Czechoslovak Mathematical Journal

Jiří Močkoř

Families of almost finite character

Czechoslovak Mathematical Journal, Vol. 24 (1974), No. 3, 462–466

Persistent URL: http://dml.cz/dmlcz/101261

Czechoslovak Mathematical Journal, 24 (99) 1974, Praha

FAMILIES OF ALMOST FINITE CHARACTER

JIŘÍ MOČKOŘ, Brno

(Received June 20, 1973)

1. Introduction. In [5], Pirtle introduced the notion of almost Krull domain and proved that almost Krull domains in general have many of the properties of Krull domains. Every Krull domain is defined by a family of valuations of finite character. Hence it seems natural to look for the proper generalization of domains defined by a family of valuations of finite character to domains defined by a family of almost finite character.

2. Families of almost finite character. In what follows D denotes a commutative integral domain with identity and K denotes the quotient field of D. A family Ω of valuations of the field K is said to be of *finite character* if for every $x \in K$, $x \neq 0$ the set $\{w \in \Omega \mid w(x) \neq 0\}$ is finite. If $w \in \Omega$ has a ring R_w and a maximal ideal $M(w)$, then Ω is said to be a *defining family for D* if $D = \bigcap_{w \in \Omega} R_w$, and $M(w) \cap D$ is a prime ideal called the *centre of w on D* and is denoted by $P(w)$. If $R_w = D_{P(w)}$, then w is said to be *essential for D*.

We use the following notation from [2]. If w, w' are valuations of K with the rings R_w, $R_{w'}$ with $R_w \subseteq R_{w'}$ we say that w' is *coarser than* w and write $w' \leqq w$. If Ω, Ω' are families of valuations of K and if every valuation $w' \in \Omega'$ is coarser than a valuation w of Ω, we say that the family Ω' is *coarser than* Ω and write $\Omega' \leqq \Omega$.

Definition 2.1. A defining family Ω for a domain D is called a family of almost finite character for D if for every maximal ideal M of D there exists a subfamily $\Omega_M \subseteq \Omega$ with the following properties:

(i) Ω_M is a defining family for D_M,
(ii) Ω_M is a family of finite character.

If Ω is a defining family of almost finite character for D, then we say that D is defined by a family of almost finite character Ω.

Proposition 2.2. *Let D be defined by a family of almost finite character Ω. Let S be a multiplicative system of D. Then D_S is defined by a family of almost finite character coarser than Ω.*

Proof. Let $\{M_i\}$ be the set of maximal ideals of D_S. Let $P_i = M_i \cap D$, then $(D_S)_{M_i} = (D_S)_{P_i D_S} = D_{P_i}$. Let M_i' be a maximal ideal of D such that $P_i \subseteq M_i'$, hence $D_{P_i} = (D_{M_i'})_{P_i D_{M_i'}}$. Ω is a family of almost finite character for D; hence there exists $\Omega_{M_i'} \subseteq \Omega$ such that $\Omega_{M_i'}$ is a defining family of finite character for $D_{M_i'}$. By [3]; Lemma 11, there exists $\Omega_{P_i} \leqq \Omega_{M_i'}$ which is a defining family of finite character for D_{P_i}. Hence the family $\bigcup_i \Omega_{P_i}$ is a defining family of almost finite character for D_S and $\bigcup_i \Omega_{P_i} \leqq \bigcup_i \Omega_{M_i'} \subseteq \Omega$.

Proposition 2.3. *Let D be integrally closed in K and let L be an algebraic extension of K. Let D' denote the integral closure of D in L.*

(1) *If $[K : L] < \infty$ and D is defined by a family of almost finite character Ω, then D' is defined by a family of almost finite character coarser than the family of all extensions of valuations of Ω to the valuations of L.*

(2) *If D' is defined by a family of almost finite character, then D is defined by a family of almost finite character.*

Proof. (1) Let D be defined by a family of almost finite character Ω. Let M be a maximal ideal of D'. Let $P = M \cap D$. By [1]; Proposition 9.4, P is a maximal ideal of D. By [1]; Proposition 9.11,

$$(D_P)' = D'_{D-P} = \bigcap_{M_i \in J} D'_{M_i},$$

where J is the set of prime ideals of D' lying over P and X' denotes the integral closure of X in L. It follows that there exists $\Omega_P \subseteq \Omega$ which is a defining family of finte character for D_P. By [3]; Proposition 8, the family Ω'_P of extensions of valuations of Ω_P to the valuations of L is a defining family of finite character for $(D_P)'$. Now $M \in J$, hence $D' \subseteq (D_P)' \subseteq D'_M$. By [3]; Lemma 11, there exists a defining family of finite character $\Omega'_M \leqq \Omega_P$ for D'_M. Hence $\bigcup \Omega'_M$ is a defining family of almost finite character for D' and $\bigcup \Omega'_M \leqq \bigcup \Omega'_P \subseteq \Omega'$.

(2) Let D' be defined by a family of almost finite character Ω'. Let M be a maximal ideal of D and let M' be a prime ideal of D' lying over M. Hence M' is a maximal ideal of D'. It follows that there exists $\Omega_{M'} \subseteq \Omega'$ which is a defining family of finite character for $D'_{M'}$. Let

$$\Omega_M = \{w \mid w \text{ is a valuation of } K \text{ and } R_w = R_{w'} \cap K \text{ for some } w' \in \Omega_{M'}\}.$$

Then

$$D_M = D'_{M'} \cap K = \left(\bigcap_{w' \in \Omega_{M'}} R_{w'} \right) \cap K = \bigcap_{w' \in \Omega_{M'}} (R_{w'} \cap K) = \bigcap_{w \in \Omega_M} R_w ,$$

hence Ω_M is a defining family for D_M. Since $\Omega_{M'}$ is of finite character, Ω_M is of finite character. Hence the family $\bigcup \Omega_M$ is a defining family of almost finite character for D.

Proposition 2.4. *Let D be defined by a family Ω of almost finite character and let Ω' denote the family of canonical extensions of elements of Ω to valuations of $K(\{X_i\}_{i \in J})$ while G denotes the family of valuations of $K(\{X_i\}_{i \in J})$ defined by irreducible polynomials from $K[\{X_i\}_{i \in J}]$. Then $D[\{X_i\}_{i \in J}]$ is defined by a family of almost finite character coarser than $\Omega' \cup G$.*

Proof. Let M be a maximal ideal of $D[\{X_i\}_{i \in J}]$. Let Q be a maximal ideal of D containing a prime ideal $M \cap D$. Hence there is a family $\Omega_Q \subseteq \Omega$ which is a defining family of finite character for D_Q. By [3]; Proposition 9, the ring $D_Q[\{X_i\}_{i \in J}]$ is defined by a family of finite character $\Omega'_Q \cup G_Q$ where Ω'_Q denotes the family of canonical extensions of elements of Ω_Q to the valuations of $K(\{X_i\}_{i \in J})$ and $G_Q = \{w \in G \mid D_Q[\{X_i\}_{i \in J}] \subseteq R_w\}$. Now

$$D[\{X_i\}_{i \in J}] \subseteq D_Q[\{X_i\}_{i \in J}] \subseteq D_{M \cap D}[\{X_i\}_{i \in J}] \subseteq (D[\{X_i\}_{i \in J}])_M .$$

By [3]; Lemma 11, there exists $\psi_M \leqq \Omega_Q \cup G_Q$ and ψ_M is a defining family of finite character for $(D[\{X_i\}_{i \in J}])_M$. Hence $\psi = \bigcup \psi_M$ is a defining family of almost finite character for $D[\{X_i\}_{i \in J}]$ and

$$\psi = \bigcup \psi_M \leqq \bigcup (\Omega'_Q \cap G_Q) \subseteq \Omega' \cup G .$$

Now, we extend some results from [2].

Proposition 2.5. *Let D be defined by a family Ω of almost finite character. Let S be a multiplicative system of D. Let E be the family of prime ideals of D having empty intersection with S and assume that each prime in E contains a minimal prime in E and that there is only a finite number of minimal primes in E. Then D_S is a Prüfer ring.*

Proof. By Proposition 2.2 there exists a family of almost finite character Ω_1 for a domain D_S. Let M be a maximal ideal of D_S; thus there exists a defining family $\Omega_{1,M} \subseteq \Omega_1$ of finite character for $(D_S)_M$. Let $E_M = \{PD_S \mid P \in E \text{ and } PD_S \subseteq M\}$. It follows that the set E_M contains only a finite number of minimal primes in E_M and that each prime in E_M contains a minimal prime in E_M. Let $\{P_1 D_S, \ldots, P_n D_S\}$ be the set of minimal primes in E_M. Hence $\{P_1(D_S)_M, \ldots, P_n(D_S)_M\}$ is the set of minimal primes of $(D_S)_M$ and $\prod_{i=1}^{n} P_i(D_S)_M \neq (0)$. Let w be an element of $\Omega_{1,M}$. Since $P(w)\, D_S = (M(w) \cap D)\, D_S$ is an element of E_M, it follows that there exists a minimal prime

P_iD_S in E_M contained in $P(w)\,D_S$. It follows then that $M(w) \supseteq P(w)\,(D_S)_M \supseteq \supseteq P_i(D_S)_M \supseteq \prod_{i=1}^{n} P_i(D_S)_M \supset (0)$. Let x be a non zero element of $\prod_{i=1}^{n} P_i(D_S)_M$; thus $x \in M(w)$ for every $w \in \Omega_{1,M}$. Since $\Omega_{1,M}$ is of finite character, it is finite. Hence $(D_S)_M$ is an intersection of finite number of valuation rings, and therefore it is a Prüfer ring. Since $(D_S)_M$ is quasi-local, it is a valuation ring. Therefore, D_S is a Prüfer ring.

For every prime ideal P of D let $E(P)$ denote the set of prime ideals of D contained in P.

Proposition 2.6. *Let D be defined by a family Ω of almost finite character and suppose that every prime ideal of D contains a minimal prime ideal. If $E(P(w))$ is totally ordered for every valuation $w \in \Omega$, then Ω is a family of essential valuations for D.*

Proof. Let $E(P(w))$ be totally ordered for every $w \in \Omega$. Then $P(w)$ contains just one minimal prime ideal and applying Proposition 2.5 to multiplicative system $D - P(w)$, we obtain that the domain $D_{P(w)}$ is a Prüfer ring, and therefore also a valuation ring. Therefore, $R_w = D_{P(w)}$.

Proposition 2.7. *Let D be defined by a family Ω of almost finite character. Let P be a prime ideal of D which is such that $E(P)$ is totally ordered. Then there exists a valuation v coarser than a valuation $w \in \Omega$ such that $P(v) = P$.*

Proof. Let M be a maximal ideal of D containing P. Then there exists a defining family $\Omega_M \subseteq \Omega$ of finite character for D_M. Since $E(P)$ is totally ordered, the set $E(PD_M)$ of prime ideals of D_M contained in PD_M is totally ordered. By [2]; Lemma 11, there exists a valuation v coarser than a $w \in \Omega_M$ such that $P'(v) = M(v) \cap D_M = PD_M$. Thus,

$$P(v) = P'(v) \cap D = PD_M \cap D = P\,.$$

Corollary 2.8. *Let D be defined by a family Ω of almost finite character and let each $w \in \Omega$ be essential for D. Let P be a minimal prime ideal of D. Then there exists $w \in \Omega$ such that $P \subseteq P(w)$.*

Proposition 2.9. *Let D be an integral domain. Then the following assertions are equivalent.*

(1) *D is a Prüfer ring.*

(2) *Every ring D', $D \subseteq D' \subseteq K$, is defined by a family of almost finite character.*

(3) *D is defined by a family of almost finite character Ω of essential valuations for D and $E(P)$ is totally ordered for every maximal ideal P of D.*

Proof. Let (1) hold. By [1]; Theorem 22.1, every ring D', $D \subseteq D' \subseteq K$, is a Prüfer ring. It is clear that Prüfer ring is defined by a family of almost finite character. Thus

(1) $\Rightarrow$ (2). Now assume that (2) holds. Since every ring D_M, where M is a maximal ideal of D, is defined by a family of almost finite character, it is integrally closed. Hence, D is integrally closed. If all rings D', $D \subseteq D' \subseteq K$ are integrally closed, then D is a Prüfer ring ([1]; Theorem 22.2). Thus, (1) holds.

The implication (1) $\Rightarrow$ (3) is trivial.

Now assume that (3) holds. Let P be a maximal ideal of D. The set $E(P)$ is totally ordered, hence by Proposition 2.7, there exists a valuation v coarser than a $w \in \Omega$ and $P(v) = P$. Thus, R_v is essential for D, $R_v = D_{P(w)} = D_P$. Therefore, D is a Prüfer ring and (1) holds.

Proposition 2.10. *Let D be a one-dimensional domain. Then D is a Prüfer ring if and only if D is defined by a family of almost finite character.*

Proof. The part "only if" is trivial. Let D be defined by a family Ω of almost finite character and let M be a maximal ideal of D. Then there exists $\Omega_M \subseteq \Omega$ which is a defining family of finite character for D_M. Let w be an element of Ω_M and let $P'(w) = M(w) \cap D_M$. Since $P'(w)$ is a prime ideal of D_M and D_M is one-dimensional and quasi-local, it is $P'(w) = MD_M$. Let x be a non zero element of MD_M, so $w(x) > 0$ for all $w \in \Omega_M$. Hence Ω_M is finite. It follows that D_M is a Prüfer ring, hence D_M is a valuation ring. Thus, D is a Prüfer ring.

Proposition 2.11. *Let D be defined by a family Ω of almost finite character. If every non zero proper ideal of D is contained in only a finite number of maximal ideals of D, then Ω is a family of finite character.*

The proof of this proposition is substantially the same as that of [5]; Proposition 2.17, and will be omitted.

References

[1] *R. Gilmer,* Multiplicative ideal theory, "Queen's Papers on pure and Applied Mathematics", Kingston, Ontario, 1968.

[2] *M. Griffin,* Families of finite character, Trans. Amer. Math. Soc. *130* (1968), 75–85.

[3] *M. Griffin,* Some results on v-multiplication rings, Can. J. Math. *19* (1968), 710–722.

[4] *J. L. Mott,* On complete integral closure of a domain of Krull type, Math. Ann. *173* (1967), 238–240.

[5] *R. Pirtle,* Integral domains which are almost Krull, J. Sci. Hiroshima Univ. Ser. A-I, *32* (1968), 441–447.

Author's address: 662 95 Brno, Janáčkovo nám. 2a, ČSSR (Přírodovědecká fakulta UJEP).