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A TRACE INEQUALITY FOR FUNCTIONS OF TRIANGULAR HILBERT-SCHMIDT OPERATORS

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Introduction. CHANDLER DAVIS [1] has found trace inequalities for functions of matrices[1]). This note gives an extension of a variation of a result of C. Davis. Section 1 developes a trace inequality for simultaneously triangularizable matrices and section 2 extends these results to simultaneously triangularizable Hilbert-Schmidt operators.

Section 1. Two matrices X and Y are said to be simultaneously triangularizable iff there exists a unitary matrix U such that UXU^* and UYU^* are upper triangular. A matrix N shall be called strictly upper triangular iff N is upper triangular and nilpotent. Now if N is strictly upper triangular and S is a diagonal matrix then the following are clear:

(i) NS and SN are nilpotent;

(ii) if $f(\lambda)$ can be expanded in a power series in the circle $|\lambda - \lambda_0| < r$;

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda_n(\lambda - \lambda_0)^n ,$$

then this expansion remains valid when the scalar argument λ is replaced by the matrix $S + N$ whose spectrum, $\sigma(S + N)$, lies within the circle of convergence;

(iii) $f(S + N)$ can be written as a diagonal plus a strictly nilpotent where the diagonal part is precisely $f(S)$.

Let us adhere to the following notations: set $f^1[\beta, \alpha] = (f(\beta) - f(\alpha))/(\beta - \alpha)$ for $\beta \neq \alpha$, $\phi^*\psi = (\psi, \phi)$, and let $\psi\phi^*$ be the linear operator defined by $\psi\phi^*\theta = (\theta, \phi)\,\psi$. In this notation, we find that the trace $(XY) = \sum_{i=1}^{n} \psi_i^* XY\psi_i$ where $\{\psi_i \mid i =$

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[1]) The author has pointed out to Chandler Davis that the proof of Theorem 3 in [1] is incorrect.

$= 1, \ldots, n\}$ is any orthonormal basis and X and Y are matrices. We can now obtain a trace inequality for a function of a pair of diagonal matrices.

Lemma 1. *Let X and Y be a pair of diagonal matrices, K an open disc of radius $r > 0$ centered at the origin, and $f(\lambda)$ have a power series whose region of convergence contains $\sigma(x)$ and $\sigma(x + y)$. If $f^1[\xi, \alpha] \in K$ for $\xi \in \sigma(x + y)$ and $\alpha \in \sigma(x)$, then* $\{\text{trace}\,(YY^*)\}^{-1} \,.\, \text{trace}\,(Y(f(X + Y) - f(X))) \in K$.

Proof. Since X and Y are a diagonal pair of matrices, there exists an orthonormal basis $\{\psi_i \mid i = 1, \ldots, n\}$ such that:

$$X = \sum_{i=1}^{n} \alpha_i \psi_i \psi_i^*\,, \quad Y = \sum_{i=1}^{n} \beta_i \psi_i \psi_i^*\,; \tag{1}$$

$$f(X + Y) = \sum_{i=1}^{n} f(\alpha_i + \beta_i)\, \psi_i \psi_i^*\,, \quad f(X) = \sum_{i=1}^{n} f(\alpha_i)\, \psi_i \psi_i^*\,; \tag{2}$$

and

$$\text{trace } Y(f(X + Y) - f(X)) = \sum_{i=1}^{n} \psi_i^* Y(f(X + Y) - f(X))\, \psi_i)\,. \tag{3}$$

If we replace (1) and (2) into (3), (3) then becomes

$$\sum_{i=1}^{n} \beta_i(f(\alpha_i + \beta_i) - f(\alpha_i)) = \sum_{i=1}^{n} |\beta_i|^2\, (f(\alpha_i + \beta_i) - f(\alpha_i))/\bar{\beta}_i\,.$$

Since $f^1[\alpha_i + \beta_i, \alpha_i] \in K$ and K is an open disc of radius r, we find that

$$|(f(\alpha_i + \beta_i) - f(\alpha_i))/\bar{\beta}_i| = |(f(\alpha_i + \beta_i) - f(\alpha_i))/\beta_i| \cdot \left|\frac{\beta_i}{\bar{\beta}_i}\right| < r\,.$$

Thus,

$$|\{\text{trace}\,(YY^*)\}^{-1}\,\text{trace}\,\{Y(f(X + Y) - f(X))\}| =$$
$$= |\{\sum_{i=1}^{n} |\beta_i|^2\, (f(\alpha_i + \beta_i) - f(\alpha_i))/\bar{\beta}_i\} / \sum_{i=1}^{n} |\beta_i|^2| < r\,,$$

and this yields the result.

We can now use the lemma to obtain a similar identity for matrices which are simultaneously triangularizable.

Theorem 1. *Let X and Y be simultaneously upper triangularizable, K an open disc centered at the origin of radius r, and $f(\lambda)$ have a power series whose region of convergence contains $\sigma(X)$ and $\sigma(X + Y)$. If $f^1[\gamma, \alpha] \in K$ for $\gamma \in \sigma(X + Y)$ and $\alpha \in \sigma(X)$, then* $\{\text{trace}\, YY^*\}^{-1} \,.\, \text{trace}\, Y(f(X + Y) - f(X)) \in K$.

Proof. Since X and Y can be simultaneously upper triangularized, then there exists U unitary, S_1 and S_2 diagonal, and N_1 and N_2 strictly upper triangular, such that $UXU^* = S_1 + N_1$ and $UYU^* = S_2 + N_2$. Since the trace is invariant under similarity, it follows that

$$\begin{aligned}(4) \qquad & \{\operatorname{trace} YY^*\}^{-1} \operatorname{trace} Y(f(X+Y) - f(Y)) = \\ &= \{\operatorname{trace} UYY^*U^*\}^{-1} \operatorname{trace} UYU^*U(f(X+Y) - f(Y))\, U^* = \\ &= \{\operatorname{trace} UYU^*UY^*U^*\}^{-1} \operatorname{trace} UYU^*(f(U(X+Y)\, U^*) - f(UXU^*)) = \\ &= \{\operatorname{trace} (S_2 + N_2)(S_2^* + N_2^*)\}^{-1} \operatorname{trace} (S_2 + N_2) \\ &\qquad (f(S_1 + S_2 + N_1 + N_2) - f(S_1 + N_1)).\end{aligned}$$

Since the trace is linear, and since $N_2S_2^*$, $S_2N_2^*$ and $N_2(f(S_1 + S_2 + N_1 + N_2) - - f(S_1 + N_1))$ are nilpotent, one finds that (4) equals

$$\begin{aligned}(5) \qquad & \{\operatorname{trace} (S_2S_2^* + N_2N_2^*)\}^{-1} \operatorname{trace} S_2(f(S_1 + S_2 + N_1 + N_2) - f(S_1 + N_1)) = \\ &= \{\operatorname{trace} (S_2S_2^* + N_2N_2^*)\}^{-1} \operatorname{trace} (S_2(f(S_1 + S_2) - f(S_1)) + S_2T)\end{aligned}$$

where T is strictly upper triangular and nilpotent. Thus (5) equals

$$(6) \qquad \{\operatorname{trace} (S_2S_2^* + N_2N_2^*)\}^{-1} \operatorname{trace} S_2(f(S_1 + S_2) - f(S_1)).$$

Since trace $S_2S_2^*$ and trace $N_2N_2^*$ are positive, the absolute value of (6) is less than or equal to

$$\left|\{\operatorname{trace} S_2S_2^*\}^{-1} \operatorname{trace} S_2(f(S_1 + S_2) - f(S_1))\right|.$$

The result now follows immediately upon application of Lemma 1.

Section 2. We would like to recall a few facts about Hilbert-Schmidt and trace class operators on a separable Hilbert space H. For complete information about these operators, the reader is referred to [2]. Let $\{\psi_i\}_{i=1}^{\infty}$ be a complete orthonormal set for H. A bounded linear operator X is said to be a Hilbert-Schmidt (H. S.) operator in case the quantity $\|X\| = \{\sum_i |X\psi_i|^2\}^{1/2}$ is finite where $|X\psi_i|$ is the norm of the vector $X\psi_i$. The number $\|\cdot\|$ is sometimes referred to as the Hilbert-Schmidt norm and is independent of the orthonormal basis chosen. Every H.S. operator is compact and is the limit in the $\|\cdot\|$ norm of a sequence of operators with finite range. If X is an H.S. operator and f is a singlevalued analytic function on a domain containing $\sigma(X)$ vanishing at zero, then $f(X)$ is an H.S. operator and the map $X \to f(X)$ is continuous in the $\|\cdot\|$ norm.

Let $\{\lambda_i\}_{i=1}^{\infty}$ be the eigenvalues repeated according to multiplicity of the H.S. operator X. X is said to be a trace class if $\sum_{i=1}^{\infty} |\lambda_i| < \infty$. The trace X of an operator of trace class is defined to be trace $X = \sum_{i=1}^{\infty} \lambda_i$. Although an H.S. operator X need not be of trace class, the product of two H.S. operators are of trace class [2, p. 1093].

Two compact operators X and Y on H will be said to be simultaneously triangularized iff there exists an orthonormal bases $\{\psi_i\}_{i=1}^{\infty}$ and orthogonal projections P_n onto the subspace determined by $\psi_1, \psi_2, \ldots, \psi_n$ such that $P_n X P_n = X P_n$ and $P_n Y P_n = = Y P_n$.

In view of the above statements, it would be desirable to extend Theorem 1 to H.S. operators. With this in mind the following lemma is useful.

Lemma 2. *Let X and Y be a simultaneous triangularizable pair of Hilbert-Schmidt operators, then there exists two sequences of operators $\{X_n\}$, $\{Y_n\}$ such that the following hold:*

(i) *for each n, X_n and Y_n are simultaneously triangularizable operators of finite rank,*

(ii) *the sequence $\{X_n\}$ and $\{Y_n\}$ converge in the Hilbert-Schmidt norm to X and Y respectively.*

Proof. By the hypothesis there exists an orthonormal basis $\{\psi_i\}_{i=1}^{\infty}$ and a sequence of orthogonal projections $\{P_n\}$ onto the subspace spanned by $\psi_1, \psi_2, \ldots, \psi_n$ such that $P_n X P_n = X P_n$ and $P_n Y P_n = Y P_n$ for each n. Set $X_n = P_n X P_n$ and $Y_n = P_n Y P_n$. Clearly (i) is satisfied.

To show (ii), observe that

$$\|X - X_n\|^2 = \sum_{i=1}^{\infty} |(X - P_n X P_n)\,\psi_i|^2 = \sum_{i=1}^{\infty} |(X - X P_n)\,\psi_i|^2 = \sum_{i=n+1}^{\infty} |X\psi_i|^2 .$$

Since $\sum_{i=1}^{\infty} |X\psi_i|^2$ equals $\|X\|$ and converges, the proof of the lemma is completed.

The above lemma, together with the earlier discussion, the continuity of trace [2, p. 1100], and Theorem 1 yields the following theorem.

Theorem 2. *Let X and Y be simultaneously triangularizable Hilbert-Schmidt operators on a separable Hilbert space, K an open disc, and $f(\lambda)$ an analytic function vanishing at zero with a power series whose region of convergence contains $\sigma(X)$ and $\sigma(X + Y)$. If $f'[\gamma, \alpha] \in K$ for $\gamma \in \sigma(X + Y)$ and $\alpha \in \sigma(X)$, then* $\{\text{trace}\,(YY^*)\}^{-1}\,\text{trace}\,\{Y(f(X + Y) - f(X))\} \in K$.

References

[1] *Davis, C.*, An Inequality for Traces of Matrix Functions, Czech. Math. J., vol. *15* (1965), pp. 37–41.

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