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TORSION THEORY FOR LATTICE-ORDERED GROUPS

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0. NOTATIONAL CONVENTION AND PRIMITIVE TERMS

The terminology from the theory of lattice-ordered groups (henceforth l-groups) is standard (and mostly that of [4]). All our classes are assumed to contain along with a given l-group all its isomorphic copies.

A *Brouwer lattice* L is one in which the set $\{x \in L \mid x \wedge a \leqq b\}$ has a unique largest element for all $a, b \in L$. If L is complete this is equivalent to saying that $a \wedge (\bigvee_\lambda b_\lambda) = \bigvee_\lambda (a \wedge b_\lambda)$ for all $a, b_\lambda \in L (\lambda \in \Lambda)$; see [1].

For subsets A and B of the set X $(A \subset B)$ $A \subseteq B$ stands for (proper) containment. The complement of B in A is denoted $A \setminus B$.

1. ARCHIMEDEAN-LIKE CLASSES REVISITED

The starting point of this work is found in [6] where the author studied classes of l-groups with closure properties resembling the classes of archimedean and hyper-archimedean l-groups. The motivation there was obvious but the development was still rather artificial and incomplete. The latter was especially true for two reasons: first, it was restricted to representable l-groups, and therefore did not handle certain classes with similar closure properties, and secondly, it clearly did not tell the story fully on these classes from a universal standpoint.

To review then, one started from a class of o-groups (totally ordered groups) $\mathscr{C}$ closed under taking subgroups and o-homomorphic images; one then formed Res $(\mathscr{C})$, the class of all subdirect products of groups in $\mathscr{C}$, and Hyp $(\mathscr{C})$, the class of all l-groups in Res $(\mathscr{C})$ having all their l-homomorphic images in Res $(\mathscr{C})$. It is Hyp $(\mathscr{C})$ that will concern us in this work, so we shall isolate on it right away.

Hyp $(\mathscr{C})$ is always closed under quotients, and if $\mathscr{C}$ is closed under unions of convex subgroups, Hyp $(\mathscr{C})$ is closed under joins of convex l-subgroups. It was stated in [6], proposition 1.5(i) that Hyp $(\mathscr{C})$ is always closed under l-subgroups; the proof given there only shows it to be closed under l-ideals. It may well be that a stronger closure condition is possible, but the author is unable to establish this eitherway.

So even as we (probably) lose some of the Hyp ($\mathscr{C}$) classes of [6] we shall zero in on classes of l-groups closed under taking 1) convex l-subgroups, 2) l-homomorphic images and 3) joins of convex l-subgroups in the class. Such a class $\mathscr{T}$ will be called a *torsion class*. This terminology makes sense, if one thinks of torsion theories on modules over a ring.

Examples of torsion classes abound: all the Hyp ($\mathscr{C}$) classes [6] are torsion classes, provided $\mathscr{C}$ is closed under joins of convex subgroups and Hyp ($\mathscr{C}$) is closed under convex l-subgroups. Among these are all the varieties (equationally closed) of representable l-groups; see [7]. We shall list some that are not of this type:

a) The class of finite valued l-groups.

b) The class of l-groups with property (F): each positive element exceeds at most finitely many pairwise disjoint elements.

c) The class of cardinal sums of o-groups.

d) The class of cardinal sums of copies of $\boldsymbol{R}$, the group of additive reals with the usual order.

e) Ditto, with $\boldsymbol{Z}$, the group of additive integers with the usual order.

f) The class of S-groups: l-groups generated by their singular elements. (An element $0 < s$ is *singular* if $0 \leqq g \leqq s$ implies that $g \wedge (s - g) = 0$.)

g) The class of (real) vector lattices.

h) The class of divisible abelian l-groups.

Note: the classes in c), f), g) and h) are not closed under all l-subgroups.

So let $\mathscr{T}$ be a torsion class and G be an l-group. Let $\mathscr{T}(G)$ be the join of all the convex l-subgroups of G belonging to $\mathscr{T}$. $\mathscr{T}(G)$ is invariant under each l-automorphism of G, ie. $\mathscr{T}(G)$ is a characteristic l-ideal of G. We call $\mathscr{T}(G)$ the *$\mathscr{T}$-torsion radical* of G.

The following proposition outlines the basic properties of the radical $\mathscr{T}$.

1.1. Proposition. *Let $\mathscr{T}$ be a torsion class and G be an l-group.*

a) *If A is a convex l-subgroup of G $\mathscr{T}(A) = A \cap \mathscr{T}(G)$.*

b) *If $\phi : G \to H$ is an onto l-homomorphism, then $[\mathscr{T}(G)]\, \phi \subseteq \mathscr{T}(H)$.*

c) $\mathscr{T}(\mathscr{T}(G)) = \mathscr{T}(G)$.

d) *If $\{A_\lambda \mid \lambda \in \Lambda\}$ is a family of convex l-subgroups of G then $\mathscr{T}(\bigvee_\lambda A_\lambda) = \bigvee_\lambda \mathscr{T}(A_\lambda)$.*

Proof. a) $A \cap \mathscr{T}(G)$ is a convex l-subgroup of $\mathscr{T}(G)$, so that $A \cap \mathscr{T}(G) \in \mathscr{T}$; hence $A \cap \mathscr{T}(G) \subseteq \mathscr{T}(A)$. On the other hand $\mathscr{T}(A)$ is a convex l-subgroup of G belonging to $\mathscr{T}$, implying that $\mathscr{T}(A) \subseteq A \cap \mathscr{T}(G)$.

b) Since $\mathscr{T}$ is quotient closed and $[\mathscr{T}(G)]\, \phi$ is an l-ideal of H the result follows.

c) Obvious.

d) Using the fact that the lattice of convex l-subgroups is Brouwer we have $\mathscr{T}(\bigvee_\lambda A_\lambda) = (\bigvee_\lambda A_\lambda) \cap \mathscr{T}(G) = \bigvee_\lambda (A_\lambda \cap \mathscr{T}(G)) = \bigvee_\lambda \mathscr{T}(A_\lambda)$.

(The reader should note that c) and d) follow from a).)

We need a converse to 1.1; ie. we should be able to pick up a torsion class from a radical. Thus:

1.2. Proposition. *Suppose we assign to each l-group G an l-ideal $\overline{\mathscr{T}}(G)$ subject to a) and b) in proposition* 1.1. *Let $\mathscr{T} = \{G \mid \overline{\mathscr{T}}(G) = G\}$; then $\mathscr{T}$ is a torsion class, and $\overline{\mathscr{T}}(G)$ is the $\mathscr{T}$-torsion radical of G, for each l-group G.*

Proof. As we have remarked c) in proposition 1.1 is a consequence of a). Now, suppose $G \in \mathscr{T}$ and A is a convex l-subgroup of G; $\overline{\mathscr{T}}(A) = A \cap \overline{\mathscr{T}}(G) = A \cap G = A$, so $A \in \mathscr{T}$, and $\mathscr{T}$ is closed under convex l-subgroups. b) obviously implies that $\mathscr{T}$ is quotient closed. To establish the remaining closure property it suffices to prove that whenever C is a convex l-subgroup of the l-group G, then $C \in \mathscr{T}$ if and only if $C \subseteq \overline{\mathscr{T}}(G)$. Now, $C \in \mathscr{T}$ if and only if $C = \overline{\mathscr{T}}(C)$; ie. if and only if $C = C \cap \overline{\mathscr{T}}(G)$, by a). This proves what we want.

Thus we've shown that $\mathscr{T}$ is a torsion class; let us now look at its $\mathscr{T}$-torsion radical. By c) $\overline{\mathscr{T}}(G) \in \mathscr{T}$, so we have immediately that $\overline{\mathscr{T}}(G) \subseteq \mathscr{T}(G)$. Moreover, $\mathscr{T}(G) \in \mathscr{T}$ so $\mathscr{T}(G) = \overline{\mathscr{T}}(\mathscr{T}(G)) = \mathscr{T}(G) \cap \overline{\mathscr{T}}(G)$; consequently $\mathscr{T}(G) \subseteq \overline{\mathscr{T}}(G)$ and hence equality follows.

Summarizing then: there is a one-to-one correspondence between torsion classes and *torsion radicals* (functions satisfying a) and b) in proposition 1.1), and we shall view matters from either point of view, as dictated by convenieince.

Let us indicate two further properties of torsion radicals.

1.3. Proposition. *Suppose $\mathscr{T}$ is a torsion radical.*

a) *If $\{A_\lambda \mid \lambda \in \Lambda\}$ is a family of convex l-subgroups of the l-group G, then $\mathscr{T}(\bigcap_\lambda A_\lambda) = \bigcap_\lambda \mathscr{T}(A_\lambda)$.*

b) *If $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of l-groups, then $\mathscr{T}(\boxplus_\lambda G_\lambda) = \boxplus_\lambda \mathscr{T}(G_\lambda)$.*

(Note: $\boxplus_\lambda G_\lambda$ denotes the cardinal sum of the G_λ.)

Proof. The letter $\mathscr{T}$ identifies the radical as well as the torsion class.

a) $\mathscr{T}(\bigcap_\lambda A_\lambda) \in \mathscr{T}$ and a convex l-subgroup of each A_λ. Hence, $\mathscr{T}(\bigcap_\lambda A_\lambda) \subseteq \bigcap_\lambda \mathscr{T}(A_\lambda)$. On the other hand, $\bigcap_\lambda \mathscr{T}(A_\lambda)$ is an l-ideal of each $\mathscr{T}(A_\lambda)$; hence $\bigcap_\lambda \mathscr{T}(A_\lambda) \in \mathscr{T}$. It is also an l-ideal of $\bigcap_\lambda A_\lambda$, so that $\bigcap_\lambda \mathscr{T}(A_\lambda) \subseteq \mathscr{T}(\bigcap_\lambda A_\lambda)$.

b) The sum of the $\mathscr{T}(G_\lambda)$ is indeed a cardinal sum by part a); the equality follows from d) in proposition 1.1.

There are easy examples of torsion classes which are not *closed under extensions*: that is, if $\mathscr{T}$ is a torsion class and G is an l-group with an l-ideal $A \in \mathscr{T}$ such that

$G/A \in \mathscr{T}$, then G may not be in $\mathscr{T}$. If a torsion class does have this closure with respects to extensions we shall say it is a *complete* torsion class. (Notice that the intersection of complete torsion classes is again complete.) Below we describe how to manufacture complete torsion classes from given non-complete classes.

First we spend some time developing a binary operation on the class $\boldsymbol{T}$ of all torsion classes. Let $\mathscr{T}$ and $\mathscr{U}$ be torsion classes and define $\mathscr{T} . \mathscr{U}$ as follows: $G \in \mathscr{T} . \mathscr{U}$ if G has an l-ideal $A \in \mathscr{U}$ such that $G/A \in \mathscr{T}$. Let us verify that $\mathscr{T} . \mathscr{U}$ is indeed a torsion class; instead let's find its corresponding radical. For a given l-group G let $\mathscr{X}(G)$ be the unique l-ideal for which $\mathscr{X}(G)/\mathscr{U}(G) = \mathscr{T}(G/\mathscr{U}(G))$; it should be obvious that $G \in \mathscr{T} . \mathscr{U}$ if and only if $\mathscr{X}(G) = G$.

Suppose A is a convex l-subgroup of G; to show that $\mathscr{X}(A) = A \cap \mathscr{X}(G)$ we prove that $(A \cap \mathscr{X}(G))/\mathscr{U}(A) = \mathscr{T}(A/\mathscr{U}(A))$.

$$(A \cap \mathscr{X}(G))/\mathscr{U}(A) = (A \cap \mathscr{X}(G))/(A \cap \mathscr{U}(G)) \cong [A \cap \mathscr{X}(G)] \vee \mathscr{U}(G)/\mathscr{U}(G) =$$
$$= [A \vee \mathscr{U}(G)] \cap \mathscr{X}(G)/\mathscr{U}(G) = A \vee \mathscr{U}(G)/\mathscr{U}(G) \cap \mathscr{X}(G)/\mathscr{U}(G) =$$
$$= A \vee \mathscr{U}(G)/\mathscr{U}(G) \cap \mathscr{T}(G/\mathscr{U}(G)) =$$
$$= \mathscr{T}(A \vee \mathscr{U}(G)/\mathscr{U}(G)) \cong \mathscr{T}(A/A \cap \mathscr{U}(G)) = \mathscr{T}(A/\mathscr{U}(A))\,.$$

As for condition b) in proposition 1.1, it suffices to establish that $\mathscr{T} . \mathscr{U}$ is quotient closed – and this is trivial. Once this is done, suppose $\phi : G \to H$ is an l-epimorphism; $[\mathscr{X}(G)]\,\phi$ is an l-ideal of H and is in $\mathscr{T} . \mathscr{U}$, so $[\mathscr{X}(G)]\,\phi = \mathscr{X}([\mathscr{X}(G)]\,\phi) = [\mathscr{X}(G)]\,\phi \cap \mathscr{X}(H)$, ie. $[\mathscr{X}(G)]\,\phi \subseteq \mathscr{X}(H)$.

By proposition 1.2 $\mathscr{T} . \mathscr{U}$ is a torsion class with $\mathscr{X}$ as torsion radical. We summarize this as follows.

1.4. Proposition. *$\mathscr{T} . \mathscr{U}$ is a torsion class whenever $\mathscr{T}$ and $\mathscr{U}$ are; if G is an l-group, the torsion radical $\mathscr{T} . \mathscr{U}(G)$ is defined by the equation: $\mathscr{T} . \mathscr{U}(G)/\mathscr{U}(G) = \mathscr{T}(G/\mathscr{U}(G))$.*

The proof of the next proposition is straightforward, and will be omitted.

1.5. Proposition. *The binary operation . defined on torsion classes is associative.*

Let $\mathscr{T}$ be a torsion class and σ be an ordinal number. We wish to define an ascending: $\mathscr{T}, \mathscr{T}^2, \ldots, \mathscr{T}^\sigma, \ldots$ of torsion classes. Suppose $\mathscr{T}^\alpha$ has been defined for each ordinal $\alpha < \sigma$; if σ is not a limit ordinal define $\mathscr{T}^\sigma = \mathscr{T} . \mathscr{T}^{\sigma-1}$. If σ is a limit ordinal $\mathscr{T}^\sigma$ is defined as $\{G \mid \bigcup_{\alpha<\sigma} \mathscr{T}^\alpha(G) = G\}$; we verify that in this instance $\mathscr{T}^\sigma$ is a torsion class.

Define (for an l-group G) $\mathscr{T}^\sigma(G) = \bigcup_{\alpha<\sigma} \mathscr{T}^\alpha(G)$; then $\mathscr{T}^\sigma$ is a torsion radical: for if A is a convex l-subgroup of G $\mathscr{T}^\sigma(A) = \bigcup_{\alpha<\sigma} \mathscr{T}^\alpha(A) = \bigcup_{\alpha<\sigma} [A \cap \mathscr{T}^\alpha(G)] = A \cap$

$\cap [\bigcup_{\alpha<\sigma} \mathscr{T}^{\alpha}(G)] = A \cap \mathscr{T}^{\sigma}(G)$. Next, if $\phi : G \to H$ is an l-epimorphism $[\mathscr{T}^{\sigma}(G)]\, \phi = = \bigcup_{\alpha<\sigma} [\mathscr{T}^{\alpha}(G)]\, \phi \subseteq \bigcup_{\alpha<\sigma} \mathscr{T}^{\alpha}(H) = \mathscr{T}^{\sigma}(H)$. This shows that for a limit ordinal σ $\mathscr{T}^{\sigma}$ is a torsion class with the indicated torsion radical. We have therefore proved most of the following.

1.6. Theorem. *Let $\mathscr{T}$ be a torsion class; there is an ascending sequence of torsion classes $\{\mathscr{T}^{\sigma}\}$, σ an ordinal, such that $\mathscr{T}^{\sigma} = \mathscr{T} . \mathscr{T}^{\sigma-1}$ if σ is not a limit ordinal, and $\mathscr{T}^{\sigma}(G) = \bigcup_{\alpha<\sigma} \mathscr{T}^{\alpha}(G)$ for each l-group G, if σ is a limit ordinal.*

Let $\mathscr{T}^{*} = \bigcup_{\sigma} \mathscr{T}^{\sigma}$; then $\mathscr{T}^{*}$ is a complete torsion class; it is the smallest complete torsion class containing $\mathscr{T}$.

Proof. We turn directly to $\mathscr{T}^{*}$, identifying its associated radical. Let G be an l-group; by a simple cardinality argument there is an ordinal σ (depending on G) such that $\mathscr{T}^{\sigma}(G) = \mathscr{T}^{\sigma+1}(G) = \ldots$. So we define $\mathscr{T}^{*}(G) = \mathscr{T}^{\sigma}(G)$; clearly $G \in \mathscr{T}^{*}$ if and only if $\mathscr{T}^{*}(G) = G$. We leave it to the reaster to verify the condition in proposition 1.1.

An easy transfinite induction argument will show that $\mathscr{T}^{\sigma} . \mathscr{T}^{*} = \mathscr{T}^{*}$; from this it is clear that $\mathscr{T}^{*}$ is complete. If $\mathscr{U}$ is a complete torsion class containing $\mathscr{T}$ then by another induction approach $\mathscr{T}^{\sigma} \subseteq \mathscr{U}$ for each ordinal σ. Thus $\mathscr{T}^{*} \subseteq \mathscr{U}$ as claimed.

If $\mathscr{T}$ is a torsion class $\mathscr{T}^{*}$ is the *completion* of $\mathscr{T}$. Notice that $\mathscr{T}(G) = 0$ if and only if $\mathscr{T}^{*}(G) = 0$. Further, $\mathscr{T}$ is complete if and only if $\mathscr{T}^{*} = \mathscr{T}$. The sequence $\mathscr{T}, \mathscr{T}^{2}, \ldots, \mathscr{T}^{\sigma}, \ldots$ is called the *Loewy-sequence associated with* $\mathscr{T}$.

The final theorem of this section has some interesting consequences.

1.7. Theorem. *Let $\mathscr{T}$ be a torsion class, and G be an l-group. Then $\mathscr{T}^{*}(G) \subseteq \subseteq \mathscr{T}(G)''$.*

Proof. It suffices to show that $\mathscr{T}^{\sigma}(G) \cap \mathscr{T}(G)' = 0$ for each ordinal σ; this of course proceeds by induction. The case $\sigma = 1$ is trivial; now suppose $\mathscr{T}^{\alpha}(G) \cap \cap \mathscr{T}(G)' = 0$ for all ordinals $\alpha < \sigma$. If σ is a limit ordinal it follows immediately that $\mathscr{T}^{\sigma}(G) \cap \mathscr{T}(G)' = 0$.

If σ has a predecessor $\mathscr{T}^{\sigma-1}(\mathscr{T}(G)') = 0$, since $\mathscr{T}^{\sigma-1}(G)$ is the largest $\mathscr{T}^{\sigma-1}$ convex l-subgroup of G and $\mathscr{T}^{\sigma-1}(G) \cap \mathscr{T}(G)' = 0$. Thus $\mathscr{T}^{\sigma}(G) \cap \mathscr{T}(G)'$ is in $\mathscr{T}^{\sigma}$ and $\mathscr{T}^{\sigma-1}(\mathscr{T}^{\sigma}(G) \cap \mathscr{T}(G)') = 0$, forcing $\mathscr{T}^{\sigma}(G) \cap \mathscr{T}(G)'$ to be in $\mathscr{T} \subseteq \mathscr{T}^{\sigma-1}$. From this it is clear that $\mathscr{T}^{\sigma}(G) \cap \mathscr{T}(G)' = 0$, and we're done.

Recall that an l-subgroup H of G is *dense* in G if for each $0 < g \in G$ there is an $h \in H$ such that $0 < h \leqq g$. A convex l-subgroup C of G is dense in G if and only if $C'' = G$; (see [4].)

1.7.1. Corollary. *Let $\mathscr{T}$ be a torsion class. If G is an l-group having a finite sequence of l-ideals $0 = K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots \subseteq K_r = G$ such that $K_{i+1}/K_i \in \mathscr{T}$ for each $i = 0, 1, \ldots, r - 1$ then G has a dense l-ideal in $\mathscr{T}$.*

1.7.2. Corollary. *If G is an l-group which is the union of an ascending sequence $0 = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n \subseteq \ldots$ of l-ideals with the property that each quotient A_{n+1}/A_n has property* (F), *then G has a basis.*

Proof. The class of l-groups with property (F) is a torsion class. By theorem 1.7 G has a dense l-ideal with property (F), call it A; then if $0 < g \in G$, there is an $a \in A$ such that $0 < a \leqq g$. But l-groups with property (F) have a basis (see [4]), hence a exceeds a basic element of A (and hence of G), which establishes that G has a basis.

1.7.3. Corollary. *If G is an extension of a representable l-group by another, then G has a dense representable l-ideal. The same statement holds if one replaces "representable" by "abelian".*

2. THE LATTICE OF TORSION CLASSES

The variety of all l-groups is clearly a torsion class, and it is easily verified that the intersection of a family of torsion classes is a torsion class. This is enough to make the class $\boldsymbol{T}$ of torsion classes into a complete lattice. But we can describe the join operation explicitly; let $\mathscr{T}$ and $\mathscr{U}$ be torsion classes, and define a radical as follows: $\mathscr{X}(G) = \mathscr{T}(G) + \mathscr{U}(G)$. It is straightforward to verify that $\mathscr{X}$ defines a torsion radical; we claim that its corresponding torsion class is $\mathscr{T} \vee \mathscr{U}$. Par abuse de language, let $\mathscr{X}$ denote the torsion class associated with the radical $\mathscr{X}$; thus $G \in \mathscr{X}$ if and only if $G = \mathscr{T}(G) + \mathscr{U}(G)$. If $\mathscr{V}$ is a torsion class containing $\mathscr{T}$ and $\mathscr{U}$ and $G \in \mathscr{X}$ then $\mathscr{V}(G) = \mathscr{V}(\mathscr{T}(G)) + \mathscr{V}(\mathscr{U}(G) = \mathscr{T}(G) + \mathscr{U}(G) = G$, ie. $G \in \mathscr{V}$. It follows that $\mathscr{X} = \mathscr{T} \vee \mathscr{U}$.

If $\{\mathscr{T}_\lambda \mid \lambda \in \Lambda\}$ is any family of torsion classes it is easy to extend the above argument to show that $\mathscr{T} = \bigvee_\lambda \mathscr{T}_\lambda$ if and only if $\mathscr{T}$ is the torsion class of all l-groups G which are the join of their $\mathscr{T}_\lambda$-torsion radicals. In general $\mathscr{T}(H) = \Sigma_\lambda \mathscr{T}_\lambda(H)$.

This discussion makes the next result transparent.

2.1. Theorem. *$\boldsymbol{T}$ is a complete, Brouwer lattice.*

The following result relates the composition of the previous section to the lattice operations on $\boldsymbol{T}$.

2.2. Proposition. *Let $\mathscr{U}$ and $\{\mathscr{T}_\lambda \mid \lambda \in \Lambda\}$ be torsion classes.*

a) $\bigcap_\lambda \mathscr{T}_\lambda \,.\, \mathscr{U} = (\bigcap_\lambda \mathscr{T}_\lambda) \,.\, \mathscr{U}$ *and* $\mathscr{U} \,.\, (\bigcap_\lambda \mathscr{T}_\lambda) \subseteq \bigcap_\lambda \mathscr{U} \,.\, \mathscr{T}_\lambda$.

b) $\bigvee_\lambda \mathscr{T}_\lambda \,.\, \mathscr{U} = (\bigvee_\lambda \mathscr{T}_\lambda) \,.\, \mathscr{U}$ *and* $\bigvee_\lambda \mathscr{U} \,.\, \mathscr{T}_\lambda \subseteq \mathscr{U} \,.\, (\bigvee_\lambda \mathscr{T}_\lambda)$.

Proof. We'll prove the equation in b) and give an example where equalit fails in the containment of b).

Let G be an l-group; it is sufficient to verify: $(\bigvee_\lambda \mathcal{T}_\lambda \, . \, \mathcal{U})(G)/\mathcal{U}(G) = [(\bigvee_\lambda \mathcal{T}_\lambda) \, . \, \mathcal{U}](G)/\mathcal{U}(G)$. Now,

$$[(\bigvee_\lambda \mathcal{T}_\lambda) \, . \, \mathcal{U}](G)/\mathcal{U}(G) = (\bigvee_\lambda \mathcal{T}_\lambda)(G/\mathcal{U}(G)) = \Sigma_\lambda \mathcal{T}_\lambda(G/\mathcal{U}(G)) =$$

$$= \Sigma_\lambda \mathcal{T}_\lambda \, . \, \mathcal{U}(G)/\mathcal{U}(G) = (\bigvee_\lambda \mathcal{T}_\lambda \, . \, \mathcal{U})(G)/\mathcal{U}(G) \, .$$

To get the example, let $\mathcal{T}_{\mathbf{Z}}$ and $\mathcal{T}_{\mathbf{R}}$ be the torsion classes of cardinal sums of copies of $\mathbf{Z}$ and $\mathbf{R}$ respectively. Then $\mathcal{T}_{\mathbf{Z}} \cap \mathcal{T}_{\mathbf{R}} = 0$, (the smallest torsion class). Let G be the direct lexicographic extension of $\mathbf{Z} \boxplus \mathbf{R}$ by $\mathbf{Z}$; then $\mathcal{T}_{\mathbf{Z}}(G) = \mathbf{Z} \boxplus 0$, $\mathcal{T}_{\mathbf{R}}(G) = 0 \boxplus \mathbf{R}$ and $G \in \mathcal{T}_{\mathbf{Z}} \, . \, (\mathcal{T}_{\mathbf{Z}} \vee \mathcal{T}_{\mathbf{R}})$. However, $\mathcal{T}_{\mathbf{Z}} \, . \, \mathcal{T}_{\mathbf{Z}}(G) = \mathcal{T}_{\mathbf{Z}}(G)$ while $\mathcal{T}_{\mathbf{Z}} \, . \, \mathcal{T}_{\mathbf{R}}(G) = \mathbf{Z} \boxplus \mathbf{R}$, so $G \notin \mathcal{T}_{\mathbf{Z}} \, . \, \mathcal{T}_{\mathbf{Z}} \vee \mathcal{T}_{\mathbf{Z}} \, . \, \mathcal{T}_{\mathbf{R}}$.

The author assumes that it is possible to find just as simple a counter example to equality in the second part of a). The equality does hold if $\mathcal{U}$ is closed under l-subgroups.

As $\mathcal{T}$ is a complete, Brouwer lattice there is a natural polarization: given $\mathcal{T} \in \mathbf{T}$ there is a unique largest torsion class $\mathcal{T}'$ whose intersection with $\mathcal{T}$ is trivial: $\mathcal{T}' = \bigvee\{\mathcal{U} \in \mathbf{T} \mid \mathcal{U} \cap \mathcal{T} = 0\}$. Let us try and describe $\mathcal{T}'$ a bit more carefully.

2.3. Theorem. *Let $\mathcal{T}$ be a torsion class, and $\mathcal{X} = \{G \mid$ if C is a convex l-subgroup of G and H is a quotient of C then $\mathcal{T}(H) = 0\}$. Then $\mathcal{X} = \mathcal{T}'$.*

Proof. Before proving that $\mathcal{X}$ is a torsion class, let's establish that if it is, it's the right thing. For suppose $\mathcal{U}$ is a torsion class so that $\mathcal{U} \cap \mathcal{T} = 0$, and $\phi : C \to H$ is an l-epimorphism, where C is a convex l-subgroup of the l-group $G \in \mathcal{U}$. $C \in \mathcal{U}$ whence H is also, so that $\mathcal{T}(H) = 0$, that is, $G \in \mathcal{X}$ and $\mathcal{U} \subseteq \mathcal{X}$.

Now to show $\mathcal{X}$ is a torsion class: it is trivially closed under convex l-subgroups. So suppose $G \in \mathcal{X}$ and H is an l-homomorphic image of G, say by $\phi : G \to H$. Let C be a convex l-subgroup of H and $\theta : C \to K$ be an l-epimorphism; let D be the inverse image of C by ϕ. Then K is a quotient of D and hence $\mathcal{T}(K) = 0$, ie. $H \in \mathcal{X}$.

Finally, suppose G is an l-group so that $G = \bigvee_\lambda G_\lambda$, convex l-subgroups of G in $\mathcal{X}$. Pick a convex l-subgroup C of G and let $\phi : C \to K$ be an l-epimorphism. $C = \bigvee_\lambda(G_\lambda \cap C)$ and $K = \bigvee_\lambda[G_\lambda \cap C]\,\phi$, while for each $\gamma \mathcal{T}([G_\lambda \cap C]\,\phi) = 0$. Thus $\mathcal{T}(K) = 0$ and hence $G \in \mathcal{X}$.

Note that for each torsion class $\mathcal{T}$, $\mathcal{T} \subseteq \mathcal{T}''$; we say that $\mathcal{T}$ is a *polar torsion class* if $\mathcal{T}'' = \mathcal{T}$. The polar torsion classes form a complete Boolean lattice under inclusion (see [1]), in which meets agree with those of $\mathbf{T}$ but joins need not.

The next theorem parallels theorem 1.7.

2.4. Theorem. *For any torsion class $\mathcal{T}$, $\mathcal{T}^* \subseteq \mathcal{T}''$; thus, if $\mathcal{T}$ is a polar torsion class it is complete.*

Proof. We proceed by induction: ie. suppose $\mathscr{T}^{\alpha} \cap \mathscr{T}' = 0$ for each ordinal $\alpha < \sigma$. If σ is a limit ordinal and $G \in \mathscr{T}^{\sigma} \cap \mathscr{T}'$, then $G = \bigcup_{\alpha<\sigma} \mathscr{T}^{\alpha}(G)$ and each $\mathscr{T}^{\alpha}(G) \in \mathscr{T}'$, whence $\mathscr{T}^{\alpha}(G) = 0$, for all $\alpha < \sigma$; it follows that $G = 0$ and $\mathscr{T}^{\sigma} \cap \mathscr{T}' = 0$.

If σ is not a limit ordinal and $G \in \mathscr{T}^{\sigma} \cap \mathscr{T}'$ then G has an l-ideal $A \in \mathscr{T}^{\sigma-1}$ such that $G/A \in \mathscr{T}$. As both A and G/A are in $\mathscr{T}'$ we conclude that $A = 0$ and $G/A = 0$, ie. $G = 0$. Thus $\mathscr{T}^{\sigma} \cap \mathscr{T}' = 0$, the induction is complete, and $\mathscr{T}^* \subseteq \mathscr{T}''$. The last statement now follows trivially.

We now seek to give a more precise description of the polar torsion classes. It will be helpful to make a definition first: if G is an l-group and $A \subset B$ are convex l-subgroups, we call the pair (A, B) a *normal interval* if A is an l-ideal of B. If (A, B) and (C, D) are normal intervals for G we write $(A, B) \leqq (C, D)$ if $C \subseteq A$ and $B \subseteq D$; we say that (C, D) *exceeds* (A, B). If $\mathscr{T}$ is a torsion class and (A, B) is a normal interval in an l-group G so that $B/A \in \mathscr{T}$, we call (A, B) a *normal* $\mathscr{T}$*-interval.*

2.5. Theorem. *Let $\mathscr{T}$ be a torsion class and $\mathscr{X}$ be the class of l-groups with the property that each normal interval of G exceeds a normal $\mathscr{T}$-interval. Then $\mathscr{X} = \mathscr{T}''$; thus $\mathscr{T}$ is a polar torsion class if and only if for each $G \in \mathscr{T}$ and each $0 < x \in G$ there is an element $0 < y \leqq nx$ (for a suitable integer n) so that $G(y)$ has a non-trivial quotient in $\mathscr{T}$.*

(Note: $G(y)$ denotes the convex l-subgroup generated by y.)

Proof. According to theorem 2.3 $G \in \mathscr{T}''$ if and only if $\mathscr{T}'(H) = 0$ for each quotient of a convex l-subgroup of G. In general, $\mathscr{T}'(G)$ is the largest convex l-subgroup with the property that if C is a convex l-subgroup and (A, C) is a normal interval, then $\mathscr{T}(C/A) = 0$. So, to say that $\mathscr{T}'(G) = 0$ is to say that each convex l-subgroup C of G has a normal interval (A, D) so that $\mathscr{T}(D/A) \neq 0$. By shrinking D if necessary we can decide that $\mathscr{T}'(G) = 0$ if and only if each convex l-subgroup of G contains a normal $\mathscr{T}$-interval.

So now we see that $G \in \mathscr{T}''$ if and only if for each normal interval (A, C) of G $\mathscr{T}'(C/A) = 0$, ie. for each convex l-subgroup D of C containing A, D/A contains a normal $\mathscr{T}$-interval. The intermediate convex l-subgroup D can clearly be eliminated to give finally; $G \in \mathscr{T}''$ if and only if each normal interval (A, C) of G exceeds a normal $\mathscr{T}$-interval, ie. $G \in \mathscr{X}$. This proves the theorem.

Suppose 1 denotes the variety of all l-groups; a reasonable question here would be: is $1 = \mathscr{T} \vee \mathscr{T}'$ for each polar torsion class $\mathscr{T}$? (If the answer were yes then the Boolean lattice of polar torsion classes would be a sublattice of $\boldsymbol{T}$; see [1].) We show that this question has a negative answer; for now we only give an example where the equation fails, but later we shall see that it always does, unless $\mathscr{T} = 0$.

Let $\mathscr{T}_{\boldsymbol{Z}}$ (resp. $\mathscr{T}_{\boldsymbol{R}}$) once more denote the torsion class of cardinal sums of copies of $\boldsymbol{Z}$ (resp. $\boldsymbol{R}$). Then $\mathscr{T}_{\boldsymbol{Z}} \cap \mathscr{T}_{\boldsymbol{R}} = 0$, ie. $\mathscr{T}_{\boldsymbol{R}} \subseteq \mathscr{T}'_{\boldsymbol{Z}}$. So consider the group of all sequences $(s_1, s_2, \ldots)$ with $s_n \in \boldsymbol{R}$ and integral if n is odd; the order is lexicographic from left to right. Let $\mathcal{T} = \mathcal{T}''_{\mathbf{Z}}$; if $G \in \mathcal{T} \vee \mathcal{T}'$ then since G is an o-group $G \in \mathcal{T}$ or $G \in \mathcal{T}'$. However $G \notin \mathcal{T}$ since the second coordinate of the sequences provides a quotient of G whose $\mathcal{T}$-radical is zero. On the other hand, the first coordinate gives a quotient of G in $\mathcal{T}_{\mathbf{Z}}$, so that $G \notin \mathcal{T}'$. This is a contradiction, $G \notin \mathcal{T} \vee \mathcal{T}'$ and $\mathcal{T} \vee \mathcal{T}' \subset 1$.

In [6] we introduced the notion of a hyper c-archimedean l-group: G is *hyper c-archimedean* if each convex l-subgroup of G is normal in G. By theorem 1.2 in [6], G is hyper c-archimedean if and only if for each $0 < x \in G$ and $g \in G$ there is a positive integer n so that $x^g \leqq nx$. For reasons that will be clear presently, let $1°$ denote this class of hyper c-archimedean l-groups. It was shown in [6], (and it is easy to prove directly), that $1°$ is a torsion class which is closed under all l-subgroups. Let $\mathbf{T}°$ denote the class of torsion classes contained in $1°$; $\mathbf{T}°$ is a complete sublattice of $\mathbf{T}$, and is therefore a Brouwer lattice whose natural polarization is the one induced from $\mathbf{T}$: thus, if $\mathcal{T} \in \mathbf{T}°$ let $\mathcal{T}^{\pi}$ stand for the polar of $\mathcal{T}$ in $\mathbf{T}°$; then $\mathcal{T}^{\pi} = \mathcal{T}' \cap 1°$ and $\mathcal{T}^{\pi\pi} = \mathcal{T}'' \cap 1°$. Relative to $\mathbf{T}°$ we speak of *$\mathbf{T}°$-polar* torsion classes. In this context the $\mathbf{T}°$-polar torsion classes are easy to classify.

2.6. Theorem. *Let $\mathcal{R}$ be a family of subgroups of $\mathbf{R}$ and $\mathbf{X}(\mathcal{R}) = \mathcal{X}$ denote the class of all hyper c-archimedean l-groups G with the property that if N is a regular subgroup of G and $\overline{N}$ is its cover, then $\overline{N}/N \in \mathcal{R}$. Then $\mathcal{X}$ is a $\mathbf{T}°$-polar torsion class.*

Conversely, if $\mathcal{T} \subseteq 1°$ is a torsion class and $\mathbf{Y}(\mathcal{T})$ is the set of subgroups of $\mathbf{R}$ that occur as quotients of l-groups in $\mathcal{T}$, then $\mathbf{X}(\mathbf{Y}(\mathcal{T})) = \mathcal{T}^{\pi\pi}$.

The mappings $\mathbf{X}$ and $\mathbf{Y}$ are mutually inverse isomorphisms between the Boolean lattice of $\mathbf{T}°$-polar torsion classes and the field of all subsets of the set of isomorphy classes of subgroups of $\mathbf{R}$.

If $\mathcal{T}$ is closed under l-subgroups so is $\mathcal{T}^{\pi\pi}$.

Proof. $\mathcal{X}$ is clearly closed under l-homomorphic images. If $G \in \mathcal{X}$ and C is a convex l-subgroup of G, select a regular convex l-subgroup N of C and its cover $\overline{N}$. There is a regular convex l-subgroup M of G so that $N = M \cap C$, and the cover $\overline{M}$ satisfies $\overline{M} \cap C = \overline{N}$; then $\overline{N}/\overline{N} \simeq \overline{M}/M \in \mathcal{R}$ and so $C \in \mathcal{X}$.

Suppose now that $G = \Sigma_{\lambda} G_{\lambda}$, where the G_{λ}'s are convex l-subgroups of C belonging to $\mathcal{X}$. Pick a regular subgroup N of G and its cover $\overline{N}$. $N = \Sigma_{\lambda} G_{\lambda} \cap N$ and $G_{\lambda} \cap N \neq \neq G_{\lambda}$ for some λ (otherwise $N = G$); $G_{\lambda} \cap \overline{N}$ is the cover of $G_{\lambda} \cap N$ and $\overline{N}/N \simeq \simeq G_{\lambda} \cap \overline{N}/G_{\lambda} \cap N \in \mathcal{R}$. It follows that $G \in \mathcal{X}$ and $\mathcal{X}$ is a torsion class.

Let $\mathcal{R}'$ stand for the set of all pairwise non-isomorphic subgroups of $\mathbf{R}$ not in $\mathcal{R}$. Clearly $\mathcal{X} \cap \mathbf{X}(\mathcal{R}') = 0$; further, if $\mathcal{U}$ is a torsion class of hyper c-archimedean l-groups, disjoint from $\mathcal{X}$, and if $G \in \mathcal{U}$, then for each regular subgroup N of G with cover $\overline{N}$, $\overline{N}/N \in \mathcal{R}'$. It follows immediately that $\mathcal{U} \subseteq \mathbf{X}(\mathcal{R}')$; on the other hand, it is clear that if $\mathcal{U} \subseteq \mathbf{X}(\mathcal{R}')$ then $\mathcal{U} \cap \mathcal{X} = 0$, so we may deduce that $\mathcal{X} = \mathcal{X}^{\pi\pi}$.

As for the converse, let us suppose that $\mathcal{T} \subseteq 1°$; it is evident that $\mathcal{T} \subseteq \mathbf{X}(\mathbf{Y}(\mathcal{T}))$, and by the first part $\mathcal{T}^{\pi\pi} \subseteq \mathbf{X}(\mathbf{Y}(\mathcal{T}))$. However, by theorem 2.5 it is easy to see that $\mathbf{X}(\mathbf{Y}(\mathcal{T})) = \mathcal{T}'' \cap 1°$. Since $\mathcal{T}'' \cap 1° = \mathcal{T}^{\pi\pi}$, we're done.

Trivially $\mathbf{Y}(\mathbf{X}(\mathscr{R})) = \mathscr{R}$, for each subset $\mathscr{R}$ of (pairwise non-isomorphic) subgroups of $\mathbf{R}$. Thus the properties claimed about $\mathbf{X}$ and $\mathbf{Y}$ are obvious.

Finally, it is straightforward to verify that if $\mathscr{X} = \mathscr{X}^{\pi\pi}$, then $\mathscr{X}$ is closed under l-subgroups if and only if $\mathbf{Y}(\mathscr{X})$ is closed under subgroups. Therefore, if $\mathscr{T}$ is l-subgroup closed, so is $\mathscr{T}^{\pi\pi}$.

As an illustration then, if $\mathscr{A}\mathfrak{r}$ denotes the class of hyper archimedean l-groups, then $\mathscr{A}\mathfrak{r}^{\pi\pi} = 1^\circ$ while $\mathscr{A}\mathfrak{r}'' \supseteq \mathscr{N}$, the variety of normal valued l-groups. In fact, according to 2.5, $G \in \mathscr{A}\mathfrak{r}''$ if and only if each $0 < x \in G$ exceeds an element with a normal value.

The following result is a curious observation about joins of disjoint torsion classes.

2.7. Proposition. *If $\mathscr{T}$ and $\mathscr{U}$ are disjoint torsion classes then $\mathscr{T} . \mathscr{U} \cap \mathscr{U} . \mathscr{T} = \mathscr{T} \vee \mathscr{U}$. So if $\mathscr{T}$ and $\mathscr{U}$ are closed with respect to l-subgroups so is $\mathscr{T} \vee \mathscr{U}$.*

Proof. It is evident that $\mathscr{T} \vee \mathscr{U} \subseteq \mathscr{T} . \mathscr{U} \cap \mathscr{U} . \mathscr{T}$. Next, if G is in $\mathscr{T} . \mathscr{U} \cap \mathscr{U} . \mathscr{T}$ then $G/\mathscr{U}(G) \in \mathscr{T}$ and $G/\mathscr{T}(G) \in \mathscr{U}$. By quotient closure $G/(\mathscr{T}(G) + \mathscr{U}(G)) \in \mathscr{T} \cap \mathscr{U}$ which implies that $G = \mathscr{T}(G) + \mathscr{U}(G)$, ie. $G \in \mathscr{T} \vee \mathscr{U}$.

If $\mathscr{T}$ and $\mathscr{U}$ are closed relative to l-subgroups, so are $\mathscr{T} . \mathscr{U}$ and $\mathscr{U} . \mathscr{T}$, and hence $\mathscr{T} \vee \mathscr{U}$ are also l-subgroup closed. (Note: in the last section we give an example of two torsion classes which are closed with respect to l-subgroups, whose join does not have this closure property.)

2.8. Lemma. *Suppose $\mathscr{T}$, $\mathscr{U}$, $\mathscr{X}$ and $\mathscr{Y}$ are torsion classes, and $\mathscr{T}$ and $\mathscr{U}$ are both disjoint to $\mathscr{X}$ and $\mathscr{Y}$; then $\mathscr{T} . \mathscr{U} \cap \mathscr{X} . \mathscr{Y} = 0$.*

Proof. Suppose $G \in \mathscr{T} . \mathscr{U}$, $\mathscr{X} . \mathscr{Y}$; then $G/\mathscr{U}(G) \in \mathscr{T}$ and $G/\mathscr{Y}(G) \in \mathscr{X}$. As in the proof of 2.7 (by quotient closure) $\mathscr{U}(G) + \mathscr{Y}(G) = G$. Now $\mathscr{U}(G) \simeq G/\mathscr{Y}(G) \in \mathscr{X}$ and hence $\mathscr{U}(G) = 0$; by symmetry $\mathscr{Y}(G)$, and hence G is 0.

2.9. Proposition. *Suppose $\mathscr{T}$ and $\mathscr{U}$ are torsion classes and $\mathscr{T} \cap \mathscr{U} = 0$. Then $\mathscr{T}^\sigma \cap \mathscr{U}^\sigma = 0$, for each ordinal σ, and thus $\mathscr{T}^* \cap \mathscr{U}^* = 0$.*

Remark concerning the proof of 2.9. The proposition may be proved via transfinite induction using lemma 2.8, or else from theorem 2.4, since if $\mathscr{T} \cap \mathscr{U} = 0$ it follows that $\mathscr{T}'' \cap \mathscr{U}'' = 0$.

Recall that an element z of a lattice is *finitely meet* (*join*) *irreducible* if $z = a \wedge b$ ($z = a \vee b$) implies that $z = a$ or $z = b$. Join and meet irreducibility refer respectively to the obvious properties relative to arbitrary joins and meets. If the lattice has 0, we say that z is *indecomposable* if $z = a \vee b$ with $a \wedge b = 0$ implies that $a = 0$ or $b = 0$. We can say the following basic things about indecomposability in $\mathbf{T}$.

2.10. Proposition. *If $\mathscr{T}$ is a complete torsion class containing at least one nontrivial o-group, then $\mathscr{T}$ is indecomposable.*

Proof. Write $\mathcal{T} = \mathcal{X} \vee \mathcal{Y}$ with $\mathcal{X} \cap \mathcal{Y} = 0$. Let A be a nontrivial o-group in $\mathcal{T}$. Without loss of generality suppose $A \in \mathcal{X}$. If $\mathcal{Y} \neq 0$ pick $0 \neq B \in \mathcal{Y}$ and form G, the direct lexicographic extension of B by A. Since $\mathcal{T}$ is complete $G \in \mathcal{T}$; further, G is indecomposable so that $G \in \mathcal{X}$ or $G \in \mathcal{Y}$. If $G \in \mathcal{X}$ then $B \in \mathcal{X}$, and if $G \in \mathcal{Y}$ then $A \in \mathcal{Y}$; either way we get a contradiction. We must therefore conclude that $\mathcal{Y} = 0$ and $\mathcal{T}$ is indecomposable.

(The reader should notice that only closure under direct lexicographic extensions was used in the above proof.)

2.11. Proposition. *Suppose $\mathcal{T}$ is a torsion class, finitely meet irreducible in $\mathcal{T}$. Then $\mathcal{T}$ is indecomposable.*

Before proving this we should recall the definition of wreath products. Let A and B be l-groups, and suppose B is represented as an l-group of order preserving permutation (o-permutations) of a totally ordered set S. (This can always be done by Holland's embedding theorem.) When we think of an l-group as a group of o-permutations we write it multiplicatively, and denote it (B, S), S being the chain its acting on.

The (*restricted*) *wreath product* of A by $(B; S)$, denoted $((B; S) \text{ wr } A)$ $(B; S) \text{ Wr } A$, is the semi-direct product of $((A^S)_0)$ A^S, the group of (finitely non-zero) functions of S into A, by B. The group operation is the following:

$$(b, \phi) + (c, \theta) = (bc, \phi^c + \theta),$$

where $b, c \in B$, $\phi, \theta \in A^S$ (resp. $(A^S)_0$), and $s\phi^c = (sc^{-1})\,\phi$ for all $s \in S$. By definition $(b, \phi) \geqq 0$ if $b \geqq 0$ and $s\phi \geqq 0$ whenever $sb = s$. $(B; S) \text{ Wr } A$ becomes an l-group with $(B; S) \text{ wr } A$ as an l-subgroup.

If B is an o-group and $(B; B)$ represents the ordinary Cayley representation, then we speak of the standard wreath products: $B \text{ Wr } A$ and $B \text{ wr } A$ respectively; they are lexicographic extensions of A^B and A_0^B respectively, by B.

Proof of 2.11. Suppose $\mathcal{T} = \mathcal{X} \vee \mathcal{Y}$ while $\mathcal{X} \cap \mathcal{Y} = 0$; then according to proposition 2.7 $\mathcal{T} = \mathcal{X} \,.\, \mathcal{Y} \cap \mathcal{Y} \,.\, \mathcal{X}$. Since $\mathcal{T}$ is finitely meet irreducible we may suppose $\mathcal{T} = \mathcal{X} \,.\, \mathcal{Y}$. If $\mathcal{X}$ and $\mathcal{Y}$ are nontrivial select $0 \neq B \in \mathcal{X}$ and $0 \neq A \in \mathcal{Y}$ and relative to a suitable permutation representation $(B; S)$ form $G = (B; S) \text{ wr } A$. $G \in \mathcal{T}$, so $G = \mathcal{X}(G) \boxplus \mathcal{Y}(G)$; $G/(A^S)_0 \simeq B \in \mathcal{X}$, so $\mathcal{Y}(G) \subseteq (A^S)_0$. Moreover, $(A^S)_0 \in \mathcal{Y}$ whence $\mathcal{Y}(G) = (A^S)_0$. This is a contradiction since $(A^S)_0$ is not a summand; thus $\mathcal{X} = 0$ or $\mathcal{Y} = 0$, and $\mathcal{T}$ is indecomposable.

Remarks 1. Proposition 2.11 indicates how badly noncommutative the operation . on torsion classes is. If $\mathcal{X}$ and $\mathcal{Y}$ are disjoint torsion classes then $\mathcal{X} \,.\, \mathcal{Y} = \mathcal{Y} \,.\, \mathcal{X}$ implies that $\mathcal{X} = 0$ or $\mathcal{Y} = 0$.

2. The use of wreath products also allows us to drop the assumption in 2.10 that the class $\mathcal{T}$ contain a non-trivial o-group. The proof of 2.11 shows that if $\mathcal{T}$ is a complete torsion class then it is indecomposable. However, the earlier proof for 2.10 will come in handy quite soon.

3. In particular, it becomes obvious that $1 = \mathcal{T}'' \vee \mathcal{T}'$ if and only if $\mathcal{T} = 0$ or $\mathcal{T}' = 0$.

The model for our discussion here is the primary decomposition theorem for abelian torsion groups. First we must decide which torsion classes we shall consider as being primary. There are several likely choices: the atoms of $\mathbf{T}$, classes whose torsion subclasses form a chain, indecomposable torsion classes, finitely join irreducible torsion classes. The first two seem a bit restricted, the other two too broad.

There is another, perhaps more efficient choice: we call $\mathcal{T}$ a *primary* torsion class if for each pair of torsion classes $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{T}$, $\mathcal{X} \cap \mathcal{Y} \neq 0$. With this definition we get the desirable feature of indecomposability, plus a good deal more, as will be apparent soon.

3.1. Lemma. *For the torsion class $\mathcal{T}$ the following are equivalent.*

(a) $\mathcal{T}$ *is primary;*

(b) $\mathcal{T}''$ *is primary;*

(c) $\mathcal{T}'$ *is finitely meet irreducible;*

(d) $\mathcal{T}'$ *is a maximal polar torsion class;*

(e) $\mathcal{T}''$ *is an atom in the Boolean lattice of polar torsion classes.*

This lemma appears in [5] as lemma 2.1; the reader should pause for a second to think that we are asserting something here about a structure on what could very well be a proper class. At any rate, the key to the proof of lemma 3.1 is that the correspondence $\mathcal{X} \to \mathcal{X} \cap \mathcal{T}$ is a one to one correspondence between the finite meet irreducible elements of $\mathbf{T}$ not containing $\mathcal{T}$, and the finite meet irreducibles of the lattice of torsion subclasses of $\mathcal{T}$.

If $\mathcal{U}$ is any torsion class we say that $\mathcal{U}$ *admits a primary decomposition* if there exist primary torsion classes $\{\mathcal{T}_\lambda \mid \lambda \in \Lambda\}$ such that $\mathcal{T}_\lambda \cap \mathcal{T}_\mu = 0$ for $\lambda \neq \mu$, and $\mathcal{U} = \bigvee_\lambda \mathcal{T}_\lambda$.

The lattice $\mathbf{T}_{\mathcal{U}}$ of torsion classes contained in $\mathcal{U}$ is again a Brouwer lattice under the same lattice operations, so that a natural polarization arises; (in fact, the polar of $\mathcal{T} \subseteq \mathcal{U}$ in $\mathbf{T}_{\mathcal{U}}$ is $\mathcal{U} \cap \mathcal{T}'$). It is obvious that lemma 3.1 is valid for $\mathbf{T}_{\mathcal{U}}$ as well.

3.2. Proposition. *If $\mathcal{U}$ admits a primary decomposition, it admits only one.*

Proof. Suppose $\{\mathcal{T}_\lambda \mid \lambda \in \Lambda\}$ and $\{\mathcal{U}_i \mid i \in I\}$ yield primary decompositions for $\mathcal{U}$; then each $\mathcal{T}_\lambda = \bigvee_i \mathcal{T}_\lambda \cap \mathcal{U}_i$. Since $\mathcal{T}_\lambda$ is indecomposable $\mathcal{T}_\lambda = \mathcal{T}_\lambda \cap \mathcal{U}_{i_0} \subseteq \mathcal{U}_{i_0}$, for a suitable $i_0 \in I$. By symmetry, $\mathcal{U}_{i_0}$ is contained in some $\mathcal{T}_\mu$, which forces $\mu = \lambda$. Thus $\mathcal{T}_\lambda = \mathcal{U}_{i_0}$ and it's clear that the families are identical.

From the arguments involving proposition 2.10 we conclude immediately:

3.3. Theorem. *If $\mathscr{U}$ is a torsion class which is closed under direct lexicographic extensions or restricted wreath products, then $\mathscr{U}$ admits a primary decomposition if and only if it is primary.*

Proof. $\mathscr{U}$ is indecomposable under either assumption.

If we restrict our attention to torsion classes of normal valued l-groups, a clear picture of primary classes emerges. Let $\mathscr{N}$ be the class of normal valued l-groups; we are concerned with $\mathbf{T}_{\mathscr{N}}$.

The reader might wonder whether $\mathscr{N}$ is indeed a torsion class. It is evidently closed under convex l-subgroups and quotients as it is a variety. To see it is also closed under joins of convex l-subgroups in N, note that if $G = \bigvee_\gamma G_\gamma$, where each G_γ is a normal valued convex l-subgroup, then the elements of the G_γ are normal valued as elements of G. Further G is the subgroup generated by the G_γ, hence each element of G is normal valued.

We need some notation: if R is a subgroup of $\mathbf{R}$, let $\mathscr{T}_R$ denote the torsion class of cardinal sums of copies of R. It is clear that if $\mathscr{T} \in \mathbf{T}_{\mathscr{N}}$ then $\mathscr{T} \cap \mathscr{T}_R \neq 0$ for a suitable subgroup R of $\mathbf{R}$. Also, each $\mathscr{T}_R$ is an atom in $\mathbf{T}$, hence each torsion class of $\mathbf{T}_{\mathscr{N}}$ exceeds an atom $\mathscr{T}_R$; clearly then, the $\mathscr{T}_R$ are the only atoms of $\mathbf{T}_{\mathscr{N}}$.

We denote the polar in $\mathbf{T}_{\mathscr{N}}$ of $\mathscr{T} \subseteq \mathscr{N}$ by $\mathscr{T}^p = \mathscr{T}' \cap \mathscr{N}$. From the comments above we conclude that these relative polars form an atomic Boolean algebra. Also, the primary classes in $\mathbf{T}_{\mathscr{N}}$ are easily identified.

3.4. Proposition. *If $\mathscr{U}$ is a torsion subclass of $\mathscr{N}$ then $\mathscr{U}$ is primary if and only if $\mathscr{U} \subseteq \mathscr{T}_R^{pp} = \mathscr{T}_R'' \cap \mathscr{N}$ for a suitable subgroup R of $\mathbf{R}$.*

As a special consequence of this we get:

3.4.1. Corollary. *Suppose $\mathscr{U}$ is a torsion class in which each l-group is a subdirect product of copies of a fixed subgroup R of $\mathbf{R}$. Then $\mathscr{U}$ is primary.*

Note. For a given subgroup R of $\mathbf{R}$ the class Hyp (R) of all subdirect products of copies of R for which each quotient is once again of this form, is the largest torsion class in which every l-group is a subdirect product of copies of R. In the case of $R = \mathbf{Z}$ we have already identified one torsion class between $\mathscr{T}_{\mathbf{Z}}$ and Hyp $(\mathbf{Z})$, namely, the class $\mathscr{S}$ of l-groups which are generated by their singular elements. There are incomparable torsion classes between $\mathscr{T}_{\mathbf{Z}}$ and Hyp $(\mathbf{Z})$: for instance, $\mathscr{S}$ and $\mathscr{T}_{\mathbf{Z}}^2 \cap$ $\cap$ Hyp $(\mathbf{Z})$. For if G is the group of bounded integral sequences then $G \in \mathscr{S}$; (see [3]). On the other hand $\mathscr{T}_{\mathbf{Z}}(G)$ is the l-ideal of finitely non-zero sequences, and it can easily be shown that $G/\mathscr{T}_{\mathbf{Z}}(G)$ has no basic elements so that $\mathscr{T}_{\mathbf{Z}}(G) = \mathscr{T}_{\mathbf{Z}}^2(G)$. The l-group H of integral sequences which are eventually constant and even is in $\mathscr{T}_{\mathbf{Z}}^2 \cap$ Hyp $(\mathbf{Z})$, (in fact $H/\mathscr{T}_{\mathbf{Z}}(H) \simeq \mathbf{Z}$), but H is obviously not in $\mathscr{S}$, ($\mathscr{S}(H) =$ $= \mathscr{T}_{\mathbf{Z}}(H) \neq H$). We simply mention in passing that the intersection $\mathscr{S} \cap \mathscr{T}_{\mathbf{Z}}^2$ is easily seen to be larger than $\mathscr{T}_{\mathbf{Z}}$.

The author hopes to take up the torsion subclasses of Hyp ($\mathbf{Z}$) in a later article.

Directly from proposition 3.4 we can conclude that the class of vector lattices is primary; that of divisible abelian l-groups is not.

4. HOMOGENEOUS l-GROUPS AND OTHER EYE-CATCHERS

An intriguing class of l-groups emerges from the concepts discussed in this paper. Before describing it let us settle some general questions. Let G be an l-group; the family of torsion classes with $\mathscr{T}(G) = 0$ forms a complete sublattice of $\mathbf{T}$. In particular, there is a unique largest torsion class $\mathscr{X}^G$ so that $\mathscr{X}^G(G) = 0$; it is easy to verify that $\mathscr{X}^G$ is complete. It would be convenient if $\mathscr{X}^G$ were meet irreducible in $\mathbf{T}$, but in general it is not clear what happens with classes that contain $\mathscr{X}^G$ properly.

We therefore turn to the following definition: G is *homogeneous* if for each torsion class $\mathscr{T}$, either $G \in \mathscr{T}$ or else $\mathscr{T}(G) = 0$. It is clear that if G is a homogeneous l-group then $\mathscr{X}^G$ is meet irreducible; conversely, if $\mathscr{X}^G$ is meet irreducible, it has a cover $\mathscr{Y}$. Let $K = \mathscr{Y}(G)$; if $\mathscr{T}$ is a torsion class and $\mathscr{T}(K) \neq 0$ then $\mathscr{T}(G) \neq 0$. Thus $\mathscr{Y} \subseteq \mathscr{X}^G \vee \mathscr{T}$ and hence $K = \mathscr{Y}(K) = \mathscr{X}^G(K) + \mathscr{T}(K) = (\mathscr{X}^G(G) \cap \mathscr{K}) + \mathscr{T}(K) = \mathscr{T}(K)$, ie. $K \in \mathscr{T}$. We conclude that K is homogeneous, and a similar argument shows that $\mathscr{X}^K = \mathscr{X}^G$.

The complete, meet irreducible torsion classes of $\mathscr{T}$ give rise to homogeneous l-groups: if $\mathscr{X}$ is complete and meet irreducible and $\mathscr{Y}$ is its cover, select $G \in \mathscr{Y} \setminus \mathscr{X}$. Letting $G_0 = G/\mathscr{X}(G)$, we see that $\mathscr{X}(G_0) = 0$, and if $\mathscr{T}$ is a torsion class with $\mathscr{T}(G_0) \neq 0$, then $\mathscr{Y} \subseteq \mathscr{X} \vee \mathscr{T}$, and as in the previous paragraph $\mathscr{T}(G_0) = G_0$. It is also evident that $\mathscr{X} = \mathscr{X}^G$.

We summarize the above discussion as follows:

4.1. Theorem. *Let G be an l-group. If G is homogeneous then $\mathscr{X}^G$ is a complete, meet irreducible torsion class. Conversely, if $\mathscr{X}^G$ is meet irreducible, G has a no-trivial homogeneous l-ideal. On the other hand, if $\mathscr{X}$ is any complete, meet irreducible torsion class, there is a homogeneous l-group H so that $\mathscr{X} = \mathscr{X}^H$.*

From the point of view of the lattice $\mathbf{T}$ there is yet another way to look at homogeneous l-groups. Suppose G is homogeneous and $\mathscr{X}_G$ is the torsion class it generates; then $\mathscr{X}^G \vee \mathscr{X}_G$ is the cover of $\mathscr{X}^G$ and $\mathscr{X}^G \cap \mathscr{X}_G$ is the largest subclass of $\mathscr{X}_G$ that fails to contain G. In effect, $\mathscr{X}_G$ is join irreducible.

Conversely, suppose $\mathscr{U}$ is join irreducible, $\mathscr{U}^{\sim}$ the largest torsion class properly contained in $\mathscr{U}$, and finally, suppose there is an l-group $H \in \mathscr{U}$ such that $\mathscr{U}^{\sim}(H) = 0$. $\mathbf{T}$ is a Brouwer lattice, so there is a unique largest torsion class $\mathscr{X}$ so that $\mathscr{X} \cap \mathscr{U} = \mathscr{U}^{\sim}$. Put $\mathscr{Y} = \mathscr{X} \vee \mathscr{U}$; then $H \in \mathscr{Y} \setminus \mathscr{X}$, and if $\mathscr{T}$ is a torsion class that misses H we get $\mathscr{T} \cap \mathscr{U} \subseteq \mathscr{U}^{\sim}$, and hence $\mathscr{T} \subseteq \mathscr{X}$. Notice that $\mathscr{X}(H) = \mathscr{X}(H) \cap \mathscr{U}(H) = \mathscr{U}^{\sim}(H) = 0$; that is, $\mathscr{X} = \mathscr{X}^H$ and so it is clear that H is homogeneous.

We've proved the following:

4.2. Proposition. *If G is a homogeneous l-group the torsion class $\mathscr{X}_G$ generated by G is join irreducible. Conversely, if $\mathscr{U}$ is a join irreducible torsion class and covers $\mathscr{U}^{\sim}$, and there is an l-group $H \in \mathscr{U}$ so that $\mathscr{U}^{\sim}(H) = 0$, then H is homogeneous, and $\mathscr{X}^H$ is the largest torsion class satisfying $\mathscr{X}^H \cap \mathscr{U} = \mathscr{U}^{\sim}$.*

Now for some examples of homogeneous l-groups: evidently, each characteristically simple l-group is homogeneous: eg. free abelian l-groups, cardinal sums of copies of a fixed subgroup R of $\mathbf{R}$; $C(X)$ when X is a compact space with a transitive group of homoomorphisms; the group of periodic real sequences; etc.

If G is any l-group with the property that each characteristic l-ideal H of G ($H \neq 0$) is l-isomorphic to G then G is homogeneous. For suppose $\mathscr{T}$ is a torsion class and $\mathscr{T}(G) \neq 0$. $\mathscr{T}(G)$ is characteristic, and therefore $G \simeq \mathscr{T}(G)$, which in turn implies that $G \in \mathscr{T}$; hence G is indeed homogeneous. An example of such an l-group is the o-group of all real sequences $(s_1, s_2, \ldots)$, lexicographically ordered from left to right. Notice that each non-zero l-ideal is characteristic and isomorphic to G. Thus by our remarks G is homogeneous.

What is needed here is a theorem that indicates how to verify whether an l-group G is homogeneous, by checking a few key classes. It does not seem very easy to compute X^G and then check whether it is meet irreducible.

We conclude with a few remarks and examples:

If $\mathscr{T}$ is a torsion class then $\mathscr{T}(G)$ is always characteristic. It is in fact invariant under every l-endomorphism of G *onto* itself. The crucial word is *onto*, for torsion radicals are not necessarily fully invariant; that is, they are not always invariant under all l-endomorphisms. Let $G = \mathbf{Z} \boxplus \mathbf{R}$ and $\mathscr{T} = \mathscr{T}_{\mathbf{Z}}$; define the l-endomorphism ϕ of G by: $(m, r)\,\phi = (0, m)$, for $m \in \mathbf{Z}$ and $r \in \mathbf{R}$. It should be obvious that $[\mathscr{T}(G)]\,\phi \nsubseteq \mathscr{T}(G)$; in fact, $[\mathscr{T}(G)]\,\phi \cap \mathscr{T}(G) = 0$.

We've also seen an example (in the group of real sequences with the lexicographic order) of a characteristic l-ideal which is not a torsion radical. Here's another: let $G = C([0, 1])$ and A be the l-ideal of all continuous functions that vanish off an open interval (a, b) in $[0, 1]$. It is shown in corollary 6.11 of [2] that A is the smallest non-trivial characteristic l-ideal of G. We define an l-endomorphism ϕ of G by $(g\phi)(t) = g(\frac{1}{2}t)$; it is clear that ϕ is an l-endomorphism of G onto itself. Let $h(t) = \sin(\pi t)$; then $h \in A$ while $(h\phi)(t) = \sin(\frac{1}{2}\pi t)$ is not in A. Thus A is not a torsion radical; the author suspects that G is actually a homogeneous l-group.

Finally, an example of two torsion classes: $\mathscr{A}r$, the class of hyper-archimedean l-groups, and $\mathscr{F}$ the class of finite valued l-groups, both closed under taking l-subgroups, so that $\mathscr{A}r \vee \mathscr{F}$ is not closed under taking l-subgroups. Let $G = A \boxplus B$, where A is the group of eventually constant real sequences, and $B = \mathbf{R} \overrightarrow{\times} \mathbf{R}$, the lexicographic product of $\mathbf{R}$ with itself. $\mathscr{A}r(G) = \{((s_n), (b_1, b_2)) \mid b_1 = 0\}$ and $\mathscr{F}(G) = \{((s_n), (b_1, b_2)) \mid s_n \text{ is finitely non-zero}\}$, so $G = \mathscr{A}r(G) + \mathscr{F}(G)$ and

$G \in \mathscr{A}r \vee \mathscr{F}$. But let $H = \{(s_n), (b_1, b_2)) \mid \lim_{n \to \infty} s_n = b_1\}$; then H is an l-subgroup of G and $\mathscr{F}(H) = \{(s_n), (0, b)) \mid s_n$ is finitely non-zero$\}$ while $\mathscr{A}r(H) = \mathscr{F}(H)$, so that $H \notin \mathscr{A}r \vee \mathscr{F}$. The author wishes to thank Charles Holland for his help with this example.

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