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UNORIENTED GRAPHS OF MODULAR LATTICES

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A lattice L is called discrete if each bounded chain of L is finite. All lattices dealt with in this note are assumed to be discrete. For $a, b \in L$, $a \leqq b$, the interval $[a, b]$ is the set $\{x \in L : a \leqq x \leqq b\}$. If $a < b$ and $[a, b] = \{a, b\}$, then $[a, b]$ is said to be a prime interval; this situation is also described by saying that b covers a or that a is covered by b.

To each lattice L there corresponds in a natural way an unoriented graph (an unoriented diagram) $G(L)$. The vertices of $G(L)$ are the elements of L; two vertices a, b are connected by an edge if and only if either a is covered by b or b is covered by a.

G. BIRKHOFF ([1], Problem 8) proposed the question to find necessary and sufficient conditions on a lattice L, in order that every lattice M whose unoriented graph is isomorphic with the graph of L be lattice-isomorphic with L. For the case when the lattices L and M are supposed to be distributive (or modular, respectively), this problem was solved in [3] (resp. [4]). Isomorphisms of unoriented diagrams of modular lattices were investigated also in [5].

The purpose of the present note is to show that if L and M are lattices whose unoriented graphs are isomorphic and if L is modular, then M is modular as well (Thm. 1.) (For finite lattices L this was proved in [4].) An analogous statement is valid for distributive lattices (Thm. 3). This enables one to generalize some results of [4], [5] (Thms. 2, 4).

For the basic notions concerning lattices cf. Birkhoff [1] and GRÄTZER [2]. The lattice operations will be denoted by $\wedge$ and $\vee$. A discrete lattice L is modular if and only if it fulfils the following "covering" condition (1) and the condition (1′) dual to (1):

(1) If a, b are elements of L such that a and b cover $a \wedge b$, then $a \vee b$ covers both elements a and b.

Let L be a modular lattice and let L' be a lattice such that there exists an isomorphism φ of $G(L)$ onto $G(L')$. Let a, b, u be distinct elements of L such that a, u are connected by an edge in $G(L)$ and b, u are connected by an edge in $G(L)$. Then $\varphi(a)$, $\varphi(u)$ are connected by an edge in $G(L')$, and similarly for $\varphi(b)$, $\varphi(u)$.

Let us remark that if x, y, z are elements of a discrete lattice X and if x is covered by y, z (or x covers y, z), then $x = y \wedge z$ (resp. $x = y \vee z$).

Lemma 1. *Let*

$$u < a\,, \quad u < b\,, \quad a \vee b = v\,,$$

$$\varphi(u) < \varphi(a)\,, \quad \varphi(u) < \varphi(b) < \varphi(v)\,.$$

Then $\varphi(a) \vee \varphi(b) = \varphi(v)$ and $\varphi(v)$ covers both elements $\varphi(a)$ and $\varphi(b)$.

Proof. According to (1), v covers a and b. Hence $\varphi(a)$, $\varphi(v)$ are connected by an edge in $G(L')$ and similarly for $\varphi(b)$, $\varphi(v)$. Hence $\varphi(b)$ is covered by $\varphi(v)$. Suppose that $\varphi(v)$ is covered by $\varphi(a)$. Then we would have

$$\varphi(u) < \varphi(b) < \varphi(v) < \varphi(a)$$

and this is a contradiction, because $\varphi(u)$ is covered by $\varphi(a)$. Thus $\varphi(a)$ is covered by $\varphi(v)$. Therefore $\varphi(a) \vee \varphi(b) = \varphi(v)$.

Lemma 2. *Let*

$$u < a\,, \quad u < b\,, \quad a \vee b = v\,,$$

$$\varphi(u) < \varphi(a)\,, \quad \varphi(u) < \varphi(b)\,.$$

Then $\varphi(a) \vee \varphi(b) = \varphi(v)$ and $\varphi(v)$ covers both elements $\varphi(a)$ and $\varphi(b)$.

Proof. Analogously as in the proof of Lemma 1 we conclude that $\varphi(a)$, $\varphi(v)$ are connected by an edge in $G(L')$ and similarly for $\varphi(b)$, $\varphi(v)$. Obviously $\varphi(a) \wedge \varphi(b) = \varphi(u)$. If

$$\varphi(a) > \varphi(v) \quad \text{and} \quad \varphi(b) > \varphi(v)\,,$$

then $\varphi(a) \wedge \varphi(b) = \varphi(v) \neq \varphi(u)$, which is a contradiction. Hence either $\varphi(a) < \varphi(v)$ or $\varphi(b) < \varphi(v)$. For completing the proof it suffices to apply Lemma 1.

The proof of the following lemma is analogous to that of Lemma 2.

Lemma 2′. *Let*

$$u > a\,, \quad u > b\,, \quad a \wedge b = v\,,$$

$$\varphi(u) < \varphi(a)\,, \quad \varphi(u) < \varphi(b)\,.$$

Then $\varphi(a) \vee \varphi(b) = \varphi(v)$ and $\varphi(v)$ covers $\varphi(a)$ and $\varphi(b)$.

Lemma 3. *Let*

$$u < a\,, \quad u < b\,, \quad a \vee b = v\,,$$

$$\varphi(a) < \varphi(u) < \varphi(b)\,.$$

Then $\varphi(a)$ is covered by $\varphi(v)$ and $\varphi(v)$ is covered by $\varphi(b)$.

Proof. The elements $\varphi(a)$, $\varphi(v)$ are connected by an edge in $G(L')$, hence $\varphi(a)$, $\varphi(v)$ are comparable, and similarly for $\varphi(b)$, $\varphi(v)$. If $\varphi(v) < \varphi(a)$, then

$$\varphi(v) < \varphi(a) < \varphi(u) < \varphi(b)\,,$$

hence neither $\varphi(v)$ is covered by $\varphi(b)$ nor $\varphi(b)$ is covered by $\varphi(v)$, which is a contradiction. Thus $\varphi(a) < \varphi(v)$. Analogously, if $\varphi(b) < \varphi(v)$, then

$$\varphi(a) < \varphi(u) < \varphi(b) < \varphi(v)\,,$$

which is impossible, because $\varphi(a)$ and $\varphi(v)$ are connected by edge in $G(L')$. Thus $\varphi(v) < \varphi(b)$. Therefore $\varphi(a)$ is covered by $\varphi(v)$ and $\varphi(v)$ is covered by $\varphi(b)$.

Dually, we can prove

Lemma 3'. *Let*

$$u > a\,, \quad u > b\,, \quad a \wedge b = v\,,$$
$$\varphi(b) < \varphi(u) < \varphi(a)\,.$$

Then $\varphi(b)$ is covered by $\varphi(v)$ and $\varphi(v)$ is covered by $\varphi(a)$.

Lemma 4. *Let $a_0, a_1, \ldots, a_n \in L$, $b \in L$,*

$$a_0 < b\,, \quad a_i < a_{i+1} \quad (i = 1, \ldots, n-1)\,,$$
$$\varphi(a_0) < \varphi(b)\,, \quad \varphi(a_i) > \varphi(a_{i+1}) \quad (i = 1, \ldots, n-1)\,.$$

Assume that all intervals $[a_0, b]$, $[a_i, a_{i+1}]$ $(i = 1, \ldots, n-1)$ are prime. Put $t_i = a_i \vee b$ $(i = 0, \ldots, n)$. Then $\varphi(a_i)$ is covered by $\varphi(t_i)$ for $i = 0, \ldots, n$ and $\varphi(t_i)$ covers $\varphi(t_{i+1})$ for $i = 0, \ldots, n-1$.

Proof. We proceed by induction on n. For $n = 1$ the assertion is valid according to Lemma 3. Let $n > 1$ and assume that the assertion is valid for $n - 1$. By Lemma 3, the element $\varphi(a_1)$ is covered by $\varphi(t_1)$ and $\varphi(t_1)$ is covered by $\varphi(b) = \varphi(t_0)$. Now consider the elements $a_1, \ldots, a_n$, t_1. The element a_1 is covered by t_1 and $\varphi(a_1) < \varphi(t_1)$. Moreover, for $i = 1, \ldots, n$ we have

$$t_i = a_i \vee b = (a_1 \vee a_i) \vee b = (a_1 \vee b) \vee (a_i \vee b) =$$
$$= t_1 \vee a_i \vee b = a_i \vee t_1\,.$$

Therefore, according to the assumption, $\varphi(a_i)$ is covered by $\varphi(t_i)$ for $i = 1, \ldots, n$ and $\varphi(t_i)$ covers $\varphi(t_{i+1})$ for $i = 1, \ldots, n-1$. The proof is complete.

Lemma 5. *Let $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m \in L$, $a_0 = b_0$, $a_n = b_m$. Suppose that*

(i) *a_i is covered by a_{i+1} and $\varphi(a_i)$ is covered by $\varphi(a_{i+1})$ for $i = 1, \ldots, n-1$;*

(ii) $\varphi(b_j)$ *is covered by* $\varphi(b_{j+1})$ *for* $j = 1, \ldots, m - 1$. *Then* $m = n$ *and* $b_j < b_{j+1}$ *holds for* $j = 1, \ldots, m - 1$.

Proof. We proceed by induction on n. If $n = 1$. then the assertion is obviously valid. Assume that $n > 1$ and that the assertion holds for $n - 1$. Clearly $m > 1$. Let us distinguish two cases.

(a) Let $b_0 < b_1$. Denote $a_1 \vee b_1 = c_2$. Then c_2 covers both elements a_1, b_1 and according to Lemma 1, $\varphi(c_2)$ covers both elements $\varphi(a_1), \varphi(b_1)$; moreover, $\varphi(c_2) \leqq \leqq \varphi(a_n)$. By the assumption there are elements $c_3, \ldots, c_n \in L$ such that $c_n = a_n$, c_i is covered by c_{i+1} and $\varphi(c_i)$ is covered by $\varphi(c_{i+1})$ for $i = 1, \ldots, n - 1$. Because $\{b_1, c_2, c_3, \ldots, c_n\}$ is a maximal chain in L, by the assumption we have $n - 1 = = m - 1$ and $b_j < b_{j+1}$ for $j = 1, \ldots, m - 1$.

There are elements $c_3, c_4, \ldots, c_k \in L$ such that $c_k = a_n$ and $\varphi(c_i)$ is covered by $\varphi(c_{i+1})$ for $i = 3, \ldots, k - 1$. By using the induction assumption for the elements $a_1, a_2, \ldots, a_n$; $c_2, c_3, \ldots, c_k$, we obtain that $k = n$ and that c_i is covered by c_{i+1} for $i = 3, 4, \ldots, k - 1$. Now we use the induction assumption for the elements $b_1, c_2, c_3, \ldots, c_n$; $b_2, \ldots, b_m$ and we infer that $m = n$ and that b_i is covered by b_{i+1} for $i = 1, \ldots, n - 1$.

(b) Suppose that $b_0 > b_1$. If $b_j > b_{j+1}$ for $j = 0, \ldots, m - 1$, then $a_0 = b_0 > > b_m = a_n$, which contradicts (i). Thus there exists a minimal j, $1 < j < m$, with $b_j < b_{j+1}$; we denote this j by j_0. Denote $x = b_0 \vee b_{j_0+1}$, $t_1 = b_{j_0+1} \vee b_1$. According to lemma 4,

$$x \text{ covers } a_0 \text{ and } \varphi(x) \text{ covers } \varphi(a_0)\,,$$

$$x \text{ covers } t_1 \text{ and } \varphi(x) \text{ is covered by } \varphi(t_1)\,,$$

$$\varphi(t_1) \leqq \varphi(b_{j_0+1})\,.$$

If $x = a_1$, then we consider the chain $C_1 = \{\varphi(a_1), \ldots, \varphi(a_n)\}$ in L'. There exists a maximal chain C_2 in $[\varphi(a_1), \varphi(a_n)]$ such that $\varphi(t_1), \varphi(b_{j_0+1}), \ldots, \varphi(b_m) \in C_2$. Since card $C_1 < n + 1$, by the induction assumption (by considering the chains C_1 and C_2) we obtain that the element $a_1 = x$ is covered by the element t_1, which is a contradiction.

If $x \neq a_1$, then we put $x \vee a_1 = y$. By Lemma 1, $\varphi(y)$ covers $\varphi(a_1)$ and $\varphi(x)$. Clearly $\varphi(y) \leqq \varphi(a_n)$. By considering the chain C_1 we infer (by the induction assumption) that there are elements $y_2, \ldots, y_n = a_n$ in L, $y_2 = y$ such that $C_3 = = \{\varphi(a_1), \varphi(y_2), \ldots, \varphi(y_n)\}$ is a maximal chain in $[\varphi(a_1), \varphi(a_n)]$ and y_i is covered by y_{i+1} for $i = 2, \ldots, n - 1$. Thus $C_4 = \{\varphi(x), \varphi(y_2), \ldots, \varphi(y_n)\}$ is a maximal chain in $[\varphi(x), \varphi(a_n)]$. Because card $C_4 < n + 1$, and $\varphi(x) < \varphi(t_1) \leqq \varphi(b_{j_0+1}) \leqq \varphi(a_n)$, by the induction assumption we must have $x < t_1$, which is a contradiction.

Analogously we can prove

Lemma 5'. *Let* $a_0, \ldots, a_n, b_0, \ldots, b_m$ *be as in Lemma 5 with the distinction that* a_i *covers* a_{i+1} *for* $i = 0, \ldots, n - 1$. *Then* $m = n$ *and* $b_j > b_{j+1}$ *for* $j = 1, \ldots, m - 1$.

Lemma 6. *Let $a, u, b, v \in L$ and assume that a is covered by u and u is covered by b, $\varphi(u)$ is covered by $\varphi(a)$ and $\varphi(b)$; $\varphi(v)$ covers $\varphi(a)$ and $\varphi(b)$. Then a is covered by v and v is covered by b.*

Proof. The elements v and a are comparable. If $v < a$, then $v < a < u < b$, hence v is not covered by b and b is not covered by a, a contradiction. Thus $a < v$ and hence a is covered by v. The remaining part of the proof is analogous.

Lemma 7. *Let $a_0, b_0, u_0, a_1, b_1, u_1, v_1 \in L$ such that x_1 covers x_0 for each $x \in \in \{a, b, u\}$, $\varphi(x_1)$ covers $\varphi(x_0)$ for each $x \in \{a, b, u\}$;*

a_i is covered by u_i and u_i is covered by b_i for $i = 0, 1$;

$\varphi(u_i)$ is covered by $\varphi(a_i)$ and $\varphi(b_i)$ for $i = 0, 1$;

$\varphi(v_1)$ covers $\varphi(a_1)$ and $\varphi(b_1)$.

Then there is $v_0 \in L$ such that $\varphi(v_0)$ covers $\varphi(a_0)$ and $\varphi(b_0)$.

Proof. By Lemma 6, a_1 is covered by v_1 and v_1 is covered by b_1. Put $v_0 = b_0 \wedge v_1$. According to Lemma 3′, $\varphi(b_0)$ is covered by $\varphi(v_0)$ and $\varphi(v_0)$ is covered by $\varphi(v_1)$. We have $a_0 < v_1$, $a_0 < b_0$, thus $a_0 \leqq v_0$. Because L is modular and $\{a_0, a_1, v_1\}$ is a maximal chain in $[a_0, v_1]$, we obtain that $\{a_0, v_0, v_1\}$ must be a maximal chain in $[a_0, v_1]$, hence a_0 is covered by v_0. If $\varphi(v_0) < \varphi(a_0)$, then $\varphi(v_0)$ is not covered by $\varphi(v_1)$, which is a contradiction. Thus $\varphi(v_0) > \varphi(a_0)$ and hence $\varphi(v_0)$ covers $\varphi(a_0)$. The proof is complete.

An element $p \in L$ will be said to have the property (α) with respect to elements $q, r \in L$ if

(i) $r \neq q$;

(ii) $\varphi(p)$ is covered by both elements $\varphi(r)$ and $\varphi(q)$;

(iii) either $\varphi(r)$ or $\varphi(q)$ is not covered by $\varphi(r) \vee \varphi(q)$.

Lemma 8. *Suppose that $u \in L$ has the property (α) with respect to elements $a, b \in L$. Then there are elements $u_1, a_1, b_1 \in L$ such that u_1 has the property (α) with respect to a_1, b_1, the element $\varphi(u)$ is covered by $\varphi(u_1)$ and*

$$\varphi(u_1) < \varphi(a) \vee \varphi(b) = \varphi(a_1) \vee \varphi(b_1) .$$

Proof. If a and b cover u then according to Lemma 2, u cannot have the property (α) with respect to a and b. If both a and b are covered by u then the same holds by Lemma 2′. Hence we may suppose that a is covered by u, and u is covered by b.

Let $x_0, x_1, \ldots, x_n \in L$ with $x_0 = u$, $x_1 = a$ such that $\{\varphi(x_0), \varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n)\}$ is a maximal chain in $[\varphi(u), \varphi(a) \vee \varphi(b)]$. If x_i covers x_{i+1} for $i = 0, \ldots, n - 1$, then by Lemma 5′ we would have $u > b$, which is a contradiction. Hence there is

$i_0 > 0$ such that x_i is covered by x_{i_0+1}. Let i_0 be the first index with this property. Put $t_i = x_i \vee x_{i_0+1}$ for $i = 0, \ldots, i_0$. According to Lemma 4,

t_i covers x_i and $\varphi(t_i)$ covers $\varphi(x_i)$ for $i = 0, \ldots, i_0$;

t_i covers t_{i+1} and $\varphi(t_i)$ is covered by $\varphi(t_{i+1})$ for $i = 0, \ldots, i_0 - 1$.

If $t_0 = b$, then $\varphi(t_1) = \varphi(x_1) \vee \varphi(t_0) = \varphi(a) \vee \varphi(b)$ and because $\varphi(t_1)$ covers $\varphi(x_1) = \varphi(a)$ and $\varphi(t_0)$, we have a contradiction. Thus $t_0 \neq b$. Denote $t_0 = u_1$, $t_1 = a_1$, $t_0 \vee b = b_1$. Then b_1 covers both b and u_1. By Lemma 2 we have $\varphi(b_1) = \varphi(t_0) \vee \varphi(b)$. Hence $\varphi(b_1)$ covers both $\varphi(b)$ and $\varphi(u_1)$. Moreover,

$$\varphi(a) \leqq \varphi(a_1) = \varphi(t_1) \leqq \ldots \leqq \varphi(t_{i_0+1}) = \varphi(x_{i_0+1}) \leqq \varphi(x_n) = \varphi(a) \vee \varphi(b),$$

$$\varphi(t_0) \leqq \varphi(t_1) = \varphi(a_1),$$

hence $\varphi(t_0) \leqq \varphi(a) \vee \varphi(b)$ and therefore

$$\varphi(b) < \varphi(b_1) = \varphi(t_0) \vee \varphi(b) \leqq \varphi(a) \vee \varphi(b),$$

$$\varphi(a_1) \vee \varphi(b_1) = \varphi(a) \vee \varphi(b).$$

The elements $\varphi(a_1)$, $\varphi(b_1)$ are uncomparable, thus $\varphi(a_1) < \varphi(a) \vee \varphi(b)$ and $\varphi(b_1) < \varphi(a) \vee \varphi(b)$. Suppose that the element $\varphi(a) \vee \varphi(b)$ covers both elements $\varphi(a_1)$ and $\varphi(b_1)$. Then according to Lemma 7 (applied to the elements a, x_0, b; $t_1 = a_1, t_0, b_1$, and $v_1 = \varphi^{-1}(\varphi(a) \vee \varphi(b))$ there is $v_0 \in L$ such that $\varphi(v_0)$ covers $\varphi(a)$ and $\varphi(b)$. Obviously $\varphi(v_0) = \varphi(a) \vee \varphi(b)$, hence $\varphi(a) \vee \varphi(b)$ covers both $\varphi(a)$ and $\varphi(b)$, which is a contradiction. Therefore either $\varphi(a_1)$ or $\varphi(b_1)$ is not covered by $\varphi(a) \vee \varphi(b) = \varphi(a_1) \vee \varphi(b_1)$. Thus u_1 has the property (α) with respect to a_1, b_1 and $\varphi(u_1) < \varphi(a) \vee \varphi(b)$.

Lemma 9. *There does not exist elements $u, a, b \in L$ such that u has the property (α) with respect to a, b (i.e., the covering condition (1) is valid for L').*

Proof. Suppose that there are elements $u, a, b \in L$ such that u has the property (α) with respect to a, b. From Lemma 8 it follows by induction, that there are elements $u_n, a_n, b_n \in L$ ($n = 1, 2, \ldots$) such that

(i) u_n has the property (α) with respect to a_n, b_n,

(ii) $\varphi(u) < \varphi(u_1) < \varphi(u_2) < \ldots < \varphi(a) \vee \varphi(b)$.

The relation (ii) cannot hold because L' is discrete. Hence we have a contradiction.

By a dual argument we can verify

Lemma 10. *Let $a, b, u \in L$ such that $\varphi(u)$ covers $\varphi(a)$ and $\varphi(b)$. Then $\varphi(a) \wedge \varphi(b)$ is covered by $\varphi(a)$ and $\varphi(b)$.*

From Lemma 9 and Lemma 10 we obtain:

Theorem 1. *Let L and L' be discrete lattices such that the unoriented graphs $G(L)$ and $G(L')$ are isomorphic. If L is modular, then L' is modular as well.*

For any lattice A, we denote by $A^{\sim}$ the lattice dual to A.
The following theorem generalizes Thm. 7.8, [3] and Thm. 1, [4].

Theorem 2. *Let L be a discrete modular lattice. Let L' be a discrete lattice such that $G(L)$ is isomorphic to $G(L')$. Then there are lattices A, B such that L is isomorphic with the direct product $A \times B$ and L' is isomorphic with $A^{\sim} \times B$.*

The proof follows from Thm. 1, and Thm. 1, [4].

Theorem 3. *Let L be a discrete distributive lattice and let L' be a discrete lattice such that $G(L)$ is isomorphic with $G(L')$. Then L' is distributive.*

Proof. Let A, B be as in Thm. 2. Since L is distributive, the lattices A, B must be distributive and hence $A^{\sim}$ is distributive. By Thm. 2, L' is distributive.
Let us remark that if L and L' are discrete lattices such that $G(L)$ is isomorphic with $G(L')$ and L is semimodular, then L' need not be semimodular.

Theorem 4. *Let L be a discrete modular lattice. Then the following conditions are equivalent:*

(a) *If L' is a discrete lattice such that $G(L')$ is isomorphic to $G(L)$, then L' is isomorphic to L.*

(b) *Each direct factor of L is self-dual.*

(c) *Each undecomposable direct factor of L is self-dual.*

The proof follows from Thm. 1 and [5], Thm. 3.7.

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