Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 25 (1975), No. 1, 20–23

Persistent URL: http://dml.cz/dmlcz/101289

Czechoslovak Mathematical Journal, 25 (100) 1975, Praha

ON WEAKLY COMMUTATIVE SEMIGROUPS

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(Received January 31, 1973)

In [1] N. P. MUKHERJEE studies some properties of quasicommutative semigroups. In this note we shall extend his results on quasicommutative semigroups to weakly commutative or duo semigroups. Finally, we consider the maximal separative homomorphic image of a duo semigroup.

Definition 1. A semigroup S is called *weakly commutative* if for any $a, b \in S$, we have $(ab)^k = xa = by$ for some $x, y \in S$ and a positive integer k. See Definition 6.4 [2].

Remarks 1. Evidently, every *commutative semigroup* is weakly commutative.

2. A semigroup S is *duo* if every one-sided ideal of S is two-sided. It is known (Theorem 2 and Remark 2 [3]) that a semigroup S is duo if and only if for any $a \in S$ we have $aS \cup a = a \cup Sa$. Evidently, every duo semigroup si weakly commutative.

3. A semigroup S is *normal* if for any $a \in S$, we have $aS = Sa$. Every normal semigroup is duo and so every normal semigroup is weakly commutative.

4. Every *group* is a normal semigroup and so every group is weakly commutative.

5. A semigroup S is called *quasicommutative* if for any $a, b \in S$, we have $ab = b^r a$ for a positive integer r. See [1]. Evidently, every quasicommutative semigroup is normal and so every quasicommutative semigroup is weakly commutative.

Remark 6. It follows from Lemma 6 [3] that every regular duo semigroup S is a semilattice of groups. Hence every regular qusicommutative semigroup S is a semilattice of groups. See Theorem 2 [1].

Theorem 1. *Let a weakly commutative semigroup S be a union of groups. Then S is a semilattice of groups.*

Proof. Let S be a weakly commutative semigroup and a union of groups. We first prove that S is normal. Let $x \in aS$. Then $x = as$ for some $s \in S$. Since S is weakly commutative, we have $(as)^k = ua$ for some $u \in S$ and a positive integer k. Since S is a union of groups, we have $as = v(as)^k$ for some $v \in S$. Hence $x = as = v(as)^k = vua \in Sa$ and so $aS \subset Sa$. Similarly we obtain that $Sa \subset aS$ and therefore $aS = Sa$. Thus S is a normal semigroup. The rest of the proof follows from Remarks 3, 2 and 6.

Definition 2. We define a relation η on a semigroup S as follows: $a \eta b$ if and only if $ax = b^m$, $by = a^n$ for some $x, y \in S$ and some positive integers m, n.

Theorem 2. *If S is a weakly commutative semigroup then η is a congruence on S, S/η is a maximal semilattice homomorphic image of S and every equivalence class of S mod η is an Archimedean semigroup.*

Proof. Let S be a weakly commutative semigroup. It follows from Theorem 3.2 [2] and dual Corollary 6.6 [2] that η is a congruence on S and S/η is a maximal semilattice homomorphic image of S. We shall prove that every equivalence class C of S mod η is an Archimedean semigroup.

We first prove that C is a subsemigroup of S. If $a, b \in C$ then $a \eta b$ and $b^2 \eta b$. Consequently $ab \eta b^2 \eta b$ and hence $ab \in C$. We show next that C is Archimedean. If $a, b \in C$ then $ax = b^m$, $by = a^n$ for some $x, y \in S$ and some positive integers m, n. Since S is weakly commutative, there exist $u, v \in S$ and positive integers k, r such that $(xb)^k = bu$, $(axb)^r = (xb)v$ and so $(xb)v = (axb)^r = (b^{m+1})^r = b^{rm+r}$. Hence $xb \eta b$ and thus $xb \in C$. Similarly we obtain that $ya \in C$. Hence $a(xb) = b^{m+1}$, $b(ya) = a^{n+1}$, where $xb, ya \in C$ and so C is an Archimedean semigroup.

Definition 3. We define a relation σ on a semigroup S as follows: $a \sigma b$ if and only if $ab^n = b^{n+1}$ and $ba^n = a^{n+1}$ for a positive integer n. See [1].

Remark 7. It is clear that $ab^n = b^{n+1}$ implies $ab^m = b^{m+1}$ for every positive integer $m \geqq n$. Thus $a \sigma b$ if and only if $ab^r = b^{r+1}$ and $ba^s = a^{s+1}$ for some positive integers r, s.

Definition 4. A semigroup S is called *left (right) weakly commutative* if for every $a, b \in S$, there exist $x \in S$ and a positive integer k such that $(ab)^k = bx$ $((ab)^k = xa)$. See Definition 3.3 [4].

Remark 8. A semigroup S is weakly commutative if and only if it is left weakly commutative and right weakly commutative. See Lemma 10 [5].

Lemma 1. *Let S be a left weakly commutative semigroup and let $a, b \in S$. Then to every positive integer n there exists a positive integer m such that $(ab)^m = b^n x$ for some $x \in S$.*

Proof. The result is true for $n = 1$ by Definition 4. Assume now that the result holds for $n \geqq 1$. Then $(ab)^m = b^n x$ for some $x \in S$ and a positive integer m. It follows from Definition 4 that there exist $y \in S$ and a positive integer k such that $(xb^n)^k = b^n y$. Thus $(ab)^{km+m} = ((ab)^m)^{k+1} = (b^n x)^{k+1} = b^n(xb^n)^k x = b^n(b^n y) x = b^{n+1} u$ for some $u \in S$. Hence the result follows by induction.

Theorem 3. *If S is a left weakly commutative semigroup then σ is a separative congruence on S.*

Proof. Let S be a weakly commutative semigroup. It follows from the first part of the proof of Theorem 5 [1] that σ is an equivalence on S.

We shall show that σ is a congruence on S. Suppose $a \sigma b$ and $c \in S$. Then $ab^n = b^{n+1}$ and $ba^n = a^{n+1}$. It follows from Lemma 1 that $(cb)^m = b^n x$ for some $x \in S$ and a positive integer m. Hence $(ac)(bc)^m = a(cb)^m c = a(b^n x) c = b(b^n x) c = b(cb)^m c = (bc)^{m+1}$ and $(ca)(cb)^m = (ca)(b^n x) = (cb)(b^n x) = (cb)(cb)^m = (cb)^{m+1}$. Similarly we obtain $(bc)(ac)^k = (ac)^{k+1}$ and $(cb)(ca)^k = (ca)^{k+1}$ for a positive integer k and so, by Remark 7, $ac \sigma bc$ and $ca \sigma cb$.

Next we prove that σ is separative. Let $a^2 \sigma ab \sigma b^2$. It follows from $ab \sigma b^2$ that $(ab)(b^2)^m = (b^2)^{m+1}$ for a positive integer m and so $ab^{2m+1} = b^{2m+2}$. Since σ is a congruence, we have $ba^3 \sigma (ba) a^2 \sigma (ba) b^2 \sigma b(ab) b \sigma b(b^2) b \sigma b^2 . b^2 \sigma a^2 . a^2 \sigma a^4$ and so $ba^3 \sigma a^4$. This implies $ba^3(a^4)^k = (a^4)^{k+1}$ for a positive integer k. Thus $ba^{4k+3} = a^{4k+4}$ and so $a \sigma b$.

Definition 5. Dually, we define a relation τ on a semigroup S as follows: $a \tau b$ if and only if $b^n a = b^{n+1}$ and $a^n b = a^{n+1}$ for a positive integer n.

Put $\pi = \sigma \cap \tau$.

Remark 9. It follows from Definition 3, Remark 7 and Definition 5 that $a \pi b$ if and only if $ab^n = b^{n+1} = b^n a$ and $ba^n = a^{n+1} = a^n b$ for a positive integer n.

Theorem 4. *If S is a weakly commutative semigroup then σ, τ and π are separative congruences on S.*

Proof. This follows from Theorem 3, its dual and Remark 8.

Theorem 5. *If S is a duo semigroup then S/π is a maximal separative homomorphic image of S.*

Suppose ϱ is an arbitrary separative congruence on a duo semigroup S. We first prove the following Lemma:

Lemma 2. *Let $a, b \in S$. If $ab^n \varrho b^{n+1} \varrho b^n a$ and $ba^n \varrho a^{n+1} \varrho a^n b$ for a positive integer n then $a \varrho b$.*

Proof. Since ϱ is a separative congruence on S, hence the result is true for $n = 1$. Assume now that the result holds for $n \geqq 1$. Let $ab^{n+1} \varrho b^{n+2} \varrho b^{n+1}a$ and $ba^{n+1} \varrho \varrho a^{n+2} \varrho a^{n+1}b$. It follows from Remark 2 that $ab^n = b^n$ or $ab^n = b^n x$ for some $x \in S$. If $ab^n = b^n$ then $(ab^n)^2 \varrho (ab^n) b^n \varrho b^{n+1} . b^n \varrho b^{n+1}(ab^n) \varrho (b^{n+1}a) b^n \varrho b^{n+2} . . b^n \varrho (b^{n+1})^2$ and so $(ab^n)^2 \varrho b^{n+1}(ab^n) \varrho (b^{n+1})^2$. Since ϱ is a separative congruence on S, we have $ab^n \varrho b^{n+1}$. Similarly, if $ab^n = b^n x$ then $(ab^n)^2 \varrho (ab^n)(b^n x) \varrho b^{n+1}(b^n x) \varrho b^{n+1}(ab^n) \varrho (b^{n+1})^2$ and so $(ab^n)^2 \varrho b^{n+1}(ab^n) \varrho (b^{n+1})^2$. This implies $ab^n \varrho b^{n+1}$. We can prove $b^{n+1} \varrho b^n a$ and $ba^n \varrho a^{n+1} \varrho a^n b$ in a similar way. Hence we get $a \varrho b$. The result therefore follows by induction.

The proof of Theorem 5. Let ϱ be an arbitrary separative congruence on a duo semigroup S. If $a \pi b$ then, by Remark 9, we have $ab^n = b^{n+1} = b^n a$ and $ba^n = a^{n+1} = a^n b$ for a positive integer n and so $ab^n \varrho b^{n+1} \varrho b^n a$, $ba^n \varrho a^{n+1} \varrho a^n b$. It follows from Lemma 2 that $a \varrho b$ and thus $\pi \subseteq \varrho$.

Corollary. *If S is a quasicommutative semigroup then S/π is a maximal separative homomorphic image of S.*

Author's Note. When the article had already been in print, the author's attention was drawn to the book by Petrich M.: *Introduction to semigroups*, Merrill, Columbus, Ohio, 1973, where some of the results of the present paper are contained in the other form.

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