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ON COMPRESSED IDEALS IN TOPOLOGICAL SEMIGROUPS[1])

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A topological semigroup is a non-empty Hausdorff space together with a continuous associative multiplication, denoted by juxtaposition, $(x, y) \to xy$. When there is no possible ambiguity, we shall simply refer to S as a topological semigroup. In addition, if S is a compact space, we call S a compact semigroup. Let I be a non-empty subset of S. I is called an ideal of S if $IS \subset I$ and $SI \subset I$. In 1957, GABRIEL THIERRIN [14] introduced a new class of ideals in ring theory, namely the compressed ideals. An ideal B in a ring R is called compressed, if and only if for any positive integer, n, $a_1^2 a_2^2 \ldots a_n^2 \in B$ implies $a_1 a_2 \ldots a_n \in B$, where $a_1, a_2, \ldots, a_n$ need not be distinct elements of R. KYOSHI ISEKI [5] then noticed that the majority of the results obtained by G. Thierrin can be carried over to semi-rings without change. According to K. Iseki, the smallest compressed ideal containing an ideal A is called the Thierrin radical of A. The aim of this paper is to study the compressed ideals in compact semigroups. We first show that a compressed ideal of a semigroup S is in fact a completely semi-prime ideal of S, and thus the Thierrin radical of an ideal A in a commutative semi-group is the algebraic radical of A defined in [4]. If B is a compressed ideal of S, then for any element a of S, the set $\{s \in S \mid as \in B\}$ is called the topological B-divisor of a, and we denote it by $\mathrm{Tod}_B\, a$. Some properties of the set $\mathrm{Tod}_B\, a$ will be studied in this paper, and some results obtained in [12] dealing with the topological zero divisors in compact commutative semigroups with zero will be amplified. We shall also give some characterizations of the Thierrin radical of an open ideal A in a compact semigroup S, which in a sense transports some results obtained by V. A. ANDRUNAKEVIC - JU. M. RJABUHIN [1] and W. H. CORNISH - P. N. STEWART [2] from ring theory to compact semigroups.

Throughout the paper we will use the terminology of A. B. PAALMAN DE-MIRANDA [9]. Unless otherwise stated, S is an arbitrary semigroup and the word ideal shall mean two-sided ideal of S.

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I. COMPRESSED IDEALS

Definition 1.1. A non-empty ideal P of S is said to be prime if $AB \subset P$ implies that $A \subset P$ or $B \subset P$, A, B being ideals of S. An ideal Q of S is said to be completely prime if $ab \in Q$ implies that $a \in Q$ or $b \in Q$, a and b being elements of S.

Remark 1. An ideal which is completely prime is prime, but the converse is not generally true. (See the example on page 51 in [9].) However, these concepts coincide in the case of normal semigroups. (A semigroup is called normal if $aS = Sa$ for all $a \in S$.)

Remark 2. N. H. McCoy [7] proved that P is a prime ideal of S if and only if $aSb \subset P$ implies that $a \in P$ or $b \in P$. In this paper, we shall use McCoy's characterisation for prime ideals in semigroups.

Definition 1.2. A non-empty ideal B of S is said to be completely semi-prime if and only if $a^2 \in B$ implies that $a \in B$.

Definition 1.3. *Let B be an ideal of S. The algebraic radical of B is defined to be the set $R(B) = \{x \in S \mid x^n \in B$ for some integers $n \geqq 1\}$.*

Remark. If S is a commutative semigroup, then it was proved in [4] that B is a completely semi-prime ideal of S if and only if $B = R(B)$. Moreover, if B is open, then $R(B)$ is open [13].

Theorem 1.4. *B is a compressed ideal of S if and only if B is completely semi-prime.*

Before proving Theorem 1.4, we need the following lemma.

Lemma 1.5. *Let B be a completely semi-prime ideal of S, then the following statements hold:*

(i) *If $xy \in B$, then $yx \in B$.*

(ii) *If $a^2b^2 \in B$, then $ab \in B$.*

Proof. (i) Since B is an ideal of S, thus $xy \in B$ implies that $y(xy)\,x = (yx)^2 \in B$. Since B is completely semi-prime, so $yx \in B$.

(ii) The proof of (ii) depends on the result obtained in (i). By (i), we have $a^2b^2 \in B \Rightarrow b^2a^2 \in B \Rightarrow a(b^2a) \in B$. Since B is a completely semi-prime ideal of S, thus $a(b^2a) \in B \Rightarrow (b^2a)^2 \in B \Rightarrow b^2a \in B$. By (i), we obtain that $(ba)\, b \in B$, and hence $(ba)^2 \in B \Rightarrow ba \in B$. Applying (i) once more, we prove that $ab \in B$.

We now prove theorem 1.4. It is quite clear that every compressed ideal is completely semi-prime. We only need to show that every completely semi-prime is compressed. We proceed by induction. In lemma 1.5 (ii), we have already noted that the theorem is true when $n = 2$. Supposing that the theorem is true for n, we want to prove the theorem is true for $n+1$ as well. So suppose $a_1^2a_2^2 \ldots a_n^2a_{n+1}^2 \in B$, then since B is an ideal of S, we have $a_1^2a_2^2 \ldots a_{n-1}^2(a_n^2a_{n+1}^2)^2 \in B$. By induction hypothesis, we thus have $a_1a_2 \ldots a_{n-1}a_n^2a_{n+1}^2 \in B$, and so $(a_1a_2 \ldots a_{n-1}a_n^2)^2\, a_{n+1}^2 \in B$. By induction hypothesis and lemma 1.5 (i), we therefore obtain that $a_{n+1}a_1 \ldots a_{n-1}a_n^2 \in B$. Since B is an ideal of S, so we have $(a_{n+1}a_1 \ldots a_{n-1})^2\, a_n^2 \in B$. Again, by induction hypothesis and lemma 1.5 (i), we obtain that $a_1 \ldots a_{n-1}a_na_{n+1} \in B$. Thus B is a compressed ideal of S. The proof is complete.

Corollary 1. *If B is a compressed ideal of S, then $a_1a_2 \ldots a_n \in B$ implies $a_1^{l_1}a_2^{l_2} \ldots a_n^{l_n} \in B$ for any positive integers $l_1, l_2, \ldots, l_n$ and $a_1^{l_1}a_2^{l_2} \ldots a_n^{l_n} \in B$ implies $a_1a_2 \ldots a_n \in B$* (see K. Iseki [5]).

Remark 1. Theorem 1.4 is not true if B is not a two-sided ideal of S. Let for instance $S = \{0, e_1, e_2, a, b\}$ with the multiplication table

$\cdot$	0	e_1	e_2	a	b
0	0	0	0	0	0
e_1	0	e_1	0	0	b
e_2	0	0	e_2	a	0
a	0	a	0	0	e_2
b	0	0	b	e_1	0

The set $B = \{0, b, e_1\}$ is a right ideal of S and is clearly completely semi-prime. However, $a^2b^2 = 0 \in B$, but $ab = e_2 \notin B$. So B is not compressed.

Remark 2. A subset C of S is called compressed subset of S if and only if $a^2 \in C$ implies $a \in C$ for any element $a \in S$. In general, C need not be an ideal of S as can be seen by the above example. The subsets $\{0, a\}$ and $\{0, b\}$ are compressed subsets of S, and neither one is an ideal of S.

Definition 1.6. Let B be an ideal of S. The Thierrin radical of B is defined to be the intersection of all compressed ideals containing B, denoted by $T(B)$. If S does not contain any proper compressed ideals containing B, then $T(B) = S$.

Given an ideal B of S, by applying Theorem 1.4 we can construct the Thierrin radical of B by the following way: we call an element x a t-element of B if $x \notin B$ but $x^2 \in B$. B together with the set of all t-elements of B is denoted by $R^*(B)$, that is, $R^*(B)$ is the set $\{x \in S \mid x^n \in B$ for some integer $n \geqq 1\}$. If S is a commutative semigroup, then $R^*(B)$ is a compressed ideal of S, and hence $R^*(B)$ is the Thierrin radical of B. If S is not a commutative semigroup, then let $T_1(B) = J(R^*(B))$, which is the principal ideal generated by the set $R^*(B)$. Write $T_2(B) = J(R^*(T_1(B))$. By induction, we have $T_n(A) = J(R^*(T_{n-1}(B))$. Then each $T_n(B)$ is an ideal of S and $T_n(B) \subset \subset T_{n+1}(B)$. The Thierrin radical $T(B)$ of B is the union of all sets $T_n(B)$ $(n = 1, 2, \ldots)$, that is, $T(B) = \bigcup_{n=1}^{\infty} T_n(B)$.

K. Iseki proved in [5] that the algebraic radical of B is contained in $T(B)$. If S is a commutative semigroup, then since $R(B) = R(R(B))$, so it follows $R(B)$ is a completely semi-prime ideal and hence compressed. But $T(B)$ is defined to be the intersection of all compressed ideals of S containing B, that is, $T(B)$ is the smallest compressed ideal containing B, thus we must have $R(B) = T(B)$. Summing up the above information, we have the following:

Theorem 1.7. *Let B be an ideal of a commutative semigroup S. Then, the Thierrin radical of B coincides with the algebraic radical of B.*

Theorem 1.8. *Let J be a maximal proper ideal of a compact semigroup S. Then J is a compressed ideal of S if and only if J is a completely prime ideal of S.*

Proof. If J is a compressed ideal, then J is completely semi-prime. Hence $a \in \in S - J$ implies $a^2 \in S - J$. By a result of W. H. FAUCETT - R. J. KOCH - K. NUMAKURA [3; p. 656], we then have that J is a completely prime ideal. The converse part of this theorem is obvious.

Definition 1.9. An ideal B^* in a semigroup S is called a weakly compressed ideal of S if for any pairs of distinct elements a_1, a_2 of S such that $a_1^2a_2^2 \in B^*$ we have $a_1a_2 \in B^*$.

Clearly, a compressed ideal is weakly compressed, but a weakly compressed ideal need not be compressed. For instance, let $S = \{0, a\}$ with multiplication $0^2 = a^2 = = 0a = a0 = 0$, then S is a semigroup and $\{0\}$ is a weakly compressed ideal of S, $\{0\}$ is not compressed since $a^2 = 0 \in \{0\}$, but $a \notin \{0\}$.

Theorem 1.10 *Let S be a compact connected semigroup such that $S \neq S^2$, then S is the union of proper weakly compressed ideals of S, each of which is dense in S.*

Proof. Since $S^2 \neq S$, then we can pick an element $a \in S - S^2$. Clearly $S - \{a\}$ is a maximal proper ideal of S. Suppose if possible, $a_1^2a_2^2 \in S - \{a\}$ but $a_1a_2 \notin S - - \{a\}$. Then $a_1a_2 = a$. But this means that $a \in S^2$, which contradicts the choice of a. $S - \{a\}$ is therefore a weakly compressed ideal of S. We claim that $|S - S^2| \geqq 2$.

This is because S^2 is compact, hence closed and S is connected, hence $S - S^2$ has to have more than one element. So there are at least two distinct elements a, b in $S - S^2$. As $S = (S - \{a\}) \cup \{a\} \subset (S - \{a\}) \cup (S - \{b\})$, we conclude that $S = \bigcup_{a_i}(S - \{a_i\})$, where a_i runs through $S - S^2$. The compactness and connectedness of S imply that each of the ideals $(S - \{a_i\})$ is dense in S. Our proof is completed.

II. TOPOLOGICAL B-DIVISORS

In this section, B will denote a compressed ideal of a semigroup S. The topological B-divisor of a subset A of S with respect to B is defined to be the set $\mathrm{Tod}_B\, A = \{x \in S \mid ax \in B \text{ for all } a \in A\}$. If $A = \{a\}$, then we abbreviate $\mathrm{Tod}_B\, \{a\}$ by $\mathrm{Tod}_B\, a$ and a will be called a topological B-factor if $a \in S - B$. In view of lemma 1.5(i), we know that $\mathrm{Tod}_B\, a$ is an ideal of S, and $\mathrm{Tod}_B\, a = \{x \in S \mid ax \in B\} = \{x \in S \mid xa \in B\}$.

The following results on topological B-divisors are straightforward and proofs are therefore omitted.

Proposition 2.1. *For any $x, y \in S$, we have*

(i) $\mathrm{Tod}_B\, x$ *is a compressed ideal of* S.

(ii) $\mathrm{Tod}_B\, b = S$ *for any* $b \in B$.

(iii) $\mathrm{Tod}_{\mathrm{Tod}_B x}\, x = \mathrm{Tod}_B\, x$.

(iv) $\mathrm{Tod}_B\, (\mathrm{Tod}_B\, xy) = \mathrm{Tod}_B\, (\mathrm{Tod}_B\, x) \cap \mathrm{Tod}_B\, (\mathrm{Tod}_B\, y)$.

Proposition 2.2. (i) *If B is an open compressed ideal of S, then* $\mathrm{Tod}_B\, x$ *is an open compressed ideal of* S.

(ii) *If B is a closed compressed ideal of S, then* $\mathrm{Tod}_B\, x$ *is a closed compressed ideal of* S.

Proof. It is known that if B is open, then $\mathrm{Tod}_B\, x$ is open, and if B is closed, then $\mathrm{Tod}_B\, x$ is closed. See [4].

Proposition 2.3. *For any $x, y, z \in S$, we have*

(i) $\mathrm{Tod}_B\, x = \mathrm{Tod}_B\, x^2$.

(ii) $\mathrm{Tod}_B\, xy = \mathrm{Tod}_B\, yx$.

(iii) $\mathrm{Tod}_B\, x \subset \mathrm{Tod}_B\, z$ *and* $\mathrm{Tod}_B\, y \subset \mathrm{Tod}_B\, z$, *if and only if* $\mathrm{Tod}_B\, xy \subset \mathrm{Tod}_B\, z$.

(iv) *If* $\mathrm{Tod}_B\, x = \mathrm{Tod}_B\, y$, *then* $\mathrm{Tod}_B\, zx = \mathrm{Tod}_B\, zy$.

(v) *If* $\mathrm{Tod}_B\, x \subset \mathrm{Tod}_B\, y$, *then* $\mathrm{Tod}_B\, zx \subset \mathrm{Tod}_B\, zy$.

Proof.

(i) Obvious.

(ii) Let $a \in \mathrm{Tod}_B\, xy$. Then $xya \in B$. Since B is a compressed ideal, we have

$$yxya \in B \Rightarrow a(yxy) \in B \Rightarrow a(yx)^2 \in B \Rightarrow a \in \mathrm{Tod}_B\,(yx)^2 = \mathrm{Tod}_B\, yx\,.$$

Similarly, $\mathrm{Tod}_B\, yx \subset \mathrm{Tod}_B\, xy$. Thus, $\mathrm{Tod}_B\, xy = \mathrm{Tod}_B\, yx$.

(iii) Let

$$a \in \mathrm{Tod}_B\, xy\,. \quad \text{Then} \quad xya \in B \Leftrightarrow xa \in \mathrm{Tod}_B\, y \subset \mathrm{Tod}_B\, z \Leftrightarrow zxa \in B \Leftrightarrow$$

$$\Leftrightarrow azx \in B \Leftrightarrow az \in \mathrm{Tod}_B\, x \subset \mathrm{Tod}_B\, z \Leftrightarrow az^2 \in B \Leftrightarrow a \in \mathrm{Tod}_B\, z^2 = \mathrm{Tod}_B\, z\,.$$

Hence, $\mathrm{Tod}_B\, xy \subset \mathrm{Tod}_B\, z$. For the converse part, we first notice that $\mathrm{Tod}_B\, x \subset \mathrm{Tod}_B\, xy$, $\mathrm{Tod}_B\, y \subset \mathrm{Tod}_B\, xy$. Thus if $\mathrm{Tod}_B\, xy \subset \mathrm{Tod}_B\, z$, then we have

$$\mathrm{Tod}_B\, x \subset \mathrm{Tod}_B\, z\,, \quad \mathrm{Tod}_B\, y \subset \mathrm{Tod}_B\, z\,.$$

(iv) $a \in \mathrm{Tod}_B\, zx \Leftrightarrow zxa \in B \Leftrightarrow az \in \mathrm{Tod}_B\, x = \mathrm{Tod}_B\, y \Leftrightarrow azy \in B \Leftrightarrow a \in \mathrm{Tod}_B\, zy\,.$ Hence, $\mathrm{Tod}_B\, zy = \mathrm{Tod}_B\, zx$.

(v) Follows from (iv).

For elements $a, b \in S$, we define aR_Bb if and only if $\mathrm{Tod}_B\, a = \mathrm{Tod}_B\, b$. We can verify that R_B is an equivalence relation associated with B. Since proposition 2.3 (iv) holds, this equivalence relation is in fact a congruence relation defined on S.

Theorem 2.4. *Let B be a compressed ideal of a compact semigroup S. If B is simultaneously closed and open in S, then S/R_B is a compact commutative semigroup.*

Proof. Since R_B is a congruence relation on $S \times S$, the quotient S/R_B is well-known to be a semigroup [15]. The commutativity of S/R_B follows from proposition 2.3 (ii). In order to prove S/R_B is a compact semigroup, we need to verify that R_B is a closed congruence relation on $S \times S$ [15]. For this purpose, let $(x, y) \in \bar{R}_B$. Then there is a net $\{(x_i, y_i)\}_{i \in \Lambda} \in R_B$ such that $\lim (x_i, y_i) = (x, y)$, where Λ is a directed set. Suppose $\mathrm{Tod}_B\, x \neq \mathrm{Tod}_B\, y$, then we can find an element t such that $ty \in B$ but $tx \notin B$. Recall that the multiplication is continuous and a set in a topological product converges to a point p if and only if its projection in each coordinate space converges to the projection of the point p. Thus, we have $\lim tx_i = t \lim x_i = tx \notin B$ and $\lim ty_i = ty \in B$. Since B is open, $ty_{\alpha_i} \in B$ for large enough α_i. Since B is compact, there is an open neighbourhood $V(x)$ of X such that $t\,V(x) \cap B = \emptyset$. Pick x_{β_i} in the net $\{x_i\}_{i \in \Lambda}$ such that $x_{\beta_i} \in V(x)$, we have $tx_{\beta_i} \notin B$. Write $r_i = \max\{\alpha_i, \beta_i\}$. Since $\mathrm{Tod}_B\, x_i = \mathrm{Tod}_B\, y_i$ for each i, so $ty_{r_i} \in B$ if and only if $tx_{r_i} \in B$.

We thus arrive at a contradiction. R_B is therefore a closed congruence on $S \times S$, which completes the proof.

Theorem 2.5. *Let B be an open compressed ideal of a compact semigroup S. Then for each element $x \in S - B$, we have* $\mathrm{Tod}_B\, x = \mathrm{Tod}_B\, e$, *where* $e^2 = e \in \Gamma(x) = \overline{\{x, x^2, \ldots\}}$.

Proof. Let $a \in \mathrm{Tod}_B\, x$. Then $ax \in B$. Since B is an ideal, we have $axS \subset B$. Hence $ax^n \in axS \subset B$ for all positive integers $n \geqq 1$. By the compactness of S and the continuity of multiplication, we have $ae \in a\,\Gamma(x) \subset axS \subset B$, that is, $a \in \mathrm{Tod}_B\, e$, where $e^2 = e \in \Gamma(x)$. Assume that $\mathrm{Tod}_B\, x \subsetneqq \mathrm{Tod}_B\, e$. Then we can pick an element $z \in \mathrm{Tod}_B\, e - \mathrm{Tod}_B\, x$ such that $ze \in B$, with $zx \notin B$. Since B is compressed, thus $zx \notin B$ implies $zx^n \notin B$ for all integers $n \geqq 1$. Since B is open, $S - B$ is closed and hence compact, then $ze \in B$ implies that there is a neighbourhood $V(e)$ of e such that $z\,V(e) \cap (S - B) = \emptyset$. But $e \in \Gamma(x)$ implies that $x^{n_0} \in V(e)$ for some large enough integer n_0. Hence $zx^{n_0} \in z\,V(e)$. Thus, $zx^{n_0} \in B$ which is a contradiction. We therefore conclude that $\mathrm{Tod}_B\, x = \mathrm{Tod}_B\, e$.

Corollary. *In theorem 2.5, let* $G_\alpha = \{x \in S - B \mid e_\alpha^2 = e_\alpha \in \Gamma(x)\}$, *then* $G_\alpha \cap \mathrm{Tod}_B\, e_\alpha = \emptyset$.

Proof. Since B is an open compressed ideal of S, S is compact. So $S - B$ is also compact. Hence $e_\alpha \in \Gamma(x) \subset S - B$. Now suppose that $G_\alpha \cap \mathrm{Tod}_B\, e_\alpha \neq \emptyset$. Let $y \in G_\alpha \cap \mathrm{Tod}_B\, e_\alpha$. Then $e_\alpha \in \Gamma(y)$ and $e_\alpha y \in B$. Since B is an ideal of S, we have $e_\alpha yS \subset B$. Thus $K(y) = e_\alpha\,\Gamma(y) \subset B$, where $K(y) = \bigcap_{n=1}^{\infty} \overline{\{y^i \mid i \geqq n\}}$, which is a subgroup of $\Gamma(y)$ if $\Gamma(y)$ is compact. Thus $e_\alpha \in K(y) \subset B$, a contradiction.

Remark. $\mathrm{Tod}_B\, x = \mathrm{Tod}_B\, e$ does not imply that $e \in \Gamma(x)$. Let for instance $S = \{0, e, f\}$ with $e^2 = e$, $f^2 = f$, $ef = fe = e$. Then $\mathrm{Tod}_{\{0\}}\, e = \mathrm{Tod}_{\{0\}}\, f = \{0\}$, but $f \notin \Gamma(e)$.

Definition 2.6. A semigroup S is defined to be a quasi-normal semigroup if the idempotents of S are mutually commutative with each other under multiplication.

Definition 2.7. Let B be an ideal of S. An idempotent e is said to be B-primitive if $e \notin B$ and e is the only idempotent in $eSe - B$.

Remark. Let E be the set of idempotents of S. For $e, f \in E$, define $e \leqq f$ if and only if $ef = fe = e$. It is clear that $\leqq$ is a partial ordering in E. Thus, by definition 2.7, the atoms of the partially ordered set $E \cap (S - B)$ (if they exist) are B-primitive idempotents of S.

Theorem 2.8. *Let S be a compact quasi-normal semigroup and B an open compressed ideal of S. If e_α is a B-primitive idempotent of S, then* $\mathrm{Tod}_B\, e_\alpha$ *is an open prime ideal of S.*

Proof. We first claim that if I is an ideal of S which is not contained in $\mathrm{Tod}_B\, e_\alpha$, then there is an idempotent f such that $f \in I - \mathrm{Tod}_B\, e_\alpha$. Let $x \in I - \mathrm{Tod}_B\, e_\alpha$ and consider the principal ideal $J(x)$ generated by x. Clearly $\Gamma(x) \subset J(x) \subset I$. Since S is compact, there is an idempotent $f \in \Gamma(x) \subset I$. Suppose if possible that $f \in \mathrm{Tod}_B\, e_\alpha$. Then we have $K(x) = f\,\Gamma(x) \subset \mathrm{Tod}_B\, e_\alpha$. Since B is an open ideal of S, $\mathrm{Tod}_B\, e_\alpha$ is open. Thus there is an integer $n \geqq 1$ such that $x^n \in \mathrm{Tod}_B\, e_\alpha$, that is, $x^n e_\alpha \in B$. Because B is compressed, we have $x e_\alpha \in B$, which means that $x \in \mathrm{Tod}_B\, e_\alpha$, a contradiction. Our claim is established. Since $e_\alpha \notin B$, we have $\mathrm{Tod}_B\, e_\alpha \subset J_0(S - e_\alpha)$, where $J_0(S - e_\alpha)$ is the maximal ideal contained in the set $S - e_\alpha$. If $\mathrm{Tod}_B\, e_\alpha \subsetneqq J_0(S - e_\alpha)$, then by the above claim, there is an idempotent $f \in J_0(S - e_\alpha) - \mathrm{Tod}_B\, e_\alpha$. Since S is quasi-normal and e_α is B-primitive, we obtain that $e_\alpha f = e_\alpha$. Hence $e_\alpha \in J(e_\alpha)\, J(f) \subset$ $\subset J(f) \subset J_0(S - e_\alpha)$, which is clearly impossible. We thus conclude that $\mathrm{Tod}_B\, e_\alpha =$ $= J_0(S - e_\alpha)$. Applying the well-known theorem of K. Numakura [8], we know that $\mathrm{Tod}_B\, e_\alpha$ is an open prime ideal of S.

We remark that Theorem 2.8 slightly generalizes a result obtained in [12], in fact, their proofs are essentially parallel. Moreover, if e_α is not B-primitive $\mathrm{Tod}_B\, e_\alpha$ is not necessarily a prime ideal. (See [12].)

III. COMPRESSED IDEALS AND THIERRIN RADICALS

Theorem 3.1. *Let S be a compact semigroup and B a proper open compressed ideal of S. Then B can be expressed as the intersection of a family of open prime ideals of S.*

Proof. Since B is an open compressed ideal of S and $B \neq S$, then, by lemma 3.3 in [4], there exists at least one idempotent $e \in S - B$. Hence $B \subset J_0(S - e)$. By K. Numakura [8], $J_0(S - e)$ is an open prime ideal of S. Now let $Q = \bigcap_\alpha P_\alpha$ where P_α runs through all the open prime ideals containing B. It is clear that $Q \supset B$. If B itself is a prime ideal, then there is nothing to prove. If otherwise, we may assume that $Q \supsetneqq B$. Then by lemma 3.3 in [4] again, there is an idempotent $f \in Q - B$. Hence $J_0(S - f)$ is an open prime ideal containing B. Thus, $f \in Q \subset J_0(S - f)$, which is a contradiction. We therefore conclude that $Q = \bigcap_\alpha P_\alpha = B$.

Remark. This theorem gives a topological version of a theorem of J. Kist ([6; p. 33]).

Corollary 1. *If B is a proper open compressed ideal of a compact semigroup S, then $B = \bigcap_\alpha \mathrm{Tod}_B\, e_\alpha$, where e_α runs through the set $E \cap (S - B)$.*

Proof. Since $e_\alpha \notin B$, so $\mathrm{Tod}_B\, e_\alpha \subset J_0(S - e_\alpha) = P_\alpha$. By theorem 3.1, we therefore have $B \subset \bigcap_\alpha \mathrm{Tod}_B\, e_\alpha \subset \bigcap_\alpha P_\alpha = B$. Thus $B = \bigcap_\alpha \mathrm{Tod}_B\, e_\alpha$.

Note. The result of Corollary 1 can be sharpened to $B = \bigcap_\beta \mathrm{Tod}_B\, e_\beta$, where e_β runs through all non-minimal idempotents in $E \cap (S - B)$. This is because if $e_1 \leqq e_2$ then $\mathrm{Tod}_B\, e_2 \subset \mathrm{Tod}_B\, e_1$.

Corollary 1 of Theorem 3.1 has an immediate application.

Theorem 3.2. *Let S be a compact quasi-normal semigroup. Let B be an open compressed ideal of S such that B can be expressed as the intersection of all maximal ideals of S. If $E \cap (S - B)$ consists of B-primitive idempotents only, then S can be expressed as a finite union of sets $\mathrm{Tod}_B\, e_\sigma$, each e_α being some B-primitive idempotent of S.*

Proof. By Theorem 2.8, each $\mathrm{Tod}_B\, e_\alpha$ is an open prime ideal containing B. So, by Š. Schwarz [11], $\mathrm{Tod}_B\, e_\alpha$ is a maximal ideal of S. Denote $S - \mathrm{Tod}_B\, e_\alpha$ by A_α. Clearly, for each α, $e_\alpha \in A_\alpha$. Now by a result of W. M. Faucett - R. J. Koch - K. Numakura [3], each A_α is disjoint union of groups and the product of two idempotents in A_α lies in A_α. We claim that each A_α is a group. For if $e^{*2} = e^* \in A_\alpha$ then $e_\alpha e^* \in A_\alpha$. As e_α is a B-primitive idempotent, we must have $e_\alpha e^* = e_\alpha$, that is, $e^* = e_\alpha$. Our claim is established. Then, as it is proved by Š. Schwarz [10; p. 465] that for each $x \in A_\alpha$, the principal ideal generated by x must have empty intersection with A_β for every $\beta \neq \alpha$, thus $e_1 S \cap A_\beta = \emptyset$ for every β except $\beta = 1$. In particular, $e_1 e_2 \in \mathrm{Tod}_B\, e_\beta$ for all β except $\beta = 1$. Similarly, $e_2 e_1 \subset \mathrm{Tod}_B\, e_\Gamma$ for all Γ except $\Gamma = 2$. So that, by corollary 1 of theorem 3.1, we have $e_1 e_2 \in \bigcap_\alpha \mathrm{Tod}_B\, e_\alpha = B$. Hence $e_1 \in \mathrm{Tod}_B\, e_2$. As A_1 is a group, then for any $x \in A_1$, $x = e_1 x \in \mathrm{Tod}_B\, e_2$, that is, $A_1 \subset \mathrm{Tod}_B\, e_2$. Therefore S is covered by all open sets $\mathrm{Tod}_B\, e_\alpha$ for all α. Since S is compact, there is a finite number n such that $S = \bigcup_{\alpha=1}^{n} \mathrm{Tod}_B\, e_\alpha$, which completes the proof.

Let B be an open ideal of a compact semigroup S; it is natural to ask whether the Thierrin radical of B is open or not. We give here a partial answer to this problem.

Lemma 3.3. *Let S be a compact normal semigroup and B an ideal of S. Then the Thierrin radical of B contains exactly those idempotents which are contained in B.*

Proof. Let $T(B)$ denote the Thierrin radical of B. Suppose if possible that there exists an idempotent $e \in T(B) - B$. Then $B \subset J_0(S - e)$, where $J_0(S - e)$ is known to be an open prime ideal of S (by Numakura [8]). Since S is a normal semigroup, so $J_0(S - e)$ is therefore a completely prime ideal of S, and hence compressed. As $T(B)$ is defined to be the smallest compressed ideal containing B, so $T(B) \subset J_0(S - e)$. This contradicts $e \in T(B)$. Our proof is completed.

Corollary. *Let S be a normal semigroup. If C is a compressed ideal of S containing an open ideal B, then C is not the Thierrin radical of B if and only if C contains an idempotent not in B.*

Theorem 3.4. *Let S be a compact normal semigroup. If B is an open ideal of S, then the Thierrin radical of B is open.*

Proof. Let $E(T(B))$ denote the set $E \cap (S - T(B))$, let $E(B)$ denote the set $E \cap \cap (S - B)$, where E is the set of idempotents of S, which is well known to be a closed subset of S [Page 22, 9]. Since $T(B)$ is a compressed ideal of S, so $x \notin T(B)$ implies $x^n \notin T(B)$ for any integer n. Let $\Gamma(x)$ denote $\overline{\{x_n\}_{n=1}^{\infty}}$. Then since S is compact, there exists an idempotent $e^2 = e \in \Gamma(x)$. We claim that $e \notin T(B)$. For if $e \in T(B)$, then by lemma 3.3, we must have $e \in B$ and hence $e\,\Gamma(x) = K(x) \subset B \subset T(B)$, where $K(x) = \bigcap_{i=1}^{\infty} \overline{\{x^i \mid i \geqq n\}}$. Since B is open, so there is an integer n such that $x^n \in T(B)$. As $T(B)$ is compressed, hence $x \in T(B)$, a contradiction. Our claim is therefore established. Hence $T(B) \subset \bigcap\{J_0(S - e) \mid e \in E(T(B))\}$. By similar arguments in the proof of Theorem 3.1, we can prove that $T(B) = \bigcap\{J_0(S - e) \mid e \in E(T(B))\} = J_0(S - E(T(B)))$. Applying lemma 3.3 again, we then have $T(B) = J_0(S - E(B))$. Since B is open, so $E(B)$ is closed and hence $T(B)$ is open.

Definition 3.5. A subset M of a semigroup S is an M-system if and only if $a, b \in M$ imply that there is an element $x \in S$ such that $axb \in M$. By McCoy's characterisation of prime ideals, the complement of a prime ideal in S is an M-system.

Using the notion of M-system, V. A. Andrunakevic - Ju. M. Rjabuhin [1] and K. Iseki [5] proved in rings and semi-rings that if B is a compressed ideal and A is any ideal such that $A \supset B$, then there is some completely prime ideal P so that $P \supset B$ but $P \not\supset A$. We now generalize this result to compact semigroups.

Theorem 3.6. *Let S be a compact semigroup and B an open compressed ideal of S. If A is any ideal such that $A \supsetneqq B$, then there is some open completely prime ideal P so that $P \supset B$ but $P \not\supset A$.*

Before proving theorem 3.6, we need the following lemma.

Lemma 3.7. *Let S be a compact semigroup. Then the closure of an M-system of S is also an M-system.*

Proof. Let M be an M-system of S. Suppose $\overline{M}$ is not an M-system, then for any $a, b \in \overline{M}$ there does not exist element $x \in S$ such that $axb \in \overline{M}$, that is, $aSb \cap \overline{M} = \emptyset$. Since S is compact, there exist neighbourhoods V of a, W of b such that $VSW \cap \cap \overline{M} = \emptyset$. Since $a, b \in \overline{M}$, there are elements $a_1 \in V \cap M$ and $b_1 \in W \cap M$. Because M is an M-system, there is an element $x \in S$ such that $a_1xb_1 \in M$. Hence $a_1xb_1 \in \in VSW \cap M$, which is a contradiction.

We now turn to Theorem 3.6. Let a be any element in S such that $a \in A$, $a \notin B$. Since B is a compressed ideal, the M-system $M = \{a^i\}$, $i = 1. 2, \ldots$, does not intersect B. We denote by M^* a maximal M-system containing M and not intersecting the ideal B. Since B is open, $S - B$ is closed. Thus, by lemma 3.7, the maximal M-system M^* contained in $S - B$ must be closed. Let P be the complement of M^*. Then P will be an open ideal containing B. By the result of Andrunakevic-Rjabuhin [1] and K. Iseki [5], P is known to be an open completely prime ideal of S with the desired property.

Corollary 1. *If B is an open compressed ideal of a compact semigroup S, then B is the intersection of all the open completely prime ideals which contain it.*

Lemma 3.8. *Let B be a compressed ideal of S. If x is a topological B-factor, then the following assertions are equivalent.*

(i) $\mathrm{Tod}_B\, x$ *is maximal.*

(*That is, if* $\mathrm{Tod}_B\, x \subset \mathrm{Tod}_B\, y$, *then* $\mathrm{Tod}_B\, y = S$ *or* $\mathrm{Tod}_B\, y = \mathrm{Tod}_B\, x$)

(ii) $\mathrm{Tod}_B\, x$ *is a prime ideal of S.*

(iii) $\mathrm{Tod}_B\, x$ *is a minimal prime ideal containing B.*

(iv) $\mathrm{Tod}_B\, x$ *is a completely prime ideal of S.*

Proof. (i) implies (ii). Since B is an ideal, then $\mathrm{Tod}_B\, x \subset \mathrm{Tod}_B\, ax$ for any $a \in S$. If $\mathrm{Tod}_B\, x \neq \mathrm{Tod}_B\, ax$, then $\mathrm{Tod}_B\, ax = S$. In particular, $x \in \mathrm{Tod}_B\, ax$. Since B is compressed, $ax^2 \in B$ implies $ax \in B$, that is, $a \in \mathrm{Tod}_B\, x$. Thus if $a \notin \mathrm{Tod}_B\, x$, then $\mathrm{Tod}_B\, ax = \mathrm{Tod}_B\, x$. Since $x \notin \mathrm{Tod}_B\, x$ and $ax \notin \mathrm{Tod}_B\, ax = \mathrm{Tod}_B\, x$, it follows that $\mathrm{Tod}_B\, x$ is completely prime, so of course $\mathrm{Tod}_B\, x$ is prime.

(ii) implies (iii). Suppose Q is a prime ideal such that $B \subset Q \subset \mathrm{Tod}_B\, x$. We shall prove $Q = \mathrm{Tod}_B\, x$. Let $t \in \mathrm{Tod}_B\, x$, then $tx \in B$ and hence $Stx \subset B$. Since B is compressed, we have $xSt \subset B \subset Q$. As Q is assumed to be a prime ideal, thus either $x \in Q$ or $t \in Q$. Clearly $x \in Q \subset \mathrm{Tod}_B\, x$ is impossible. Therefore $t \in Q$ so $\mathrm{Tod}_B\, x = Q$.

(iii) implies (iv). Suppose that $ab \in \mathrm{Tod}_B\, x$. Then $abx \in B$. Since B is a compressed ideal, we have $bxa \in B$, $Sbxa \subset B$ and $aSbx \subset B$. Thus $aSb \subset \mathrm{Tod}_B\, x$. $\mathrm{Tod}_B\, x$ is prime, we must have $a \in \mathrm{Tod}_B\, x$ or $b \in \mathrm{Tod}_B\, x$.

(iv) implies (i). Suppose that there is a topological B-factor $y \in S - B$ such that $B \subset \mathrm{Tod}_B\, x \subset \mathrm{Tod}_B\, y$. Then pick any $a \in \mathrm{Tod}_B\, y$, hence $ay \in B \subset \mathrm{Tod}_B\, x$. Since $\mathrm{Tod}_B\, x$ is completely prime, then either $a \in \mathrm{Tod}_B\, x$ or $y \in \mathrm{Tod}_B\, y$. Clearly $y \notin \mathrm{Tod}_B\, y$. Therefore $\mathrm{Tod}_B\, y = \mathrm{Tod}_B\, x$.

Remark. The above lemma is taken from the work of W. H. Cornish and P. N. Stewart in ring theory [2].

Theorem 3.9. *Let B be a compressed ideal of a compact semigroup S. If x is an element in $S - B$ such that* $\mathrm{Tod}_B\, x$ *is a maximal proper ideal of S, then the following assertions are true and equivalent.*

(i) $S - \mathrm{Tod}_B\, x$ *contains an idempotent and the product of any two idempotents of* $S - \mathrm{Tod}_B\, x$ *lies in* $S - \mathrm{Tod}_B\, x$.

(ii) $\mathrm{Tod}_B\, x$ *is an open completely prime ideal.*

(iii) $\mathrm{Tod}_B\, x$ *is a minimal open prime ideal containing B.*

Proof. The equivalence of (i) and (ii) follows immediately from a theorem of W. M. Faucett - R. J. Koch - K. Numakura [3; p. 656]. The equivalence of (ii) and (iii) follows from lemma 3.8. As (ii) is always true under the assumption of the theorem, thus the above statements are true.

Finally we sum up the results on open compressed ideals obtained in this section.

Theorem 3.10. *Let S be a compact semigroup and B an open ideal of S. If $E \cap (S - B)$ is finite, then the following assertions are equivalent:*

(i) *B is an open compressed ideal.*

(ii) *B is the intersection of finite number of open prime ideals.*

(iii) *B is the intersection of finite number of the open ideals* $\mathrm{Tod}_B\, e_i$, *where e_i runs through $E \cap (S - B)$.*

(iv) *B is the intersection of finite number of open completely prime ideals.*

Proof. (i) implies (ii). This follows immediately from Theorem 3.1.

(ii) implies (iii). Since B is open, each $\mathrm{Tod}_B\, e_i$ is open. The fact that $B = \bigcap_{i=1}^{n} \mathrm{Tod}_B\, e_i$, $e_i \in E \cap (S - B)$ follows from Corollary 1 of Theorem 3.1.

(iii) implies (iv). If $\mathrm{Tod}_B\, e_i$ itself is a prime ideal, then by the proof of lemma 3.8 $\mathrm{Tod}_B\, e_i$ is a completely prime ideal. If $\mathrm{Tod}_B\, e_i$ is not a prime ideal, then by proposition 2.2, $\mathrm{Tod}_B\, e_i$ is always a compressed ideal. Apply theorem 3.6, $\mathrm{Tod}_B\, e_i = \bigcap_{i=1}^{n} P_i$, where P_i are open completely prime ideal of S. The finiteness follows from the fact the $E \cap (S - B)$ is finite, and each open prime ideal has the form $J_0(S - e_i)$.

(iv) implies (i). Obvious.

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