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TRANSLATION INVARIANT LINEAR OPERATORS AND GENERALIZED FUNCTIONS

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Scattered throughout the literature are various results which characterize continuous linear maps between generalized function spaces (or test spaces) which commute with translations. For example, it is well-known that if L is a continuous linear map from $\mathscr{D}$ into $\mathscr{D}'$ which commutes with translations, then there is a unique $T \in E'$ such that $L\phi = T * \phi$ for $\phi \in \mathscr{D}$ ([14]). Applications of such results are well-known; see for example [10] or [4], Corollary 2 of 5.11.3. In this paper we attempt to collect these various results and put them in a unified setting which should shed some light on the reasons for their validity. Many of the results are well-known, but by considering them in a general setting they become more transparent. A number of the results in sections 1 and 4 do not seem to have been recorded although they may be „well-known".

We adopt the approach to generalized functions ($g.f.$'s) as in GELFAND and SHILOV ([8]). For any complex-valued function f on $\mathbf{R}^n$ and $h \in \mathbf{R}^n$ denote by $\tau_h f$ the function $\tau_h f : x \to f(x + h)$. A *test space* (over $\mathbf{R}^n$) is a subspace Φ of $C^\infty(\mathbf{R}^n)$ with a locally convex Hausdorff topology such that

(i) $\mathscr{D} \subseteq \Phi$ with the injection continuous and $\mathscr{D}$ dense in Φ;

(ii) if the net $\{\phi_r\}$ converges to 0 in Φ, then for each $x \in \mathbf{R}^n$, $\phi_r(x) \to 0$;

(iii) if $P(D)$ is a partial differential operator with constant coefficients, then the map $\phi \to P(D)\, \phi$ is continuous on Φ;

(iv) for each $x \in \mathbf{R}^n$ and $\phi \in \Phi$, $\tau_x\phi \in \Phi$ and the map $\tau_x : \phi \to \tau_x\phi$ is continuous on Φ.

(Condition (iv) is usually not assumed for test spaces ([8], [7]); but since we will be dealing exclusively with problems concerning convolution, we assume this condition to simplify the terminology. See [7] and section III.3.1 of [8].) Note that (i) implies the dual, Φ', of Φ may be identified with a subspace of $\mathscr{D}'$, i.e., every element of Φ' is a distribution. We refer to the dual of a test space as a space of generalized functions (*g. f. space*).

If Φ is a test space, $\tau_h(h \in \mathbf{R}^n)$ will also denote the translation operator on Φ' defined as usual by $\langle \tau_h T, \phi \rangle = \langle T, \tau_{-h}\phi \rangle$ for $\phi \in \Phi$, $T \in \Phi'$. For $T \in \Phi'$ and $\phi \in \Phi$, the convolution of T and ϕ, $T * \phi$, is defined to be the function $T * \phi : x \to \langle T, \tau_x \phi \rangle$ $(x \in \mathbf{R}^n)$. (Here we employ the definition in [8]; unfortunately, the definition of convolution varies through the literature. See [20] for example.) If Φ and Ψ are both tests spaces, an element $T \in \Phi'$ is said to be a *convolution operator* from Φ into Ψ if $T * \phi \in \Psi$ for each $\phi \in \Phi$ and if the map $\phi \to T * \phi$ is continuous from Φ into Ψ. We denote by $\mathcal{O}'_c(\Phi, \Psi)$ the subspace of Φ' consisting of all convolution operators from Φ into Ψ. If $T \in \mathcal{O}'_c(\Phi, \Psi)$, the convolution of T and any $S \in \Psi'$ is the element $T * S \in \Phi'$ defined by $\langle T * S, \phi \rangle = \langle S, T * \phi \rangle$ for $\phi \in \Phi$. (Again, there are other definitions used in the literature.) That is, the map $S \to T * S$ from Ψ' into Φ' is just the conjugate of the map $\phi \to T * \phi$ from Φ into Ψ. From this observation it follows that the map $S \to T * S$ is continuous from Ψ' into Φ' when both of these spaces are equipped with the strong topologies (respectively, weak topologies [20], II.19.5). When $\Phi = \Psi$, we set $\mathcal{O}'_c(\Phi, \Psi) = \mathcal{O}'_c(\Phi)$.

In section 1 we consider linear continuous maps L between test spaces Φ and Ψ which commute with translations and show that any such map has the form $L\phi = T * \phi$ for a unique $T \in \mathcal{O}'_c(\Phi, \Psi)$. We then give characterizations of $\mathcal{O}'_c(\Phi, \Psi)$ for various test spaces Φ and Ψ. In section 2 a similar result is shown to hold for linear continuous maps between certain g.f. spaces. In section 3 we consider linear continuous maps from $\mathcal{D}$ into certain spaces of distributions $\mathcal{A}$ which we call convolution regular; $\mathcal{A}$ is *convolution regular* if $\mathcal{A} \subseteq \mathcal{D}'$ with continuous injection and if whenever $T * \phi \in \mathcal{A}$ for each $\phi \in \mathcal{D}$, then $T \in \mathcal{A}$. If $\mathcal{A}$ is convolution regular and L: $\mathcal{D} \to \mathcal{A}$ is linear, continuous and commutes with translations, the there is a unique $T \in \mathcal{A}$ such that $L\phi = T * \phi$. The class of convolution regular spaces of distributions is shown to be the natural class of spaces for which this conclusion is valid. In the final section we show that many of the familiar spaces of distributions are convolution regular so the results of section 3 are applicable to these spaces.

1. MAPPINGS BETWEEN TEST SPACES

Throughout this section Φ and Ψ will denote test spaces. A linear map $L : \Phi \to \Psi$ is said to *commute with translations* if $L(\tau_h \phi) = \tau_h L\phi$ for every $h \in \mathbf{R}^n$, $\phi \in \Phi$. We obtain immediately from the definition of convolution operator:

Proposition 1. *Let $S \in \mathcal{O}'_c(\Phi, \Psi)$ and denote by L the map from Φ into Ψ defined by $L\phi = S * \phi$. Then L is linear, continuous and commutes with translations.*

We next show that the converse of Proposition 1 holds, thus characterizing linear continuous maps between test spaces which commute with translations. As above, if u is a linear map $u : \Psi' \to \Phi'$, we say that *u commutes with translations* if $u(\tau_x T) = \tau_x u(T)$ for each $x \in \mathbf{R}^n$ and $T \in \Psi'$.

Lemma 2. *Let Φ, Ψ be test spaces. If a linear continuous map $L : \Phi \to \Psi$ commutes with translations, then so does its conjugate $L' : \Psi' \to \Phi'$.*

Proof. Let $x \in \mathbf{R}^n$, $T \in \Psi'$ and $\phi \in \Phi$. Then $\langle L'(\tau_x T), \phi \rangle = \langle T, L(\tau_{-x}\phi) \rangle = \langle \tau_x L'T, \phi \rangle$ which gives the desired result.

Theorem 3. *Let Φ, Ψ be test spaces. If $L : \Phi \to \Psi$ is a continuous linear map which commutes with translations, then there is a unique $S \in \mathcal{O}'_c(\Phi, \Psi)$ such that $L(\phi) = S * \phi$ for each $\phi \in \Phi$.*

Proof. Set $S = L'\delta = \delta L$. Then for $\phi \in \Phi$, $x \in \mathbf{R}^n$, we have $L(\phi)(x) = \langle \delta_x, L\phi \rangle = \langle \tau_{-x}\delta, L\phi \rangle = \langle L'(\tau_{-x}\delta), \phi \rangle = \langle \tau_{-x} L'\delta, \phi \rangle = \langle L'\delta, \tau_x \phi \rangle = \langle S, \tau_x\phi \rangle = S * \phi(x)$. This gives the representation $L\phi = S * \phi$, and the fact that L is continuous shows that $S \in \mathcal{O}'_c(\phi, \Psi)$. Uniqueness is clear.

To apply Theorem 3 it is necessary to know the space of convolution operators between the test spaces Φ and Ψ. For example, it is „well-known" that $\mathcal{O}'_c(\mathcal{D}, \mathcal{D}) = \mathcal{E}'$ and that $\mathcal{O}'_c(\mathcal{E}, \mathcal{E}) = \mathcal{E}'$. (The first equality follows from the hypocontinuity of the convolution from $\mathcal{E}' \times \mathcal{D} \to \mathcal{D}$ ([14], VI.4) and Corollary b of Theorem 1 of [2]; the second equality from the hypocontinuity of the convolution from $\mathcal{E}' \times \mathcal{E} \to \mathcal{E}$ ([14], VI.4).) Thus we obtain

Corollary 4. *If $L : \mathcal{D} \to \mathcal{D}(L : \mathcal{E} \to \mathcal{E})$ is linear, continuous and commutes with translations, then there is a unique $T \in \mathcal{E}'$ such that $L\phi = T * \phi$ for each $\phi \in \mathcal{D}$ ($\phi \in \mathcal{E}$).*

Remark 5. See [3] p. 121–122 for these results.

Consider the spaces $\mathcal{D}_+$ and $\mathcal{D}_-$ of L. Schwartz ([14], VI. 5). If $\phi \in \mathcal{D}_+$, then ϕ has support bounded on the left so that for any x, $\tau_x\phi \in \mathcal{D}_+$. Recall the dual of $\mathcal{D}_+$ is the space of distributions in $\mathcal{D}'(R)$ with support bounded on the right, denoted by $\mathcal{D}'_-$. Thus if $T \in \mathcal{D}'_-$ and $\phi \in \mathcal{D}_+$, $\langle T, \tau_x\phi \rangle = T * \phi(x)$ is an element of $\mathcal{D}_+$ and as in VI.5 of [14], the map $\phi \to T * \phi$ is continuous. (Note here we are using a different definition of $T * \phi$ than that employed in [14] so that the results are somewhat different.) Summarizing, we have $\mathcal{O}'_c(\mathcal{D}_+, \mathcal{D}_+) = \mathcal{D}'_-$. Also since $\mathcal{D}_+ \subseteq \mathcal{E}(R)$ and the injection is continuous, we must also have $\mathcal{O}'_c(\mathcal{D}_+, \mathcal{E}) = \mathcal{D}'_-$. From Theorem 3 we obtain

Corollary 6. *If $L : \mathcal{D}_+ \to \mathcal{D}_+$ ($L : \mathcal{D}_+ \to \mathcal{E}$) is continuous, linear and commutes with translations, then there is a unique $T \in \mathcal{D}'_-$ such that $L\phi = T * \phi$ for $\phi \in \mathcal{D}_+$.*

Remark 7. Of course, the analogous statements hold if $\mathcal{D}_+$ is replaced by $\mathcal{D}_-$. For the result stated in Corollary 6 see [16].

Definition 8. The test space Φ has an *equicontinuous translation* if for each $k > 0$, $\{\tau_h : |h| \leqq k\}$ is equicontinuous in $L(\Phi, \Phi)$.

Remark 9. This is somewhat like the conditions set forth in [8], III.3.1.

Definition 10. The test space Φ has a *differentiable translation* if for each j $(1 \leqq j \leqq n)$ and $\phi \in \Phi$, $\lim_{t \to 0} (\tau_{te_j}\phi - \phi)/t = D_j\phi$, where the convergence is in Φ. (Here e_j is the jth unit vector in $\boldsymbol{R}^n$ and $D_j\phi(x) = \partial\phi(x)/\partial x_j$. See [8], III.3.3.)

Remark 11. If Φ has a differentiable translation, note $T * \phi \in \mathscr{E}$ for $T \in \Phi'$, $\phi \in \Phi$ with $D^\alpha(T * \phi) = T * D^\alpha\phi$.

Proposition 12. *Let Φ have an equicontinuous and differentiable translation. Then $\mathcal{O}_c'(\Phi, \mathscr{E}) = \Phi'$.*

Proof. Let $k > 0$ and m be a positive integer. Then k, m determine a semi-norm $\| \ \|$ on $\mathscr{E}$ defined by $\|\phi\| = \sup\{|D^\alpha \phi(t)| : |t| \leqq k, \ |\alpha| \leqq m\}$. Suppose $T \in \Phi'$. Since Φ has a differentiable translation, $T * \phi \in \mathscr{E}$ for each $\phi \in \Phi$. (Remark 11 above.) Now T continuous on Φ implies there is a continuous semi-norm p on Φ such that $|\langle T, \phi \rangle| \leqq p(\phi)$ for $\phi \in \Phi$. Since $\{\tau_h : |h| \leqq k\}$ and $\{D^\alpha : |\alpha| \leqq m\}$ are equicontinuous sets in $L(\Phi, \Phi)$, there is a continuous semi-norm q on Φ such that $p(\tau_h D^\alpha \phi) \leqq q(\phi)$ for $|h| \leqq k$, $|\alpha| \leqq m$, and $\phi \in \Phi$. Thus, we obtain

$$
\begin{aligned}
(1) \qquad \|T * \phi\| &= \sup\{|D^\alpha(T * \phi)(h)| : |h| \leqq k, \ |\alpha| \leqq m\} = \\
&= \sup\{|\langle T, \tau_h D^\alpha\phi \rangle| : |h| \leqq k, \ |\alpha| \leqq m\} \leqq \\
&\leqq \sup\{p(\tau_h D^\alpha \phi) : |h| \leqq k, \ |\alpha| \leqq m\} \leqq q(\phi).
\end{aligned}
$$

From (1) we have that the map $\phi \to T * \phi$ is continuous from $\Phi \to \mathscr{E}$, and since T is arbitrary, $\Phi' \subseteq \mathcal{O}_c'(\Phi, \mathscr{E})$ so that $\Phi' = \mathcal{O}_c'(\Phi, \mathscr{E})$.

Combining Theorem 3 and Proposition 12, we obtain

Corollary 13. *Let ϕ satisfy the hypothesis of Proposition 12. If $L : \Phi \to \mathscr{E}$ is linear, continuous and commutes with translations, then there exists a unique $S \in \Phi'$ such that $L\phi = S * \phi$ for $\phi \in \Phi$.*

Remark 14. In particular if $\Phi = \mathscr{D}$ or $\mathscr{E}$ this corollary is applicable ([11], Proposition 4.3.4 and Exercise 2 of 4.3.). That is, we have

Corollary 15. *If $L : \mathscr{D} \to \mathscr{E}(L : \mathscr{E} \to \mathscr{E})$ is a continuous, linear map which commutes with translations, then there is a unique $T \in \mathscr{D}'(T \in \mathscr{E}')$ such that $L\phi = T * \phi$ for each $\phi \in \mathscr{D}(\phi \in \mathscr{E})$.*

Remark 16. See [3], p. 121 – 122 for these results.

Corollary 13 is also applicable when $\Phi = \mathscr{S}$ (See [11], Lemma 4.11.2 for a proof of the fact that $\mathscr{S}$ has a differentiable translation.) However, instead of dealing directly with $\mathscr{S}$, we will consider the larger of class of $K\{M_p\}$ spaces [8] and show that Corollary 13 is applicable to a certain subclass of such spaces.

We recall the definition of $K\{M_p\}$ spaces ([8]). Let $\{M_p\}$ be a sequence of real-valued functions defined on $\mathbf{R}^n$ and such that $1 \leqq M_1(x) \leqq M_2(x) \leqq \ldots$ for $x \in \mathbf{R}^n$. The space $K\{M_p\}$ consists of all infinitely differentiable functions ϕ on $\mathbf{R}^n$ such that

$$(2) \qquad \|\phi\|_p = \sup\{M_p(x)\,|D^\alpha \phi(x)| : x \in \mathbf{R}^n,\ |\alpha| \leqq p\} < \infty \quad (p = 1, 2, \ldots).$$

The space $K\{M_p\}$ is supplied with the locally convex topology generated by the sequence of norms $\{\|\ \|_p : p = 1, 2, \ldots\}$ defined in (2). Under this topology $K\{M_p\}$ is a Frechet space ([8], II.2.2). It should be noted that the definition of $K\{M_p\}$ space given in [8] is more general than that given above in that Gelfand and Shilov allow the M_p to take on extended real values.

We shall only consider $K\{M_p\}$ spaces where the sequence $\{M_p\}$ satisfies some additional conditions. We record some of the conditions that will be imposed on the $\{M_p\}$ below.

(P) For each $p > 0$ there is a $p' > p$ such that $M_p(x)/M_{p'}(x) \to 0$ as $|x| \to \infty$. (See [8], II.2.3 and [7], 2.2)

(M) the functions M_p are quasi-monotonic in each coordinate, i.e., if $|x_j'| \leqq |x_j''|$, then $M_p(x_1, \ldots, x_j', \ldots, x_n) \leqq C_p M_p(x_1, \ldots, x_j'', \ldots, x_n)$ for each fixed point $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ ([8], II.4.2)

(N) for each p there is a $p' > p$ such that the ratio $M_p(x)/M_{p'}(x) = m_{pp'}(x)$ tends to 0 as $|x| \to \infty$ and is a summable function on $\mathbf{R}^n$ ([8], II.4.2)

(F) each M_p is symmetric, i.e., $M_p(x) = M_p(-x)$ and for each p there is a $p' > p$ such that $M_p(x - h) \leqq C_{p'} M_{p'}(x)\, M_{p'}(h)$ for $x, h \in \mathbf{R}^n$ ([17])

If $\{M_p\}$ satisfies condition (P), then $K\{M_p\}$ is a Montel space ([8], II.2.3; the term perfect is used in [8].) and $\mathscr{D}$ is dense in $K\{M_p\}$ ([8], II.2.5). Also if $\{M_p\}$ satisfies conditions (M), (N) and (F) it is shown in Lemma 1 of [17], that $K\{M_p\}$ is a test space with an equicontinuous translation.

For later use we recall the following fact from [8]. If $\{M_p\}$ satisfies (M) and (N), the sequence of semi-norms

$$(3) \qquad \|\phi\|_p' = \sup\left\{\int_{R^n} M_p(t)\,|D^\alpha \phi(t)|\,\mathrm{d}t : |\alpha| \leqq p\right\} \quad (p = 1, 2, \ldots)$$

generates the same locally convex topology as the sequence of norms $\{\|\ \|_p : p \geqq 1\}$ defined in (2).

Some of the familar g.f. spaces are duals of $K\{M_p\}$ spaces as the following examples show.

Example 17. If $M_p(x) = (1 + |x|^2)^p$, $x \in \mathbf{R}^n$, then $K\{M_p\} = \mathscr{S}$, the space of rapidly decreasing functions ([14], VII.3). In this case $\{M_p\}$ satisfies conditions (M), (N) and (F) ([17]).

Example 18. If $M_p(x) = \exp(p\gamma(x))$, where $\gamma(x) = \sqrt{(1 + |x|^2)}$, then $K\{M_p\}'$ is the space of distributions of exponential order ([9], [22], [23]). In this case $\{M_p\}$ satisfies conditions (M), (N), and (F) ([17]).

Example 19. ([19]) Let $\{r_j\}$ be a real sequence with $0 < r_1 < r_2 < \ldots < r$ and $r_j \to r$. Set $M_p(t) = \exp(r_p|t|)$. Again (M), (N) and (F) are satisfied.

We show that Corollary 13 is applicable to $K\{M_p\}$ spaces which are such that $\{M_p\}$ satisfies (M), (N) and (F).

Lemma 20. *If $\{M_p\}$ satisfies condition* (P), *then $\Phi = K\{M_p\}$ has a differentiable translation.*

Proof. Let $\varepsilon > 0$ and p be a positive integer. For $\phi \in \Phi$, $t \in \mathbf{R}$, set $\psi_t = (\tau_{te_j}\phi - \phi)/t$. Now $M_p(x)\,|D^\alpha \phi(x)| \to 0$ as $|x| \to \infty$ for $|\alpha| \leqq p + 1$ since $M_p(x)\,|D^\alpha \phi(x)| \leqq m_{pp'}(x)\,M_{p'}(x)\,|D^\alpha \phi(x)| \leqq m_{pp'}(x)\,\|\phi\|_{p'} \to 0$ by condition (P). Therefore, there is an $R > 0$ such that $|x| > R$ implies $M_p(x)\,|D^\alpha \phi(x)| < \frac{1}{2}\varepsilon$ for $|\alpha| \leqq p + 1$. Now $D^\alpha \psi_t(x) = D_j D^\alpha \phi(x + \theta \mathrm{te}_j)$, $0 \leqq \theta = \theta(t) \leqq 1$, where $D_j\,\phi(x) = \partial\phi(x)/\partial x_j$. For $|t| \leqq 1$, $M_p(x)\,|D^\alpha \psi_t(x)| < \frac{1}{2}\varepsilon$ if $|x| > R + 1$, $|\alpha| \leqq p$; and $M_p(x)\,|D^\alpha D_j\,\phi(x)| < \frac{1}{2}\varepsilon$ if $|x| > R + 1$, $|\alpha| \leqq p$. But for $|x| \leqq R + 1$, $|\alpha| \leqq p$, $M_p(x)\,|D^\alpha(\psi_t(x) - D_j\,\phi(x))| = M_p(x)\,|D^\alpha D_j(\phi(x + \theta te_j) - \phi(x))| < \frac{1}{2}\varepsilon$ for t sufficiently small. Hence $\|\psi_t - D_j\phi\|_p < \varepsilon$ for sufficiently small t.

Remark 21. Lemma 20 is clearly applicable to $\mathscr{S}$ and $\mathscr{K}_1$. Also note (N) implies (P) so Lemma 20 holds when (N) is satisfied.

From Lemma 20 and the preceding remarks, the conclusion of Corollary 13 is valid for $K\{M_p\}$ spaces satisfying conditions (M), (N), and (F). We have

Corollary 22. *Let $\{M_p\}$ satisfy* (M), (N), *and* (F) *and $\Phi = K\{M_p\}$. If $L : \Phi \to \mathscr{E}$ is continuous, linear and commutes with translations, there is a unique $T \in \Phi'$ such that $L\phi = T * \phi$ for each $\phi \in \Phi$.*

Remark 23. For the case where $\Phi = \mathscr{S}$, see [3] p. 151. The result does not seem to have been recorded for $\Phi = \mathscr{K}_1$.

For certain test spaces Φ and Ψ, very concrete descriptions of the space of convolution operators, $\mathscr{O}'_c(\Phi, \Psi)$, are known, and the conclusion of Theorem 3 gives a precise form for the map L. For example, if $\Phi = \Psi = \mathscr{S}$, $\mathscr{O}'_c(\mathscr{S}, \mathscr{S})$ is described in [14], VII.5; if $\Phi = \Psi = \mathscr{K}_1$, $\mathscr{O}'_c(\mathscr{K}_1, \mathscr{K}_1)$ is described in [22] and [23]; and if $\Phi = \Psi = K\{M_p\}$, a description of $\mathscr{O}'_c(\Phi, \Phi)$ if given in [17] when $\{M_p\}$ satisfies conditions (M), (N) and (F). Also in [17] it is shown that $T \in \mathscr{D}'$ belongs to $\mathscr{O}'_c(K\{M_p\}, K\{M_p\})$ iff $T * \phi \in K\{M_p\}$ for each $\phi \in \mathscr{D}$ provided (M), (N) and (F) are satisfied. Thus, in this case $\mathscr{O}'_c(\mathscr{D}, K\{M_p\}) = \mathscr{O}'_c(K\{M_p\}, K\{M_p\})$ and Theorem 3 describes the continuous linear maps $L : \mathscr{D} \to K\{M_p\}$ which commute with translation.

2. MAPPINGS BETWEEN GENERALIZED FUNCTION SPACES

In this section we consider continuous linear maps between g.f. spaces which commute with translation. Again Φ and Ψ will denote test spaces; we assume that the duals Φ' and Ψ' are always equipped with the strong topologies although many of the statements would also be valid if the weak* topologies were used. From Proposition 1 and the definition of convolution, we have

Proposition 24. *Let $S \in \mathcal{O}'_c(\Phi, \Psi)$ and define $L: \Psi' \to \Phi'$ by $L(T) = S * T$. Then L is linear, continuous and commutes with translations.*

By using some of the results of section 1 we show that the converse of Proposition 24 is valid for certain spaces of generalized functions.

Lemma 25. *Let Φ and Ψ be semi-reflexive test spaces. If $L: \Phi' \to \Psi'$ is linear, continuous and commutes with translations, then its conjugate $L': \Psi \to \Phi$ commutes with translations.*

Proof. Let $T \in \Phi'$ and $\psi \in \Psi$. Then $\langle T, L'(\tau_h\psi)\rangle = \langle LT, \tau_h\psi\rangle = \langle \tau_{-h}T, L'\psi\rangle = \langle T, \tau_h L'\psi\rangle$ so that $L'(\tau_h\psi) = \tau_h L'\psi$.

Theorem 26. *Let Φ and Ψ be reflexive test spaces. If $L: \Phi' \to \Psi'$ is continuous, linear and commutes with translations, then there is a unique $S \in \mathcal{O}'_c(\Psi, \Phi)$ such that $L(T) = S * T$ for $T \in \Phi'$.*

Proof. By the reflexivity assumption, $L': \Psi \to \Phi$ is linear and continuous, and by Lemma 25, L' commutes with translations. By Theorem 3 there is a unique $S \in \mathcal{O}'_c(\Psi, \Phi)$ such that $L'(\psi) = S * \psi$ for $\psi \in \Psi$. For $T \in \Phi'$, $\psi \in \Psi$, we have $\langle LT, \psi\rangle = \langle T, L'\psi\rangle = \langle T, S * \psi\rangle = \langle S * T, \psi\rangle$ which gives $LT = S * T$. The uniqueness is clear.

Since we have computed the space of convolution operators for several test spaces in section 1, Theorem 26 is applicable in a variety of situations. We record some of these below in a single statement and include references when the corresponding result is known.

Corollary 27. *Suppose Φ and Ψ are test spaces and $L: \Phi' \to \Psi'$ is linear, continuous and commutes with translations.*

(i) *if $\Psi = \mathcal{D}$, $\Phi = \mathcal{E}$, there is a unique $S \in \mathcal{D}'$ such that $LT = S * T$ for $T \in \mathcal{E}'$* ([11], *p.* 396; [14], VI.3).

(ii) *if $\Phi = \Psi = \mathcal{D}$, there is a unique $S \in \mathcal{E}'$ such that $LT = S * T$ for $T \in \mathcal{D}'$* ([11], p. 399; [14], VI.3).

(iii) *if $\Phi = \Psi = \mathcal{D}_+$, there is a unique $S \in \mathcal{D}'_-$ such that $LT = S * T$ for $T \in \mathcal{D}'_-$.* ([1], p. 28; see also [16]).

(iv) *if* $\Psi = \mathscr{D}_+$, $\Phi = \mathscr{E}$, *there is a unique* $S \in \mathscr{D}'_-$ *such that* $LT = S * T$ *for* $T \in \mathscr{E}'$.

(v) *if* Φ *is reflexive and has an equicontinuous and differentiable translation and* $\Psi = \mathscr{E}$, *there is a unique* $S \in \Phi'$ *such that* $LT = S * T$ *for* $T \in \mathscr{E}'$.

3. MAPPING FROM $\mathscr{D}$ INTO GENERALIZED FUNCTION SPACES

In this section we consider linear continuous maps from $\mathscr{D}$ into certain spaces of distributions which commute with translations. We show that such maps are again given by convolutions. The class of spaces that we consider is given in the following definition.

Definition 28. Let $\mathscr{A}$ be a subspace of $\mathscr{D}'$ such that $\mathscr{A}$ is dense in $\mathscr{D}'$. Then $\mathscr{A}$ is said to be *convolution regular* if

(i) $\mathscr{A}$ is equipped with a locally convex Hausdorff topology such that the injection of $\mathscr{A}$ into $\mathscr{D}'$ is continuous

(ii) when $T \in \mathscr{D}'$ is such that $T * \phi \in \mathscr{A}$ for any $\phi \in \mathscr{D}$, then $T \in \mathscr{A}$.

We show in section 4 that many of the familiar spaces of generalized functions are convolution regular. However, we note that not all normal spaces of distributions are convolution regular. For example, $L^p(\mathbf{R}^n)$ $(1 \leqq p < \infty)$ is not convolution regular. For if $T \in \mathscr{D}'_{L^p}$ but $T \notin L^p(\mathbf{R}^n)$, then $T * \phi \in L^p(\mathbf{R}^n)$ for each $\phi \in \mathscr{D}$ ([14], Th. XXX, Ch. VI).

We show that if $\mathscr{A}$ is convolution regular and $L: \mathscr{D} \to \mathscr{A}$ is linear, continuous and *commutes with translations* (i.e., $L(\tau_x\phi) = \tau_x L\phi$ for $\phi \in \mathscr{D}$, $x \in \mathbf{R}^n$), then $L\phi = T * \phi$ for some (unique) $T \in \mathscr{A}$. It is more convenient to deal with maps L that commute with convolutions in the following sense.

Definition 29. Let $\mathscr{A}$ be a subspace of $\mathscr{D}'$ equipped with a locally convex Hausdorff topology. A continuous linear map $L: \mathscr{D} \to \mathscr{A}$ is said to *commute with convolution* if $L(\phi * \psi) = L(\phi) * \psi$ for $\phi, \psi \in \mathscr{D}$. (Note $L\phi \in \mathscr{D}'$ so the convolution $L(\phi) * \psi$ is defined and is an element of $\mathscr{E}$.) We have

Lemma 30. *Let* $\mathscr{A}$ *be a subspace of* $\mathscr{D}'$. *Let* $L: \mathscr{D} \to \mathscr{A}$ *be linear and continuous (with respect to the induced topology from* $\mathscr{D}'$*). Then* L *commutes with translations iff* L *commutes with convolutions.*

Proof. See 5.11.3 of [4].

As a consequence of this result it makes no difference whether we consider linear maps which commute with translations or convolution. It is usually more convenient to deal with convolutions.

Lemma 31. *Let $\mathscr{A}$ be a subspace of $\mathscr{D}'$ such that $\mathscr{A}$ is dense in $\mathscr{D}'$ and assume that $\mathscr{A}$ is equipped with a locally convex Hausdorff topology such that the injection of $\mathscr{A}$ into $\mathscr{D}'$ is continuous. If $L : \mathscr{D} \to \mathscr{A}$ commutes with convolution, then $L' : \mathscr{A}' \to \to \mathscr{D}'$ commutes with convolution in the sense that*

$$L'(\phi * \psi) = L'(\phi) * \psi = L(\phi) * \psi \quad \text{for} \quad \phi, \psi \in \mathscr{D} . \tag{4}$$

Proof. Note that $\mathscr{A}'$ is a space of distributions so the convolutions in (4) have a meaning. If $\alpha \in \mathscr{D}$, we have $\langle L'(\phi * \psi), \alpha \rangle = \langle \phi, L(\alpha) * \psi \rangle = \langle L'(\phi), \psi * \alpha \rangle = = \langle L'(\phi) * \psi, \alpha \rangle$ so that $L'(\phi * \psi) = L'(\phi) * \psi$. Similarly $\langle L'(\phi * \psi), \alpha \rangle = \langle L'(\psi), \phi * \alpha \rangle = \langle \psi, L(\phi) * \alpha \rangle = \langle L(\phi) * \psi, \alpha \rangle$ so that $L'(\phi * \psi) = L(\phi) * \psi$.

Theorem 32. *Let $\mathscr{A} \subseteq \mathscr{D}'$ be convolution regular. If $L: \mathscr{D} \to \mathscr{A}$ is a continuous linear map which commutes with convolution (translations), then there is a unique $T \in \mathscr{A}$ such that $L(\phi) = T * \phi$ for $\phi \in \mathscr{D}$.*

Proof. Let $\{\phi_n\}$ be a sequence of regularizers in $\mathscr{D}$ ([15]) so that $\phi_n \to \delta$ in $\mathscr{E}'$. Then the sequence $\{L'(\phi_n)\}$ is strongly bounded in $\mathscr{D}'$ since if $\alpha \in \mathscr{D}$, by Lemma 31 $L'(\phi_n) * \alpha = L(\alpha) * \phi_n \to L(\alpha)$ in $\mathscr{D}'$ and Theorem XXII of Chapter VI, [14], is applicable. Since $\mathscr{D}'$ is a Montel space (with respect to the strong topology), there is a $T \in \mathscr{D}'$ and a subsequence $\{L'(\phi_{n_k})\}$ such that $L'(\phi_{n_k}) \to T$ in $\mathscr{D}'$. For $\psi \in \mathscr{D}$, $L'(\phi_{n_k}) * \psi \to T * \psi$ in $\mathscr{E}$ and therefore in $\mathscr{D}'$; and by Lemma 31, $L'(\phi_{n_k}) * \psi = = L(\psi) * \phi_{n_k} \to L(\psi) * \delta = L(\psi)$ in $\mathscr{D}'$. Hence $T * \psi = L(\psi) \in \mathscr{A}$ for $\psi \in \mathscr{D}$, and since $\mathscr{A}$ is convolution regular, $T \in \mathscr{A}$. This gives the desired representation for L. That T is unique is clear.

We also have a partial converse to Theorem 32 which shows that the class of convolution regular spaces is the natural class for which the conclusion of Theorem 32 holds.

Theorem 33. *Let $\mathscr{A}$ be a dense subspace of $\mathscr{D}'$ equipped with a locally convex Hausdorff topology such that the injection of $\mathscr{A}$ into $\mathscr{D}'$ is continuous. Suppose the closed graph theorem holds for the pair $(\mathscr{D}, \mathscr{A})$. If $\mathscr{A}$ has the property that any continuous linear map $L: \mathscr{D} \to \mathscr{A}$ which commutes with convolutions (translations) has the form $L(\phi) = T * \phi$ for some (unique) $T \in \mathscr{A}$, then $\mathscr{A}$ is convolution regular.*

Proof. Let $T \in \mathscr{D}'$ be such that $T * \phi \in \mathscr{A}$ for each $\phi \in \mathscr{D}$. Define $L: \mathscr{D} \to \mathscr{A}$ by $L(\phi) = T * \phi$. Then L is linear and commutes with convolution. To show L is continuous it suffices by hypothesis to show L is closed. Suppose $\phi_r \to \phi$ in $\mathscr{D}$ and $L(\phi_r) \to S$ in $\mathscr{A}$, where $\{\phi_r\}$ is a net. Then $T * \phi_r \to T * \phi$ in $\mathscr{E}$ and therefore in $\mathscr{D}'$. Also $T * \phi_r \to S$ in $\mathscr{D}'$ since the inejction $\mathscr{A} \to \mathscr{D}'$ is continuous. Hence $T * \phi = S$ and L is closed. By hypothesis there exists $T_1 \in \mathscr{A}$ such that $L(\phi) = T_1 * \phi = T * \phi$ for $\phi \in \mathscr{D}$. Hence $T_1 = T \in \mathscr{A}$ and $\mathscr{A}$ is convolution regular.

Thus to show a linear, continuous map from $\mathscr{D}$ into a g.f. space $\mathscr{A}$ which commutes with translation is given by convolution it is only necessary to establish that the space $\mathscr{A}$ is convolution regular. In the final section we consider many of the familiar g.f. spaces and show that they are convolution regular.

4. CONVOLUTION REGULAR GENERALIZED FUNCTION SPACES

In this final section we show that many of the familiar spaces of generalized functions are convolution regular, and hence the results of section 3 are applicable to these spaces. The main tool used throughout this section in showing g.f. spaces are convolution regular is the remarkable Theorem 5 of [2].

Proposition 34. *The generalized function spaces* $\mathscr{D}'$, $\mathscr{E}'$, $\mathscr{D}'_+(\mathscr{D}'_-)$ *and* $\mathscr{D}'_{L^p}$ $(1 \leqq p < \infty)$ *are convolution regular.*

Proof. It is clear that $\mathscr{D}'$ is convolution regular. That $\mathscr{E}'$ is convolution regular is established in [5], Theorem V.5.15, but we give a proof of this fact based on Theorem 5 of [2]. Suppose $T \in \mathscr{D}'$ is such that $T * \phi \in \mathscr{E}'$ for $\phi \in \mathscr{D}$. For each $j \geqq 1$ let B_j be the set of all continuous functions $f : \mathbf{R}^n \to \mathbf{R}$ such that support $(f) \subseteq \{x : \|x\| \leqq j\}$ equipped with the sup norm. Note $B_j \subseteq B_{j+1}$ with the injection continuous. Set $B = \bigcup_{j \geqq 1} B_j$ and equip B with the inductive limit topology from the $\{B_j\}$. Then for each $\phi, \psi \in \mathscr{D}$, $T * \phi * \psi \in B$ so by Theorem 5 of [2], $T = (1 - \Delta)^l f_0 + f_1$ where $f_0, f_1 \in B$, and thus $T \in \mathscr{E}'$.

To show $\mathscr{D}'_+$ is convolution regular suppose $T \in \mathscr{D}'$ is such that $T * \phi \in \mathscr{D}'_+$ for $\phi \in \mathscr{D}$. Thus $T * \phi \in \mathscr{D}_+$ for $\phi \in \mathscr{D}$. Recall $\mathscr{D}_+ = \underset{\rightarrow}{\text{ind}}\, \mathscr{E}_{(c,\infty)}$, where $\mathscr{E}_{(c,\infty)}$ is equipped with the relative topology from $\mathscr{E}$ ([14], VI.5). The map $\phi \to T * \phi$ from $\mathscr{D}_{[-1,1]} \to \mathscr{D}_+$ has a closed graph and is therefore continuous ([4], 6.7.1). Thus, there exists $c \in \mathbf{R}$ such that $T * \phi \in \mathscr{E}_{c,\infty)}$ for each $\phi \in \mathscr{D}_{[-1,1]}$ ([4], 6.5.1). Take $\{\phi_n\}$ to be a regularizing sequence in $\mathscr{D}_{[-1,1]}$ so that $T * \phi_n \to T$ in $\mathscr{D}'$. Since support $(T * \phi_n) \subseteq (c, \infty)$ for each n, support $(T) \subseteq [c, \infty)$, or $T \in \mathscr{D}'_+$.

To show $\mathscr{D}'_{L^p}$ ([14], VI.8) is convolution regular, we again use Theorem 5 of [2]. Suppose $T \in \mathscr{D}'$ is such that $T * \phi \in \mathscr{D}'_{L^p}$ for $\phi \in \mathscr{D}$. Then $T * \phi * \psi \in L^p(\mathbf{R}^n)$ for $\phi, \psi \in \mathscr{D}$ ([14], Theorem XXV, Chapter VI). By Theorem 5 of [2], $T = (1 - \Delta)^l f_0 + f_1$ where $f_0, f_1 \in L^p(\mathbf{R}^n)$. Then $T \in \mathscr{D}'_{L^p}$ by Theorem XXV of Chapter VI, [14].

Of course, the familiar space of tempered distributions, $\mathscr{S}'$, is missing from the list of spaces in Proposition 34. Again, instead of treating this space separately, we consider a certain class of $K\{M_p\}$ spaces which includes the space $\mathscr{S}$ and also the space of test functions of exponential growth. Again our principle tool is the Theorem 5 of [2]. It should be remarked that Lemma 1 of [12] establishes the convolution regularity of $\mathscr{S}'$ as well as that of $\mathscr{D}'_{L^p}$.

Proposition 35. *Let $\{M_p\}$ satisfy conditions* (M), (N) *and (F). Then $K\{M_p\}'$ is convolution regular.*

Proof. For $p \geqq 1$, let $B_p = \{f : f \text{ continuous and } \sup |f(t)|/M_p(t) = \|f\|_p < \infty\}$ and equip B_p with the norm $\| \; \|_p$. We have $B_p \subseteq B_{p+1}$ with the injection continuous since $\| \; \|_p \geqq \| \; \|_{p+1}$. Set $B = \underrightarrow{\text{ind}}\, B_p$. Suppose $T \in \mathscr{D}'$ is such that $T * \phi \in K\{M_p\}'$ for $\phi \in \mathscr{D}$. Then by Theorem 1 of [18] $T * \phi * \psi \in B_p \subseteq B$ for each $\psi \in \mathscr{D}$. By Theorem 5 of [2], $T = (1 - \Delta)^l f_0 + f_1$ where $f_0, f_1 \in B_p$ for some l, p. Hence $T \in K\{M_p\}'$ by II.4.2 of [8] or Theorem 1 of [18], and $K\{M_p\}'$ is convolution regular.

Corollary 36. *The spaces $\mathscr{S}'$ and $\mathscr{K}'_1$, are convolution regular.*

By using Lemma 1 of [12], it can be shown that $\mathscr{O}'_c$ is convolution regular. As above, by using Theorem 5 of [2], we show that $\mathscr{O}'_c(K\{M_p\})$ is convolution regular for $\{M_p\}$ as in Proposition 35.

Proposition 37. *Let $\{M_p\}$ satisfy conditions* (M), (N), *and* (F), *then $\mathscr{O}'_c(K\{M_p\})$ is convolution regular.*

Proof. Given a positive integer k, set $B = \{f : f \text{ continuous and } \sup |f(t)\, M_k(t)| = \|\phi\|_k < \infty\}$. Then B equipped with the norm $\| \; \|_k$ is a B-space. Suppose $T \in \mathscr{D}'$ is such that $T * \phi \in \mathscr{O}'_c(K\{M_p\})$ for $\phi \in \mathscr{D}$. Then $T * \phi * \psi \in K\{M_p\}$ for $\psi \in \mathscr{D}$, and, therefore, $T * \phi * \psi \in B$. By Theorem 5 of [2], $T = (1 - \Delta)^l f_0 + f_1$, where $f_0, f_1 \in B$. By Theorem 3 of [17], $T \in \mathscr{O}'_c(K\{M_p\})$.

Corollary 38. *The spaces $\mathscr{O}'_c$* ([14], VII.5) *and $\mathscr{O}'_c(\mathscr{K}_1)$* ([23]) *are convolution regular.*

Other examples of $K\{M_p\}$ spaces are found in [8]; for example, the spaces $S_{\alpha,A}$ of Chapter IV.3 and the S_α spaces which are inductive limits of such spaces. It is not known if the conclusion of Theorem 32 is applicable to the dual of these spaces, but the following result may be useful in treating such spaces.

Proposition 39. *Suppose for each $n \geqq 0$ E_n is a normal space of distributions such that $E_n \supseteq E_0$ and $\bigcap_{n \geqq 1} E_n = E_0$ and $E_0 = \underleftarrow{\text{proj}}\, E_n$. If each E_n is convolution regular $(n \geqq 1)$, then E_0 is convolution regular.*

Proof. Let $T \in \mathscr{D}'$ be such that $T * \phi \in E_0$ for each $\phi \in \mathscr{D}$. Then $T * \phi \in E_n$ for each n so that $T \in E_n$ since E_n is convolution regular. Thus $T \in E_0$ and E_0 is convolution regular.

This result may be applicable in the following situation. Suppose K_n is a sequence of test spaces such that $K_n \subseteq K_{n+1}$ with continuous injection. Let $K_0 = \bigcup_{n \geqq 1} K_n$ and supply K_0 with the inductive limit topology. If K_0 is a test space and $\underrightarrow{\text{ind}}\, K_n$ is a regular inductive limit ([6], 23.5), then K'_0 (with the strong topology) is the projective limit of the sequence K'_n (with the strong topology) ([6], 26.2), and if each K'_n is convolution regular, then Proposition 39 implies that K'_0 is also convolution regular. This is essentially the situation encountered with respect to the S_α spaces mentioned above. However, it does not seem to be known if the inductive limit defining these spaces is regular so it remains an open question as to whether S'_α is convolution regular.

Of course, there are many other g.f. spaces which do not appear above (see [21]), and it remains an open question as to whether such spaces are convolution regular.

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