neq N_2$, $N_1 \cap N_2 = \emptyset$, exists such that $a_{ik} = 0$ whenever $i \in N_1$ and $k \in N_2$.

More generally, to the decomposition of $G(A)$ into components corresponds then the expressing the matrix A or a matrix obtained from A by permutation of rows and columns in the block-diagonal form

$$\begin{bmatrix} A_{11} & 0 & \ldots & 0 \\ 0 & A_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & A_{ss} \end{bmatrix}$$

where the principal submatrices A_{ii} are already irreducible. The number s is then uniquely determined and the number $s - 1$ will be called *degree of reducibility* of A. This degree is zero iff A is irreducible.

From the above definition if follows immediately that A is acyclic iff $G(A)$ contains no circuit. By (1,1), A is irreducible and acyclic iff $G(A)$ is a tree.

In the conclusion of this section, we shall recall two notions. We say as in [2] that a matrix $A = (a_{ik})$ is *essentially nonnegative* if $a_{ik} \geqq 0$ for all $i, k, i \neq k$.

By *inertia* of a symmetric matrix A, written In A, we mean as in [3] a row vector (p, q, z) where p denotes the number of positive eigenvalues of A, q the number of negative eigenvalues and z the number of zero eigenvalues of A (including multiplicity). As usual, we shall denote by y^T the transpose vector to the vector y, by S^T the transpose matrix to the matrix S. The identity matrix will be denoted by I, I_1, I_2 etc.

We shall use the following well known facts about inertia:

(1,9) *If A is symmetric and S nonsingular then* $\operatorname{In} SAS^T = \operatorname{In} A$.

(1,10) *If A is symmetric block-diagonal,*

$$A = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_s \end{bmatrix},$$

then

$$\operatorname{In} A = \sum_{k=1}^{s} \operatorname{In} A_k .$$

(1,11) *Let A be symmetric and let A_1 be a principal submatrix of A. If* $\operatorname{In} A = (p, q, z)$ *and* $\operatorname{In} A_1 = (p_1, q_1, z_1)$ *then*

$$p_1 \leqq p, \quad q_1 \leqq q .$$

Let us conclude this section with a lemma which we shall need in the sequel:

(1,12) **Lemma.** *Let*

$$A = \begin{bmatrix} B & c \\ c^T & d \end{bmatrix}$$

be an $n \times n$ partitioned symmetric matrix, let

$$Bu = 0, \quad c^T u \neq 0 .$$

Then

$$\operatorname{In} A = \operatorname{In} B + (1, 1, -1) .$$

Proof. Clearly $u \neq 0$. Without loss of generality, we can assume that its first coordinate $u_1 \neq 0$ so that

$$u = \begin{bmatrix} u_1 \\ \tilde{u} \end{bmatrix}, \quad u_1 \neq 0 .$$

If B and c are partitioned conformally,

$$B = \begin{bmatrix} b_{11}, & b_1 \\ b_1^T, & \tilde{B} \end{bmatrix},$$

$$c = \begin{bmatrix} c_1 \\ \tilde{c} \end{bmatrix},$$

we have for

$$\beta = c^T u$$

and

$$P = \begin{bmatrix} u_1 & \tilde{u} & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PAP^T = \begin{bmatrix} 0 & 0 & \beta \\ 0 & \tilde{B} & \tilde{c} \\ \beta & \tilde{c}^T & d \end{bmatrix}.$$

Let

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ \beta^{-1}\tilde{c} & I & 0 \\ -(2\beta)^{-1}d & 0 & 1 \end{bmatrix}.$$

Then

$$(QP)\,A(QP)^T = \begin{bmatrix} 0 & 0 & \beta \\ 0 & \tilde{B} & 0 \\ \beta & 0 & 0 \end{bmatrix}.$$

It follows that, for a nonsingular matrix R,

$$RAR^T = \begin{bmatrix} 0 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & \tilde{B} \end{bmatrix}.$$

By (1,9) and (1,10),

$$\text{In } A = \text{In } \tilde{B} + \text{In} \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}$$

so that

(1) $$\text{In } A = \text{In } \tilde{B} + (1, 1, 0)\,.$$

On the other hand,
if

$$\tilde{P} = \begin{bmatrix} u_1 & \tilde{u} \\ 0 & I \end{bmatrix},$$

we have

$$\tilde{P}B\tilde{P}^T = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{B} \end{bmatrix}.$$

Thus (1,10) implies

(2) $$\text{In } B = \text{In } \tilde{B} + (0, 0, 1)\,.$$

Combining (1) and (2), we obtain the desired result.

2. Results. We shall prove first:

(2,1) **Theorem.** *Let A be an acyclic matrix. Then there exists a diagonal matrix D for which $D^2 = I$ and such that DAD is essentially nonnegative.*

Proof. It suffices clearly to prove this theorem in the case that A is irreducible. Then we shall define the diagonal entries d_i of the matrix D as follows: We put $d_1 = 1$; if $k \neq 1$, we define d_k by

$$d_k = \operatorname{sgn}\left(a_{1j_1}a_{j_1j_2} \dots a_{j_sk}\right)$$

where $(1, j_1, j_2, \dots, j_s, k)$ is the path from 1 to k (which is unique by (1,3)).

We have to show that

$$a_{ik}d_id_k \geqq 0 \quad \text{for all} \quad i, k, i \neq k\,. \tag{3}$$

This is clear if $a_{ik} = 0$. If $a_{ik} \neq 0$, i.e. if (i, k) is an edge in $G(A)$, let G' be the graph obtained from $G(A)$ by removing the edge (i, k). By (1,5), G' has two components, one containing the vertex i, the other the vertex k. If the vertex 1 belongs to the component containing i, we have

$$d_k = \operatorname{sgn}\left(a_{1j_1}a_{j_1j_2} \dots a_{j_ti}a_{ik}\right) = d_i \operatorname{sgn} a_{ik}\,.$$

Consequently,

$$d_id_k = \operatorname{sgn} a_{ik}$$

and (3) is true. An analogous argument applies in the second case that 1 belongs to the component containing k. The proof is complete.

(2,2) **Theorem.** *Let $C = (c_{ik})$ be an $n \times n$ acyclic matrix such that $Ce = 0$ for $e = (1, 1, \dots, 1)^T$. Then the inertia of C is equal to $(p, q, n - p - q)$ where $2p$ is the total number of negative and $2q$ the total number of positive off-diagonal entries in C. The number $n - p - q - 1$ is equal to the degree of reducibility of C.*

Proof. By (1,10), it suffices to prove the first assertion for the case that C is irreducible. In this case, the graph of the matrix C is a tree T. The corresponding quadratic form is then

$$\sum_{\substack{(i,k)\in T\\ i<k}} c_{ik}(x_i - x_k)^2\,.$$

Since the linear forms $x_i - x_k$ $(i < k)$ are linearly independent by (1,8), we obtain by the Sylvester's law of inertia that the number of squares with positive coefficients is p, with negative coefficients is q. The rest is obvious.

We are now able to prove the main theorems.

(2,3) **Theorem.** *Let* $A = (a_{ik})$ *be an* $n \times n$ *acyclic matrix. Let* $y = (y_i)$ *be an eigenvector of* A *corresponding to an eigenvalue* λ. *Denote by* ω^+ *and* ω^-, *respectively, the number of eigenvalues of* A *greater than and smaller than* λ, *and let* $\omega^{(0)}$ *be the multiplicity of* λ.

Let there be first no "isolated" zero coordinate of y, *i.e. coordinate* $y_k = 0$ *such that* $a_{kj}y_j = 0$ *for all* j. *Then*

$$\omega^+ = a^+ + m\,, \quad \omega^- = a^- + m\,, \quad \omega^{(0)} = n - \omega^+ - \omega^- \tag{4}$$

where m *is the number of zero coordinates of* y, a^+ *is the number of those (unordered) pairs* (i, k), $i \neq k$, *for which*

$$a_{ik}y_iy_k < 0$$

and a^- *is the number of such pairs* (i, k), $i \neq k$, *for which*

$$a_{ik}y_iy_k > 0\,.$$

If there are isolated zero coordinates of y, *if* M *is the set of indices corresponding to such coordinates and* $\tilde{A}$ *the matrix obtained from* A *by deleting all rows and columns with indices from* M *then the numbers* ω^+, ω^- *and* $\omega^{(0)}$ *satisfy*

$$\omega^+ = \tilde{\omega}^+ + c_1\,, \quad \omega^- = \tilde{\omega}^- + c_2\,, \quad \omega^{(0)} = \tilde{\omega}^{(0)} + c_0 \tag{5}$$

where $\tilde{\omega}$ *are corresponding numbers of* $\tilde{A}$ *and* c_0, c_1, c_2 *nonnegative integers such that*

$$c_0 + c_1 + c_2 = |M|\,, \tag{6}$$

the number of elements in M.

Proof. We shall prove (5) by induction with respect to the number v of zero coordinates of y. Let first $v = 0$. Then the diagonal matrix $Y = \text{diag}\,\{y_1, \ldots, y_n\}$ is nonsingular. It follows from (1,9) that the matrix

$$B = Y(A - \lambda I)\,Y$$

has the same inertia as the matrix $A - \lambda I$ and satisfies $Be = 0$. By Theorem (2,2), In $B = (p, q, n - p - q)$ where p is the number of those (unordered) pairs (i, k), $i \neq k$, for which $a_{ik}y_iy_k < 0$, q the number of the pairs (i, k), $i \neq k$, for which $a_{ik}y_iy_k > 0$. Thus $p = a^+$, $q = a^-$ and (4) is true. If, in addition, A, and thus B, is irreducible, we have $n - p - q - 1 = 0$, by Th. (2,2). Consequently, $n - p - q = 1$ which means that λ is a simple eigenvalue of A. Since y is a solution of any $n - 1$ equations of the system $(A - \lambda I)\,x = 0$, it follows that all $(n - 1) \times (n - 1)$ submatrices of $A - \lambda I$ are nonsingular. Let us formulate this as

Proposition 1. *If A is $n \times n$ acyclic irreducible and all coordinates of an eigenvector of A are different from zero then the corresponding eigenvalue λ is simple. Moreover, all submatrices of order $n - 1$ of the matrix $A - \lambda I$ are nonsingular.*

Assume now that $\nu > 0$ and that the assertion is true for all acyclic matrices and an eigenvector y the number of zero coordinates of which does not exceed $\nu - 1$.

Denote, for a moment, by M_1 the subset of N corresponding to nonvanishing coordinates of y. Thus, $M_2 = N \setminus (M \cup M_1)$ where M was defined above as the set corresponding to isolated zero coordinates, contains exactly those indices k such that $y_k = 0$ and there exists an $i \neq k$ for which $a_{ik} y_i \neq 0$. Moreover,

$$\nu = |M| + |M_2| > 0 .$$

Let first A be reducible. We can assume that

$$A = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_s \end{bmatrix}$$

where $s \geqq 2$ and A_k are square irreducible matrices. Thus $G(A)$ is disconnected and contains s components G_k, $k = 1, \dots, s$, corresponding to the matrices A_k.

If there are zero coordinates of y corresponding to indices in at least two distinct components of $G(A)$ then none of the components G_k has more than $\nu - 1$ indices in $M \cup M_2$. If the vector y is partitioned conformally with A,

$$y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \\ y^{(s)} \end{bmatrix}$$

then $Ay = \lambda y$ implies

$$A_k y^{(k)} = \lambda y^{(k)} , \quad k = 1, \dots, s .$$

Let K be the subset of $S = \{1, \dots, s\}$ consisting of all indices k such that $y^{(k)} \neq 0$. If $j \in S \setminus K$, we have $y^{(j)} = 0$ and all indices corresponding to vertices in G_j belong clearly to M. For the inertiae of A_t,

$$\operatorname{In} A_t = (\omega_t^+, \omega_t^-, \omega_t^{(0)}) ,$$

we have by (1,10)

$$\operatorname{In} A = (\omega^+, \omega^-, \omega^{(0)}) = \sum_{t \in S} \operatorname{In} A_t .$$

By the induction hypothesis, (5) is true for all G_k, $k \in K$:

$$\omega_k^+ = \tilde{\omega}_k^+ + c_{k1} , \quad \omega_k^- = \tilde{\omega}_k^+ + c_{k2} , \quad \omega_k^{(0)} = \tilde{\omega}_k^{(0)} + c_{k0} ,$$

c_{kj}, $j = 0, 1, 2$, being nonnegative integers satisfying

$$c_{k0} + c_{k1} + c_{k2} = \mu_k ,$$

the number of vertices in G_k belonging to M. Since clearly

$$\tilde{\omega}^+ = \sum_{k \in K} \tilde{\omega}_k^+ , \quad \tilde{\omega}^- = \sum_{k \in K} \tilde{\omega}_k^- , \quad \tilde{\omega}^{(0)} = \sum_{k \in K} \tilde{\omega}_k^{(0)} ,$$

(5) is true for

$$c_1 = \sum_{k \in K} c_{k1} + \sum_{t \in S \setminus K} \omega_t^+ ,$$

$$c_2 = \sum_{k \in K} c_{k2} + \sum_{t \in S \setminus K} \omega_t^- ,$$

$$c_0 = \sum_{k \in K} c_{k0} + \sum_{t \in S \setminus K} \omega_t^{(0)} .$$

Thus it remains to prove (5) for the case that either A is irreducible or A contains only one block, say A_1, the set of indices of which has a non-void intersection with $M \cup M_2$. A similar argument as above shows it suffices to consider the case only that A is irreducible.

With this assumption, we shall prove two more propositions:

Proposition 2. *Each vertex of $G(A)$ belonging to M_2 possesses at least two vertices in M_1 joined to it by an edge.*

Proof. Let $k \in M_2$ and let there exist only one index $i \neq k$ such that $a_{ik} y_i \neq 0$. Since $Ay = \lambda y$, we have

$$(a_{kk} - \lambda) y_k = - \sum_{t \neq k} a_{tk} y_t ;$$

however, the left-hand side is zero while the right-hand side is not, a contradiction.

Remark. This implies that no end-vertex of $G(A)$ belongs to M_2.

Proposition 3. *Let $M_1 \neq N$. Then there exists in $G(A)$ a vertex $p \in M_2$ such that at least one component, say C, of the graph G' obtained from $G(A)$ by deleting the vertex p and all to p adjacent edges, contains only vertices belonging to M_1.*

Proof. Since $M_1 \neq N$ and $y \neq 0$, there is a vertex $u \in M_1$ and a vertex $v \notin M_1$. Since $G(A)$ is a tree, there is a path from u to v in $G(A)$. Let w be the first vertex of this path which does not belong to M_1. Clearly, $w \in M_2$. If w is the only vertex in M_2, the assertion is true. Thus, let M_2 contain at least two vertices. Let d be the maximum distance between any two vertices in M_2 where by the distance of two vertices the number of edges in the unique path in $G(A)$ between these vertices is understood. Let this (positive) distance d be realized by two vertices $p \in M_2$, $q \in M_2$. By Proposition 2, there exists a vertex r such that $a_{pr} y_r \neq 0$ and that r does not belong to the path from p to q in $G(A)$. Let C be that component of G' obtained from $G(A)$ by deleting p and to p adjacent edges, which contains r. Let us show that all vertices in C belong to M_1. From the maximality of d, C cannot contain a vertex in M_2. Assume C contains a vertex in M, say u. Then the first vertex in the path in C from u to r which does not belong to M (and such a vertex exists) necessarily belongs to M_2, a contradiction.

We are now able to complete the proof of the theorem. Without loss of generality, we can assume that n is the vertex denoted by p in Proposition 3 and that $\{1, 2, \ldots, m\}$ is the set of vertices of C. Thus

$$A = \begin{bmatrix} A_{11} & 0 & a^{(1)} \\ 0 & A_{22} & a^{(2)} \\ a^{(1)T} & a^{(2)T} & a_{nn} \end{bmatrix} \tag{7}$$

where A_{11} is an irreducible $m \times m$ matrix.

Let

$$y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ 0 \end{bmatrix}$$

be the corresponding partitioning of y. From the properties of C it follows that all coordinates of $y^{(1)}$ are different from zero. On the other hand, $Ay = \lambda y$ yields

$$(A_{11} - \lambda I_1)\, y^{(1)} = 0\,, \tag{8}$$

$$(A_{22} - \lambda I_2)\, y^{(2)} = 0 \tag{9}$$

where I_1, I_2 are identity matrices,

$$a^{(1)T} y^{(1)} + a^{(2)T} y^{(2)} = 0\,.$$

Since $a^{(1)}$ contains a single coordinate different from zero and all coordinates of $y^{(1)}$ are different from zero,

$$a^{(1)T} y^{(1)} \neq 0\,.$$

Consequently, the matrix

$$A - \lambda I = \begin{bmatrix} B & c \\ c^T & d \end{bmatrix}$$

where

$$B = \begin{bmatrix} A_{11} - \lambda I_1 & 0 \\ 0 & A_{22} - \lambda I_2 \end{bmatrix}$$

satisfies assumption of (1,12) together with the vector

$$u = \begin{bmatrix} y^{(1)} \\ 0 \end{bmatrix}.$$

By (1,12)

$$\operatorname{In}(A - \lambda I) = \operatorname{In} B + (1, 1, -1)$$

or,

(10) $$\operatorname{In}(A - \lambda I) = \operatorname{In}(A_{11} - \lambda I_1) + \operatorname{In}(A_{22} - \lambda I_2) + (1, 1, -1).$$

Now, the graph C is equal to $G(A_{11})$ and all coordinates of $y^{(1)}$ were shown to be different from zero. Consequently, the already proved case implies

(11) $$\operatorname{In}(A_{11} - \lambda I_1) = (a_1^+, a_1^-, 1).$$

Further, $G(A_{22})$ is the graph obtained from $G(A)$ by deleting the vertex n, the adjacent edges and the whole component C. Since $y^{(2)} \neq 0$, (9) implies $y^{(2)}$ is an eigenvector of A_{22} corresponding to λ which has already less zero components than v. By the induction hypothesis, the assertion of the theorem is true for A_{22}: thus, the inertia

(12) $$\operatorname{In}(A_{22} - \lambda I_2) = (\tilde{\omega}_2^+ + c_1', \tilde{\omega}_2^- + c_2', \tilde{\omega}_2^{(0)} + c_0')$$

where $\tilde{\omega}_2^+, \tilde{\omega}_2^-, \tilde{\omega}_2^{(0)}$ are given by formulae (5) for the matrix $\tilde{A}_2$ obtained from A_{22} by deleting all rows and columns with indices from M (and all elements from M are in the index set of A_{22}) and $c_1', c_2'\ c_0'$ are nonnegative integers satisfying

$$c_0' + c_1' + c_2' = |M|.$$

The assertion then follows easily from (10), (11), (12) and (4). The proof is complete.

(2,4) **Theorem.** *Let $A = (a_{ik})$ be an $n \times n$ acyclic matrix, let $y = (y_i)$ be an eigenvector of A. If there are not two indices i, k such that $a_{ik} \neq 0$ and $y_i = y_k = 0$ then the multiplicity of the corresponding eigenvalue is*

$$p + 1 + \sum_{k=3}^{n-1} (k - 2)\, s_k$$

where p is the degree of reducibility of A and s_k $(k = 3, \ldots, n - 1)$ is the number of those indices j for which $y_j = 0$ and $a_{jl} \neq 0$ for exactly k indices $l \neq j$. In other words, s_k is the number of vertices of $G(A)$ corresponding to zero coordinates of y and having degree k.

Proof. Denote by M_1 the set of vertices of $G(A)$ corresponding to non-zero coordinates of y, by M_2 the set corresponding to zero coordinates of y. Let e_1 denote the number of edges of $G(A)$ with both vertices in M_1, e_2 the number of edges with one vertex in M_1 and the other in M_2. By (1,4), the total number of edges $G(A)$ is $n - p - 1$ where p is the degree reducibility and thus $p + 1$ the number of components of $G(A)$. Since there is no edge in $G(A)$ with both vertices in M_2, we have

$$(13) \qquad e_1 + e_2 = n - p - 1 .$$

From the preceding theorem,

$$e_1 = a^+ + a^- , \quad \omega^+ = a^+ + |M_2| , \quad \omega^- = a^- + |M_2| ,$$

$$\omega^{(0)} = n - (a^+ + a^-) - 2|M_2| .$$

Since

$$|M_2| = \sum_{k=1}^{n-1} s_k , \quad e_2 = \sum_{k=1}^{n-1} k s_k$$

where s_k $(k = 1, \ldots, n - 1)$ is the number of vertices in M_2 of degree k in $G(A)$, we obtain, by (13),

$$\omega^{(0)} = n - e_1 - 2|M_2| = p + 1 + e_2 - 2|M_2| =$$

$$= p + 1 + \sum_{k=1}^{n-1} (k - 2)\, s_k .$$

However, $s_1 = 0$ since $G(A)$ can have (as in Proposition 2 of the proof of the preceding theorem) no end-vertex in M_2. The proof is complete.

In the conclusion, we shall formulate three corollaries which follow easily from both preceding theorems.

(2,5) **Corollary.** *Let $A = (a_{ik})$ be an $n \times n$ irreducible acyclic matrix with eigenvalues $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n$. If λ_r corresponds to an eigenvector $y = (y_i)$ with all coordinates different from zero then λ_r is simple and there are exactly $r - 1$ (unordered) pairs (i, k), $i \neq k$, for which*

$$a_{ik} y_i y_k < 0 .$$

Remark. Since A can be brought to an essentially nonnegative matrix by a diagonal orthogonal similarity by Theorem (2,1), this corollary is a generalization of the Perron-Frobenius theorem (for this particular case).

(2,6) **Corollary.** *Any eigenvector corresponding to a multiple eigenvalue of an acyclic irreducible matrix has at least one vanishing coordinate.*

R e m a r k. There exists even such a vanishing coordinate which corresponds to a vertex of degree at least three in the graph of the matrix. This implies that an irreducible symmetric tridiagonal matrix can have only simple eigenvalues.

(2,7) **Corollary.** *Let λ be a simple eigenvalue of an acyclic irreducible matrix A and $y = (y_i)$ a corresponding eigenvalue of A.*

If there are no two indices i, k such that $a_{ik} \neq 0$ and $y_i = y_k = 0$ then y_i can vanish only if i is of degree two in $G(A)$.

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