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## COMMUTATIVE SEMI-PRIMARY SEMIGROUPS

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The study of primary semigroups is initiated in [2] by M. SATYANARAYANA. Here we deal with semigroups in which every ideal is semi-primary; where in a commutative semigroup, an ideal  $A$  is called a *semi-primary* ideal if  $\sqrt{A}$  is a prime ideal. We call such semigroups to be semi-primary. We consider only commutative semigroups in this paper and for the various definitions and terms involved [2] may be consulted.

**Theorem 1.** *Let  $S$  be a commutative semigroup. Then the following statements about  $S$  are equivalent:*

- (1) *S is a semiprimary semigroup.*
- (2) *Every principal ideal of  $S$  is semi-primary.*
- (3) *Prime ideals of  $S$  are totally ordered.*

*Proof.* (1) implies (2) is obvious. We prove first (2) implies (3). Let  $P$  and  $Q$  be two prime ideals of  $S$  and assume further that  $P \not\subseteq Q$  and  $Q \not\subseteq P$ . Thus we have  $a \in P - Q$  and  $b \in Q - P$ , which means  $ab \in P \cap Q$  and  $a \notin P \cap Q$ ,  $b \notin P \cap Q$ . Let  $\sqrt{(a)} = P_1$ ,  $\sqrt{(b)} = Q_1$  and  $\sqrt{(ab)} = P'$ , where  $P_1$ ,  $Q_1$  and  $P'$  are prime ideals. This gives  $P' = P_1 \cap Q_1$ , that is,  $P_1 \subseteq P'$  or  $Q_1 \subseteq P'$ , but both are impossible. This contradiction proves (3). Assume now (3), and let  $A$  be any ideal of  $S$ . By 1.13 of [2],  $\sqrt{A} = \bigcap P_\alpha$ , where intersection is over all prime ideals  $P_\alpha \supseteq A$ . The totally ordered nature of prime ideals yields  $\sqrt{A} = P$ , for some prime  $P$ , so that  $A$  is semi-primary. Therefore  $S$  is a semi-primary semigroup.

**Corollary.** *Let  $S$  be a commutative semi-primary semigroup. Then idempotents form a chain under natural ordering.*

*Proof.* Let  $e$  and  $f$  be any two idempotents of  $S$ ; then  $\sqrt{(eS)}$  and  $\sqrt{(fS)}$  are prime ideals, so either  $\sqrt{(eS)} \subseteq \sqrt{(fS)}$  or  $\sqrt{(fS)} \subseteq \sqrt{(eS)}$ , which proves the assertion.

Converse of this corollary is false, which can be seen by the following

**Example.** Consider the semigroup of all natural numbers with respect to multiplication. In this,  $(6)$  is not prime and  $\sqrt{(6)} = (6)$ , thus  $(6)$  is not a semi-primary ideal, so that the semigroup is not semi-primary whereas the set of idempotents (there is just one in it) trivially forms a chain.

**Theorem 2.** *Let  $S$  be a regular commutative semigroup. Then the following statements about  $S$  are equivalent:*

- (1) *Every ideal in  $S$  is prime.*
- (2)  *$S$  is a primary semigroup.*
- (3)  *$S$  is a semi-primary semigroup.*
- (4) *Idempotents in  $S$  form a chain under natural ordering.*
- (5) *Principal ideals of  $S$  are totally ordered.*
- (6) *All ideals of  $S$  are totally ordered.*

**Proof.** (1)  $\Rightarrow$  (2) and (2)  $\Rightarrow$  (3) are immediate. (3)  $\Rightarrow$  (4) in view of the above corollary. We prove the remaining one by one. Assume (4). Let  $a$  and  $b$  be any two elements of  $S$ ,  $(a) = (e)$ ,  $(b) = (f)$  where  $e$  and  $f$  are idempotents in  $S$ , since  $S$  is regular. As idempotents of  $S$  form a chain, so  $(a) \subseteq (b)$  or  $(b) \subseteq (a)$ , which proves (5). Assume now (5) and let  $A, B$  be any two ideals of  $S$  with  $A \not\subseteq B$ . So we have an  $a \in A - B$ . For any  $b$  in  $B$ ,  $(b) \subseteq (a)$ . Therefore  $B \subseteq A$ . Lastly assume (6) and let  $A$  be any ideal of  $S$ . Let  $ab \in A$ . As ideals are totally ordered; so either  $(a) \subseteq (b)$  or  $(b) \subseteq (a)$ . Take for the sake of definiteness,  $(a) \subseteq (b)$ . As  $S$  is regular,  $(a) = (a)^2$  [1]. Now  $a \in (a) = (a)^2 \subseteq (a)(b) \subseteq A$ . Similarly when  $(b) \subseteq (a)$ ,  $b \in A$ . Thus  $A$  is a prime ideal. With this the proof of the theorem is completed.

**Remark.** This theorem tells us that primary semigroups and semi-primary semigroups coincide if they are regular. But this is not true in general, as can be seen by the following

**Example.** Let  $S = \{a, a^2, a^3, \dots\} \cup e$ , where  $e^2 = e$ ,  $ae = ea = a^2$ . In this semigroup,  $P = \{a, a^2, \dots\}$  is a unique proper prime ideal. So by Theorem 1,  $S$  is a semi-primary semigroup; but  $A = \{a^2, a^3, \dots\}$  is not a primary ideal in it. Thus  $S$  is not a primary semigroup. This is because  $S$  is not regular.

**Corollary 1.** *Let  $S$  be a commutative semi-primary semigroup. Then every ideal in  $S$  is prime if and only if  $S$  is regular.*

**Proof.** Let every ideal in  $S$  be prime. Now for any ideal  $A$  of  $S$ ,  $a \in A \Rightarrow a^2 \in A \Rightarrow a \in A^2$ ; since  $A^2$  is prime. Therefore  $A = A^2$  for every ideal  $A$ .  $S$ , then, is regular by a theorem of ISÉKI [1]. The other part follows from Theorem 2.

**Corollary 2.** *Let  $S$  be a commutative semigroup. Then every ideal in  $S$  is prime if and only if  $S$  is regular and idempotents in  $S$  form a chain.*

Proof. When every ideal in  $S$  is prime, as in Cor. 1,  $S$  is regular. Let now  $e$  and  $f$  be any two idempotents of  $S$  such that  $eS \not\subseteq fS$ . Then  $ef \in eS \cap fS$ ,  $e \notin eS \cap fS$  and it being a prime ideal,  $f \in eS \cap fS$ . So  $f = ef$ . And if  $eS \subseteq fS$ , the result is clear. Thus idempotents of  $S$  form a chain. Converse follows from Theorem 2.

#### *References*

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- [2] *M. Satyanarayana*, Commutative Primary Semigroups, Czech. Math. J., 22 (97), (1972), 509—516.

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