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Ernst J. Albrecht; Florian-Horia Vasilescu

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NON-ANALYTIC LOCAL SPECTRAL PROPERTIES IN SEVERAL VARIABLES

E. J. ALBRECHT, Kaiserslautern and F. - H. VASILESCU, Bucharest

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1. Introduction. Let X be a Banach space and $\mathscr{L}(X)$ the algebra of all linear continuous operators on X. In what follows we shall consider finite systems of mutually commuting operators $T = (T_1, \ldots, T_n)$ $(T_j \in \mathscr{L}(X);\ j = 1, \ldots, n)$, often called n-tuples of operators. The purpose of this paper is to find multidimensional variants of some results contained in [5], stated there for a single operator. We shall constantly use definitions and spectral properties of finite systems of commuting operators as given in [1]. For the convenience of the reader we shall recall some definitions of [1]. (For $n = 1$ see also [3].)

If $k = (k_1, \ldots, k_n)$, $l = (l_1, \ldots, l_n)$ are multi-indices then we denote

$$|k| = k_1 + \ldots + k_n\,, \quad k! = k_1! \ldots k_n!$$

and

$$\partial_{k,l} = \frac{\partial^{|k|+|l|}}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}\, \partial \bar{z}_1^{l_1} \ldots \partial \bar{z}_n^{l_n}}\,.$$

Let V be an open subset of $\mathbf{C}^n(\mathbf{C}^n = \mathbf{R}^{2n})$. Denote by $\mathscr{C}^m(V, X)$ $(m = 0, 1, 2, \ldots; \infty)$ the space of all X-valued functions, defined in V, m-times continuously differentiable, endowed with the topology of compact convergence of the functions and their derivatives up to the order m. The topology of $\mathscr{C}^m(V, X)$ $(m < \infty)$ is given by the seminorms $\{\|\cdot\|_{m,K};\ K \subset V \text{ compact}\}$, with

$$\|f\|_{m,K} = \sum_{0 \leqq |k|+|l| \leqq m} \frac{1}{k!\, l!} \sup_{z \in K} \|(\partial_{k,l} f)(z)\| \tag{1.1}$$

for any $f \in \mathscr{C}^m(V, X)$. For every compact set $K \subset \mathbf{C}^n$, $\mathscr{D}^m(K, X)$ will stand for the closed subspace of all functions f in $\mathscr{C}^m(\mathbf{C}^n, X)$ such that $\operatorname{supp} f \subset K$. If $m < \infty$ then $\mathscr{D}^m(K, X)$ is a Banach space, with the norm given by (1.1). We shall write $\mathscr{C}^m(V)$ $(\mathscr{D}^m(K))$ instead of $\mathscr{C}^m(V, \mathbf{C})$ (respectively $\mathscr{D}^m(K, \mathbf{C})$. If $f \in \mathscr{C}^m(\mathbf{C}^n, X)$ has compact support, then we shall denote its seminorm given by (1.1), for $K = \operatorname{supp} f$, simply by $\|f\|_m$.

1.1. Definition. Let $T = (T_1, \ldots, T_n)$ be a commuting finite system of operators in $\mathscr{L}(X)$. We say that T is a *$\mathscr{C}^m$-scalar n-tuple* if there exists a continuous algebra homomorphism $U : \mathscr{C}^m(\mathbf{C}^n) \to \mathscr{L}(X)$, such that $U(1) = 1_X$ and $U(z_j) = T_j$ $(j = 1, \ldots, n)$, where z_j stands for the function

$$(1.2) \qquad \mathbf{C}^n \ni z = (z_1, \ldots, z_n) \to z_j \in \mathbf{C}\,.$$

Such a homomorphism U will be called a *$\mathscr{C}^m$-functional calculus* for the n-tuple T [1].

1.2. Definition. Let $T = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators on X and $x \in X$ a fixed element. We say that $z \in \mathbf{C}^n$ is in *the local joint resolvent set* $\varrho_T(x)$ of T at x if there is a neighbourhood V of z and n X-valued functions u_j, analytic in V, such that $\sum_{j=1}^{n} (w_j - T_j)\, u_j(w) = x$ for all $w \in V$. *The local joint spectrum* $\sigma_T(x)$ of T at x is the set $\mathbf{C}^n \setminus \varrho_T(x)$, and $\sigma_T(x)$ is a compact subset of $\mathbf{C}^n$ [1].

Let T be an n-tuple of operators on X and F a closed set in $\mathbf{C}^n$. Denote by

$$(1.3) \qquad X_T(F) = \{x \in X;\ \sigma_T(x) \subset F\}\,.$$

When T is a $\mathscr{C}^m$-scalar n-tuple then any space of type (1.3) is a spectral maximal space of T (for details see [1]). In this case $X_T(F)$ is a closed subspace of X, invariant to T and to any $\mathscr{C}^m$-functional calculus of T. Moreover $\sigma(T, X_T(F)) \subset F$, where $\sigma(T, X_T(F))$ denotes the spectrum of T with regard to $X_T(F)$, in the sense of J. L. Taylor [8].

In the sequel we shall investigate first what happens if in Definition 1.2 one takes functions u_j, not necessarily analytic, T being a $\mathscr{C}^m$-scalar n-tuple. With the same conditions, we study then the algebraic character of the structure of the spaces given by (1.3).

Finally, we show that some indices can be improved in the case of n-tuples of commuting scalar operators [4] on Hilbert spaces.

2. More about the local joint spectrum. It seems that a good definition, valid for n-tuples of commuting linear operators, of the single valued extension property [4] is very difficult to be given. It is easier to define a notion of local joint spectrum $\mathrm{sp}_T(x)$, as done in [1] (see also Definition 1.2) and then to verify that the definition is "reasonable enough", i.e. it has the property:

$$(2.1) \qquad \mathrm{sp}_T(x) = \emptyset \Rightarrow x = 0\,.$$

When $n = 1$ and $\mathrm{sp}_T(x) = \sigma_T(x)$, the property (2.1) is equivalent to the single valued extension property [10]. It is fundamental for any operator T having a "good" spectral decomposition, for example if T is decomposable [3]. Let us notice that the property (2.1) is also valid for some commuting n-tuples T, if $\mathrm{sp}_T(x) = \sigma_T(x)$ (see [1, Th. I.2.11]).

2.1. Definition. Let $T = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators in $\mathscr{L}(X)$ and $x \in X$. A point $z \in \mathbf{C}^n$ is in the set $\varrho_T^{(k)}(x)$ if there is an open neighbourhood V of z and $u_j \in \mathscr{C}^k(V, X)$ $(j = 1, \ldots, n)$ such that $\sum_{j=1}^{n} (w_j - T_j)\, u_j(w) = x$ for all $w \in V$. We shall put $\sigma_T^{(k)}(x) = \mathbf{C}^n \setminus \varrho_T^{(k)}(x)$ (for the case $n = 1$ see [5]).

The method of proof of the next result has been suggested to the first author by Prof. B. Gramsch in a private discussion.

2.2. Lemma. *Let K be a compact set in $\mathbf{C}^n$. Then for any integer $q \geqq 2n + m + 1$, the space $\mathscr{D}^q(K, X)$ may be naturally embedded into the complete projective tensor product $\mathscr{D}^m(\tilde{K}) \hat{\otimes}_\pi X$, where $\tilde{K}$ is an arbitrary compact neighbourhood of K.*

Proof. Let φ be a fixed element of $\mathscr{D}^q(K, X)$. There is an $r > 0$ such that $\tilde{K} \subset \subset \prod_{1}^{2n} [-\pi r, \pi r]$, where $\tilde{K}$ stands for an arbitrary neighbourhood of K. Let us consider the Fourier expansion of φ, namely

$$\varphi(t) = \sum_{\alpha \in \mathbf{Z}^{2n}} e^{i\langle \alpha, t/r \rangle} F_\alpha(\varphi)\,, \tag{2.2}$$

where $\langle \alpha, t/r \rangle = \alpha_1 t_1/r + \ldots + \alpha_{2n} t_{2n}/r$, and

$$F_\alpha(\varphi) = (2\pi r)^{-2n} \int_{-\pi r}^{\pi r} \ldots \int_{-\pi r}^{\pi r} e^{-i\langle \alpha, s/r \rangle}\, \varphi(s)\, \mathrm{d}s\,. \tag{2.3}$$

By a straightforward calculus one gets for a fixed $\alpha \in \mathbf{Z}^{2n}$, $\alpha = (\alpha_1, \ldots, \alpha_{2n})$,

$$\Big(i + \sum_{j=1}^{2n} |\alpha_j|\Big)^q F_\alpha(\varphi) = F_\alpha\left(\left(i - ir \sum_{j=1}^{2n} \operatorname{sgn} \alpha_j \frac{\partial}{\partial s_j}\right)^q \varphi\right), \tag{2.4}$$

where "sgn" denotes the usual signum function.

From (2.3) and (2.4) we obtain

$$\Big\|\Big(i + \sum_{j=1}^{2n} |\alpha_j|\Big)^q F_\alpha(\varphi)\Big\| \leqq \sup_{s \in K} \left\|\left(1 - r \sum_{j=1}^{2n} \operatorname{sgn} \alpha_j \frac{\partial}{\partial s_j}\right)^q \varphi(s)\right\|$$

whence

$$\Big\|\Big(i + \sum_{j=1}^{2n} |\alpha_j|\Big)^q F_\alpha(\varphi)\Big\| \leqq C \|\varphi\|_{q,K}\,, \tag{2.5}$$

where $C > 0$ does not depend on α.

Now, let ψ be a function in $\mathscr{D}^q(\tilde{K})$ such that $\psi = 1$ in a neighbourhood of K. Then we have

$$\varphi(t) = \psi(t)\, \varphi(t) = \sum_{\alpha \in \mathbf{Z}^{2n}} e^{i\langle \alpha, t/r \rangle}\, \psi(t)\, F_\alpha(\varphi) \tag{2.2'}$$

and we shall show that the right side of (2.2)′ can be embedded into the space $\mathscr{D}^m(\tilde{K}) \hat{\otimes}_\pi X$. For, let us remark that on account of Leibniz' formula we get

$$\|e^{i\langle\alpha,t/r\rangle}\psi\|_{m,\tilde{K}} \leqq C_1 |i + \sum_{j=1}^{2n} |\alpha_j| |^m \|\psi\|_{m,K} . \tag{2.6}$$

According to (2.5) and (2.6) we may write

$$\sum_{\alpha\in\mathbf{Z}^{2n}} \|e^{i\langle\alpha,t/r\rangle}\psi\|_{m,\tilde{K}} \|F_\alpha(\varphi)\| \leqq C_1 \|\psi\|_{m,\tilde{K}} \sum_{\alpha\in\mathbf{Z}^{2n}} \|(i + \sum_{j=1}^{2n} |\alpha_j|)^m F_\alpha(\varphi)\| \leqq$$
$$C_2(\sum_{\alpha\in\mathbf{Z}^{2n}} |i + \sum_{j=1}^{2n} |\alpha_j| |^{m-q}) \|\varphi\|_{q,K} .$$

Since $q \geqq 2n + m + 1$ then the series

$$\sum_{\alpha\in\mathbf{Z}^{2n}} |i + \sum_{j=1}^{2n} |\alpha_j| |^{m-q}$$

is convergent, therefore

$$\tilde{\varphi} = \sum_{\alpha\in\mathbf{Z}^{2n}} e^{i\langle\alpha,t/r\rangle}\psi \otimes F_\alpha(\varphi)$$

is an element of $\mathscr{D}^m(\tilde{K}) \hat{\otimes}_\pi X$ and

$$\|\tilde{\varphi}\|_{\mathscr{D}^m(\tilde{K})\hat{\otimes}_\pi X} \leqq C_3 \|\varphi\|_{q,K} ,$$

where $C_3 > 0$ does not depend on φ, hence the map

$$\mathscr{D}^q(K, X) \ni \varphi \to \tilde{\varphi} \in \mathscr{D}^m(\tilde{K}) \hat{\otimes}_\pi X \tag{2.7}$$

is a natural embedding.

The following result has a similar proof with that of [1, Th, I.2.11].

2.3. Theorem. *Let $T = (T_1, .., T_n)$ be a $\mathscr{C}^m$-scalar n-tuple of operators on X and let U be a $\mathscr{C}^m$-functional calculus for T. Then for all $x \in X$ and $q \geqq 2n + m + 1$ we have*

$$\operatorname{supp} U(\cdot)\, x = \sigma_T^{(q)}(x) = \sigma_T(x)$$

(*here $f \to U(f)\, x$ is an X-valued distribution and* supp $U(\cdot)\, x$ *is its support*).

Proof. The inclusion $\sigma_T^{(q)}(x) \subset \sigma_T(x)$ is obvious from the definition. We shall show now the inclusion supp $U(\cdot)\, x \subset \sigma_T^{(q)}(x)$. For, let us consider a function $f \in \mathscr{C}^q(\mathbf{C}^n)$ such that supp $f \subset \varrho_T^{(q)}(x)$. According to Definition 2.1, for every $z \in$ supp f there exists an open neighbourhood V_z of z and n functions $u_{1,z}, .., u_{n,z}$ in $\mathscr{C}^q(V_z, X)$ such that $\sum_{j=1}^{n} (w_j - T_j)\, u_{j,z}(w) = x$ for all $w \in V_z$. Since supp U is compact [1, Th. I.2.9], then there is a finite number of points $z_1, \ldots, z_p$ such that $V_{z_1}, \ldots, V_{z_p}$ is a covering of the set supp $f \cap$ supp U. Let us set $V_l = V_{z_l}$, $u_{j,l} = u_{j,z_l}$ $(l = 1, \ldots, p;$

$j = 1, \ldots, n$), $V_0 = \mathbf{C}^n \setminus \operatorname{supp} U$ and $V_{p+1} = \mathbf{C}^n \setminus \operatorname{supp} f$. Then $\{V_0, V_1, \ldots, V_{p+1}\}$ is a covering of $\mathbf{C}^n$, therefore we may take a $\mathscr{C}^\infty(\mathbf{C}^n)$-partition of the unity, say $\{h_l\}_{l=0}^{p+1}$, such that $\operatorname{supp} h_l \subset V_l$ and $\sum_{l=0}^{p+1} h_l = 1$. Then we have $f = \sum_{l=0}^{p+1} h_l f = \sum_{l=0}^{p} h_l f$. On the other hand, $\sum_{j=1}^{n} (w_j - T_j)\, u_{j,l}(w) = x$ for each $w \in \operatorname{supp} h_l$ $(l = 1, \ldots, p)$, therefore for all $w \in \mathbf{C}^n$ we get

$$(2.8) \qquad f(w)\, x = h_0(w) f(w)\, x + \sum_{l=1}^{p} \sum_{j=1}^{n} (w_j - T_j) f(w)\, h_l(w)\, u_{j,l}(w)\,.$$

Notice that $h_l(w)\, u_{j,l}(w) \in \mathscr{D}^q(\operatorname{supp} h_l, X)$ $(l = 1, \ldots, p)$, therefore by Lemma 2.2 we may identify $h_l(w)\, u_{j,l}(w)$ with an element of $\mathscr{D}^m(\tilde{K}_l) \hat{\otimes}_\pi X$, where $\tilde{K}_l$ is an arbitrary compact neighbourhood of $\operatorname{supp} h_l$. Since the space $\mathscr{D}^m(\tilde{K}_l) \hat{\otimes}_\pi X$ is obviously contained in $\mathscr{C}^m(\mathbf{C}^n) \hat{\otimes}_\pi X$, then the relation (2.8) may be considered in the space $\mathscr{C}^m(\mathbf{C}^n) \hat{\otimes}_\pi X$ and it has the form:

$$(2.9) \qquad f \otimes x = h_0 f \otimes x + \sum_{l=1}^{p} \sum_{j=1}^{n} (z_j f \otimes 1_X - f \otimes T_j)(h_l u_{j,l})\,,$$

where z_j $(j = 1, \ldots, n)$ are the coordinate functions given by (1.2).

Let us remark that the map $U \otimes 1_X$, naturally defined on $\mathscr{C}^m(\mathbf{C}^n) \otimes X$, has a continuous extension $U \hat{\otimes}_\pi 1_X$ from $\mathscr{C}^m(\mathbf{C}^n) \hat{\otimes}_\pi X$ into $\mathscr{L}(X) \hat{\otimes}_\pi X$ [9, Th. 43.6]. If we denote by J the continuous extension of the map $J_0 : \mathscr{L}(X) \otimes X \to X$ on the space $\mathscr{L}(X) \hat{\otimes}_\pi X$ $(J_0(\sum_l A_l \otimes x_l) = \sum_l A_l x_l)$, then the map $J(U \hat{\otimes}_\pi 1_X)$ is continuous from $\mathscr{C}^m(\mathbf{C}^n) \hat{\otimes}_\pi X$ into X. Notice that for any $L = \sum_l g_l \otimes y_l \in \mathscr{C}^m(\mathbf{C}^n) \otimes X$ we get

$$J(U \otimes 1_X)\big((z_j f \otimes 1_X - f \otimes T_j)\, L\big) =$$
$$= J\big(\textstyle\sum_l (U(z_j)\, U(f)\, U(g_l) \otimes y_l - U(f)\, U(g_l) \otimes T_j y_l\big) = 0\,.$$

Since $\mathscr{C}^m(\mathbf{C}^n) \otimes X$ is dense in $\mathscr{C}^m(\mathbf{C}^n) \hat{\otimes}_\pi X$, it follows that

$$J(U \hat{\otimes}_\pi 1_X)\big((z_j f \otimes 1_X - f \otimes T_j)\, L\big) = 0\,,$$

for all $L \in \mathscr{C}^m(\mathbf{C}^n) \hat{\otimes}_\pi X$. By applying the operator $J(U \hat{\otimes}_\pi 1_X)$ to the relation (2.9) we obtain

$$U(f)\, x = U(f)\, U(h_0)\, x = 0\,,$$

because $\operatorname{supp} h_0 \cap \operatorname{supp} f = \emptyset$. Consequently, $\operatorname{supp} U(\cdot)\, x \subset \sigma_T^{(q)}(x)$.

Finally, let us show that $\sigma_T(x) \subset \operatorname{supp} U(\cdot)\, x$. For, let us consider a point $w^0 \notin$ $\notin \operatorname{supp} U(\cdot)\, x$. On account of [1, Lemma I.2.3] there is an open neighbourhood V_1 of $\operatorname{supp} U(\cdot)\, x$, an open neighbourhood V_2 of w^0 and n functions $u_j(z, w)$ with the properties:

(1) for any $w \in V_2$ the function $z \to u_j(z, w) \in \mathscr{C}^m(V_1)$;

(2) the map $w \to u_j(\cdot, w)$ $(V_2 \to \mathscr{C}^m(V_1))$ is analytic;
(3) the following relation holds:

$$\sum_{j=1}^{n} (w_j - z_j)\, u_j(z, w) = 1 \quad (z \in V_1,\ w \in V_2)\,. \tag{2.10}$$

Let f be a function in $\mathscr{C}^m(\mathbf{C}^n)$, $\operatorname{supp} f \subset V_1$, such that $f = 1$ in a neighbourhood of $\operatorname{supp} U(\cdot)\, x$, therefore $U(f)\, x = x$. Then, by (2.10) we obtain

$$\sum_{j=1}^{n} (w_j - T_j)\, U(f u_j(\cdot, w))\, x = x\,.$$

Since U is a continuous distribution then the map $w \to U(f u_j(\cdot, w))\, x$ is analytic in V_2 $(j = 1, \ldots, n)$, hence $w^0 \in \varrho_T(x)$. The proof is complete.

2.4. Corollary. $\sigma_T^{(q)}(x) = \emptyset$ *if and only if* $x = 0$, *for every* $q \geqq 2n + m + 1$.

Proof. Indeed, if $x = 0$ then $\operatorname{supp} U(\cdot)\, x = \emptyset$.

2.5. Remark. Theorem 2.3 is not exactly the multidimensional version of [5, Prop. 2.7]. Indeed, if $n = 1$ we obtain $\sigma_T^{(q)}(x) = \sigma_T(x)$ for any $q \geqq m + 3$, while in [5, Prop. 2.7] the same thing is true for $q \geqq m + 1$.

3. The structure of spectral maximal spaces. Most of the results of this section are generalizations of the corresponding statements in [5]. Let us also remark that such problems have been initiated in the one-dimensional case by P. VRBOVÁ [11].

In what follows $T = (T_1, \ldots, T_n)$ will be a fixed $\mathscr{C}^m$-scalar n-tuple and U a $\mathscr{C}^m$-functional calculus for T.

Denote by $D_{r,\lambda}$ the set $\{z \in \mathbf{C}^n;\ |z_j - \lambda_j| \leqq r, j = 1, \ldots, n\}$, where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a fixed point in $\mathbf{C}^n$ and $r > 0$. When $\lambda = 0$ then we put $D_{r,0} = D_r$. Denote also by $\mathrm{d}\nu$ the Lebesgue measure in $\mathbf{C}^n = \mathbf{R}^{2n}$.

For the further proofs we need the following

3.1. Lemma. *For any* $r > 0$, $r \leqq 1$ *there is a function* $\varphi_r \in \mathscr{C}^\infty(\mathbf{C}^n)$ *such that* $\operatorname{supp} \varphi_r \subset D_r$, $\|\varphi_r\|_q \leqq M r^{-q-2n}$ *for any integer* $q \geqq 0$, *where* $M > 0$ *does not depend on* r. *Moreover, the integral*

$$\int \psi(\lambda)\, U(\varphi_r(\lambda - z))\, \mathrm{d}\nu(\lambda)$$

converges to $U(\psi)$ *as* $r \to 0$ *in the norm operator topology, for any* $\psi \in \mathscr{C}^m(\mathbf{C}^n)$.

Proof. There is a function $\varphi \in \mathscr{C}^\infty(\mathbf{C}^n)$, $\varphi \geqq 0$, $\operatorname{supp} \varphi \subset D_1$ and $\int \varphi(\lambda)\, \mathrm{d}\nu(\lambda) = 1$ [6]. Consider now the function $\varphi_r(z) = r^{-2n} \varphi(z/r)$. Then we have

$$|\partial_{k,l}\, \varphi_r(z)| \leqq M_{k,l} r^{-2n-|k|-|l|}\,, \tag{3.1}$$

for any multi-indices k, l, $|k| + |l| \leqq q$. When $r \leqq 1$ we obtain easily

$$\|\varphi_r\|_q \leqq M r^{-q-2n},$$

where $M > 0$ depends only on φ.

Since the integral

$$\int \psi(\lambda)\, \varphi_r(\lambda - z)\, \mathrm{d}\nu(\lambda)$$

converges to ψ in the topology of $\mathscr{C}^m(\mathbf{C}^n)$ as $r \to 0$, then by the continuity of U one obtains

$$U\left(\int \psi(\lambda)\, \varphi_r(\lambda - z)\, \mathrm{d}\nu(\lambda)\right) =$$
$$= \int \psi(\lambda)\, U(\varphi_r(\lambda - z))\, \mathrm{d}\nu(\lambda) \to U(\psi),$$

for any $\psi \in \mathscr{C}^m(\mathbf{C}^n)$.

3.2. Lemma. *Suppose that there is an integer $q \geqq 1$ and an element $x \in X$ such that for any $\lambda \in \sigma_T(x)$ there exist $y_{j,\lambda} \in X$ $(j = 1, \ldots, n)$ such that $\sum_{j=1}^{n} (\lambda_j - T_j)^q\, y_{j,\lambda} = x$. Denote by $Y_\lambda = \{y_\lambda = (y_{1,\lambda}, \ldots, y_{n,\lambda});\ \sum_{j=1}^{n} (\lambda_j - T_j)^q\, y_{j,\lambda} = x\}$. Then there is an open polydisc D such that $D \cap \sigma_T(x) \neq \emptyset$, a constant $C > 0$ and $x_\lambda = (x_{1,\lambda}, \ldots, x_{n,\lambda}) \in Y_\lambda$ with $\max_{1 \leqq j \leqq n} \|x_{j,\lambda}\| \leqq C$, for λ running a dense subset B of $D \cap \sigma_T(x)$.*

Proof. Consider the sets

$$B_N = \text{the closure of } \{\lambda \in \sigma_T(x);\ \inf_{x_\lambda \in Y_\lambda} \max_{1 \leqq j \leqq n} \|x_{j,\lambda}\| \leqq N\}.$$

Obviously, $\sigma_T(x) = \cup_N B_N$, therefore by Baire's theorem there is a B_{N_0} which has a non-void (relative) interior. We take now $C = N_0 + 1$ and an open polydisc D such that $\emptyset \neq D \cap \sigma_T(x) \subset \operatorname{int} B_{N_0} \subset \sigma_T(x)$ and then we may choose $x_\lambda \in Y_\lambda$ such that $\max_{1 \leqq j \leqq n} \|x_{j,\lambda}\| \leqq C$, for λ running a dense subset of the set $D \cap \sigma_T(x)$.

3.3. Theorem. *Let $T = (T_1, \ldots, T_n)$ be a $\mathscr{C}^m$-scalar n-tuple. If q is any integer such that $q \geqq 2n + m + 1$ then*

$$\bigcap_{\lambda \in \mathbf{C}^n} \sum_{j=1}^{n} (\lambda_j - T_j)^q X = \{0\}.$$

Proof. Let us remark first that we have

$$\bigcap_{\lambda \notin \sigma(T,X)} \sum_{j=1}^{n} (\lambda_j - T_j)^q X = X. \tag{3.2}$$

Indeed, if $\lambda \notin \sigma(T, X)$ then the functions

$$u_{j,\lambda}(z) = \overline{(\lambda_j - z_j)^q} / \sum_{l=1}^{n} |\lambda_l - z_l|^{2q} \quad (j = 1, \ldots, n)$$

are infinitely differentiable in an open neighbourhood of $\sigma(T, X) = \operatorname{supp} U$ [1, Cor. I.2.12], for every $\mathscr{C}^m$-functional calculus U of T (which is considered fixed in the sequel). Then $U(u_{j,\lambda})$ is well defined for any j and we have

$$\sum_{j=1}^{n} (\lambda_j - T_j)^q \, U(u_{j,\lambda}) = 1_X .$$

In particular, we have (3.2).

In order to prove our theorem it is sufficient to show that

$$\bigcap_{\lambda \in \sigma(T,X)} \sum_{j=1}^{n} (\lambda_j - T_j)^q X = \{0\} .$$

Assume that there is an $x \in X$ such that $x \neq 0$ and

$$x \in \bigcap_{\lambda \in \sigma(T,X)} \sum_{j=1}^{n} (\lambda_j - T_j)^q X . \tag{3.3}$$

Denote by D the disc given by Lemma 3.2 and take $x_\lambda = (x_{1,\lambda}, \ldots, x_{n,\lambda})$ such that $\sum_{j=1}^{n} (\lambda_j - T_j)^q \, x_{j,\lambda} = x$ and $\max_{1 \leqq j \leqq n} \|x_{j,\lambda}\| \leqq C$ for all $\lambda \in B$, B chosen according to the same lemma. Let us take a function $\varphi \in \mathscr{C}^\infty(\mathbf{C}^n)$ such that $\operatorname{supp} \varphi \subset D$ and $\varphi = 1$ in a nigehbourhood of λ_0, where λ_0 is a fixed point in $D \cap \sigma_T(x)$. Therefore $U(\varphi)\, x \neq 0$ because otherwise $\lambda_0 \in \sigma_T(x)$; moreover, $\sigma_T(U(\varphi)\, x) \subset \sigma_T(x) \cap \operatorname{supp} \varphi$ (see [1, Th. I.2.14]). Denote by $y_{i,\lambda}$ the vector $U(\varphi)\, x_{j,\lambda}$ for $\lambda \in B$ and let us choose $y_{j,\lambda} \in X$ $(j = 1, \ldots, n)$ such that $\sum_{j=1}^{n} (\lambda_j - T_j)^q \, y_{j,\lambda} = U(\varphi)\, x$ for all $\lambda \in \mathbf{C}^n \setminus B$, which is possible on account of (3.2) and (3.3). Let us choose also a point $\mu(\lambda) = (\mu_1(\lambda), \ldots, \mu_n(\lambda))$ in B such that $D_{3r,\mu(\lambda)} \supset D_{r,\lambda}$, whenever $D_{r,\lambda} \cap \sigma_T(U(\varphi)\, x) \neq \emptyset$.

On account of the estimation (3.1) we have

$$\begin{aligned}
&|\partial_{k,h}((\mu_j(\lambda) - z_j)^q \, \varphi_r(\lambda - z))| \leqq \\
&\leqq \sum_{s \leqq k, t \leqq h} \binom{k}{s} \binom{h}{t} \sup_{z \in D_{r,\lambda}} |\partial_{k-s,h-t}(\mu_j(\lambda) - z_j)^q \, \partial_{s,t} \varphi_r(\lambda - z)| \leqq \\
&\leqq \sum_{s \leqq k} \binom{k}{s} \sup_{z \in D_{r,\lambda}} |\partial_{k-s}(\mu_j(\lambda) - z_j)^q \, \partial_{s,h} \varphi_r(\lambda - z)| \leqq \\
&\leqq \sum_{s \leqq k} \binom{k}{s} (3r)^{q - |k| + |s|} \, M_{s,h} r^{-2n - |s| - |h|} \leqq C_{k,h} r^{q - 2n - |k| - |h|}
\end{aligned}$$

whence we get

$$\left\| \sum_{j=1}^{n} (\mu_j(\lambda) - z_j)^q \varphi_r(\lambda - z) \right\|_m \leqq C r^{q-2n-m}, \tag{3.4}$$

where $C \geqq 0$ does not depend on r.

According to (3.4) and Lemma 3.2 we have now for $q \geqq 2n + m + 1$ the following estimation:

$$\begin{aligned} &\left\| \sum_{j=1}^{n} (\lambda_j - T_j)^q U(\varphi_r(\lambda - z)) y_{j,\lambda} \right\| = \\ &= \left\| \sum_{j=1}^{n} (\mu_j(\lambda) - T_j)^q U(\varphi_r(\lambda - z)) y_{j,\mu(\lambda)} \right\| \leqq \\ &\leqq \sum_{j=1}^{n} \left\| U((\mu_j(\lambda) - T_j)^q \varphi_r(\lambda - z)) y_{j)\mu(\lambda)} \right\| \leqq C_1 r, \end{aligned} \tag{3.5}$$

where C_1 does not depend on r.

Let us consider a function χ in $\mathscr{C}^\infty(\mathbf{C}^n)$, with compact support, such that $\chi = 1$ in a neighbourhood of $\sigma(T, X)$. Then we may write

$$\begin{aligned} &\int \chi(\lambda) \sum_{j=1}^{n} (\lambda_j - T_j)^q U(\varphi_r(\lambda - z)) y_{j,\lambda} \, \mathrm{d}\nu(\lambda) = \\ &= \int \chi(\lambda) U(\varphi_r(\lambda - z)) U(\varphi) x \, \mathrm{d}\nu(\lambda), \end{aligned} \tag{3.6}$$

and the left side is integrable as being equal to the right.

According to Lemma 3.1,

$$\lim_{r \to 0} \int \chi(\lambda) U(\varphi_r(\lambda - z)) U(\varphi) x \, \mathrm{d}\nu(\lambda) = U(\chi) U(\varphi) x = U(\varphi) x. \tag{3.7}$$

Notice that if $D_{r,\lambda} \cap \sigma_T(U(\varphi) x) = \emptyset$ then $U(\varphi_r(\lambda - z)) U(\varphi) x = 0$ (see [1, Th. I.2.14]), therefore we have for $q \geqq 2n + m + 1$ and $V_r = \{w \in \mathbf{C}^n;$ dist $(w, \sigma_T(U(\varphi) x) \leqq r\}$,

$$\begin{aligned} &\left\| \int \chi(\lambda) \sum_{j=1}^{n} (\lambda_j - T_j)^q U(\varphi_r(\lambda - z)) y_{j,\lambda} \, \mathrm{d}\nu(\lambda) \right\| \leqq \\ &\leqq \left\| \int_{V_r} \chi(\lambda) \sum_{j=1}^{n} (\lambda_j - T_j)^q U(\varphi_r(\lambda - z)) y_{j,\lambda} \, \mathrm{d}\nu(\lambda) \right\| \leqq C_2 r, \end{aligned} \tag{3.9}$$

where we have used the estimation (3.6). From (3.7), (3.8) and (3.9) we obtain $U(\varphi) x = 0$, which is a contradiction; the proof is complete.

3.6. Proposition. *Let T be a $\mathscr{C}^m$-scalar n-tuple and $q \geqq 1$ an integer such that*

$$\bigcap_{\lambda \in \mathbf{C}^n} \sum_{j=1}^{n} (\lambda_j - T_j)^q X = \{0\} .$$

Then for any closed $F \subset \mathbf{C}^n$ we have

$$X_T(F) = \bigcap_{\lambda \notin F} \sum_{j=1}^{n} (\lambda_j - T_j)^q X .$$

Proof. Since $T \mid X_T(F) = (T_1 \mid X_T(F), \ldots, T_n \mid X_T(F))$ is a $\mathscr{C}^m$-scalar n-tuple on $X_T(F)$, then we have by (3.4)

$$X_T(F) = \bigcap_{\lambda \notin \sigma(T, X_T(F))} \sum_{j=1}^{n} (\lambda_j - T_j)^q X_T(F) \subset \bigcap_{\lambda \notin F} \sum_{j=1}^{n} (\lambda_j - T_j)^q X .$$

Conversely, let x be in $\bigcap_{\lambda \notin F} \sum_{j=1}^{n} (\lambda_j - T_j)^q X$ and consider $\varphi \in \mathscr{C}^\infty(\mathbf{C}^n)$ with compact support, such that $\varphi = 1$ in a neighbourhood of F. Let U be a $\mathscr{C}^m$-functional calculus of T. Then $y = U(1 - \varphi) x \in X_T(\operatorname{supp}(1 - \varphi))$. Since $x = \sum_{j=1}^{n} (\lambda_j - T_j)^q x_{j,\lambda}$ for any $\lambda \notin F$, then we can define

$$y_{j,\lambda} = \begin{cases} U(1 - \varphi) x_{j,\lambda} & \lambda \notin F \\ U(u_{j,\lambda}) y & \lambda \in F , \end{cases}$$

where $u_{j,\lambda}(z) = \overline{(\lambda_j - z_j)^q} / \sum_{j=1}^{n} |\lambda_l - z_l|^{2q}$ (see the proof of Theorem 3.5). Then we have $\sum_{j=1}^{n} (\lambda_j - T_j)^q y_{j,\lambda} = y$, for any $\lambda \in \mathbf{C}^n$, hence from our hypothesis, $y = 0$. We get then $x = U(\varphi) x$ for any $\varphi = 1$ in a neighbourhood of F, therefore $x \in X_T(F)$ (see [1, Th. I.2.16]).

3.7. Corollary. *With the conditions of Theorem* 3.5, *for any integer $q \geqq 2m + 2n + 1$ and any closed $F \subset \mathbf{C}^n$ we have*

$$X_T(F) = \bigcap_{\lambda \notin F} \sum_{j=1}^{n} (\lambda_j - T_j)^q X .$$

3.8. Remark. When $n = 1$ then Theorem 3.5 coincides with [5, Th. 3.3].

4. The case of scalar operators on Hilbert space. Throughout this section X will be a Hilbert space and $T = (T_1, \ldots, T_n)$ an n-tuple of commuting scalar [4] operators (i.e. a $\mathscr{C}^0$-scalar n-tuple). In this case we are able to prove that the minimal index given by Theorem 3.5 can be improved. We obtain our result by using a similar method with that of [7], extended to several operators.

4.1. Theorem. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting scalar operators on the Hilbert space X. Then for any integer $q \geqq n$*

$$\bigcap_{\lambda \in \mathbf{C}^n} \sum_{j=1}^{n} (\lambda_j - T_j)^q X = \{0\} .$$

Proof. On account of [4, Th. XV.6.4], there exists an invertible operator $S \in \mathscr{L}(X)$ such that $S^{-1}T_jS = N_j$ is normal for every $j = 1, \ldots, n$. It is clear that we have only to show

$$\bigcap_{\lambda \in \mathbf{C}^n} \sum_{j=1}^{n} (\lambda_j - N_j)^q = \{0\} , \tag{4.1}$$

with N_j commuting normal operators ($j = 1, \ldots, n$). On account of [4, Th. X.2.1] there exists a spectral measure E on $\mathbf{C}^n$ such that $N_j = \int z_j \, dE(z)$ ($j = 1, \ldots, n$).

Let M be a cube in $\mathbf{C}^n$ (i.e. a Cartesian product of n equal squares in each of $\mathbf{C}$'s) containing the spectrum $\sigma(N, X)$ of $N = (N_1, \ldots, N_n)$, which is equal to the support of the spectral measure E. As in Theorem 3.5, it is sufficient to show that

$$\bigcap_{\lambda \in M} \sum_{j=1}^{n} (\lambda_j - N_j)^q X = \{0\} .$$

Assume that x belongs to $\bigcap_{\lambda \in M} \sum_{j=1}^{n} (\lambda_j - N_j)^q X$. We shall construct a sequence $\{M_k\}$ of Borel sets in $\mathbf{C}^n$, having the properties:

1° $M_0 = M$;
2° the closure of each M_k is a closed cube with the sides parallel to the axes (in $\mathbf{R}^{2n}$);
3° each $\overline{M}_k$ is one of the 2^{2n} closed cubes obtained by halving the sides of $\overline{M}_{k-1}$;
4° $\|x\|^2 \leqq 2^{2nk} \|E(M_k)\, x\|^2$.

Suppose the set M_j is constructed for all $j \leqq k$. Let $K_1, \ldots, K_{2^{2n}}$ be disjoint Borel sets such that their union is equal to M_k and the closure of each K_j is equal to one of the closed cubes obtained by halving the sides of $\overline{M}_k$.

Since

$$\|E(M_k)\, x\|^2 = \sum_{j=1}^{2^{2n}} \|E(K_j)\, x\|^2 ,$$

we have

$$2^{-2n} \|E(M_k)\, x\|^2 \leqq \|E(K_{j_0})\, x\|^2$$

for at least one j_0. Let us take $M_{k+1} = K_{j_0}$. Then we have

$$\|x\|^2 \leqq 2^{2nk} \|E(M_k)\, x\|^2 \leqq 2^{2n(k+1)} \|E(M_{k+1})\, x\|^2 .$$

Denote by w the only point of $\bigcap_{k=0}^{\infty} \overline{M}_k$. Since $w \in M$ then there exist $x_j \in X$ ($j =$

$= 1, \ldots, n)$ such that $\sum_{j=1}^{n} (w_j - N_j)^q x_j = x$. Then we can write:

$$\|x\| \leqq 2^{nk} \|E(M_k)\, x\| \leqq 2^{nk} \sum_{j=1}^{n} \|E(M_k)\,(w_j - N_j)^q\, x_j\| =$$

$$= 2^{nk} \sum_{j=1}^{n} \langle E(M_k)\,(w_j - N_j)^q\, x_j, (w_j - N_j)^q\, x_j \rangle^{1/2} =$$

$$= 2^{nk} \sum_{j=1}^{n} \left(\int_{M_k} |w_j - z_j|^{2q} \langle \mathrm{d}E(z)\, x_j, x_j \rangle \right)^{1/2} =$$

$$= 2^{nk} \sum_{j=1}^{n} \left(\int_{M_k \setminus \{w\}} |w_j - z_j|^{2q} \langle \mathrm{d}E(z)\, x_j, x_j \rangle \right)^{1/2} \leqq$$

$$\leqq 2^{nk} \sum_{j=1}^{n} (d_{k,j})^q\, (v_j(M_k \setminus \{w\}))^{1/2}$$

where $d_{k,j} = \max_{z \in M_k} |w_j - z_j| = 2^{-k} d$, d being the length of the diagonal of the projection of M onto its j^{th} complex component and $v_j(M_k \setminus \{w\})$ is the variation of the measure $\langle E(\cdot)\, x_j, x_j \rangle$ on the set $M_k \setminus \{w\}$. Since the sets $\{M_k \setminus \{w\}\}$ form a decreasing sequence with void intersection then $v_j(M_k \setminus \{w\})$ tends to zero as $k \to \infty$, for each $j = 1, \ldots, n$. Finally, let us notice that

$$2^{nk} \sum_{j=1}^{n} (d_{k,j})^q\, (v_j(M_k \setminus \{w\}))^{1/2} \leqq n \,.\, 2^{(n-q)k} d \max_{1 \leqq j \leqq n} (v_j(M_k \setminus \{w\}))^{1/2} \to 0\,,$$

therefore $x = 0$ and the proof is complete.

4.2. Corollary. *Let T be as in the previous theorem. Then for any closed subset $F \subset \mathbf{C}^n$ we have*

$$X_T(F) = \bigcap_{\lambda \notin F} \sum_{j=1}^{n} (\lambda_j - T_j)^q X\,,$$

for any $q \geqq n$.

4.3. Remark. When $n = 1$ then we obtain the result of [7].

4.4. Example. We want to show that the minimal index q for which the relation (4.1) holds is equal to n, therefore Theorem 4.1 provides the best possible index.

Let X be the Hilbert space of all square integrable scalar functions on $D_1 = \{z \in \mathbf{C}^n;\ |z_j| \leqq 1, j = 1, \ldots, n\}$, with respect to the Lebesgue measure $\mathrm{d}v$.

Consider the n-tuple $T = (T_1, \ldots, T_n)$, with $(T_j f)(z) = z_j f(z)$ $(j = 1, \ldots, n)$ and take the function $h(z) \equiv 1$. For an arbitrary $w \in \mathbf{C}^n$ let us define

$$g_{j,w}(z) = \frac{\overline{(w_j - z_j)}^{\,n-1}}{|w_j - z_j|^{n-1} \sum_{k=1}^{n} |w_k - z_k|^{n-1}} \quad (j = 1, \ldots, n)\,.$$

Let us show that $g_{j,w} \in X$. We have

$$|g_{j,w}(z)|^2 = (\sum_{k=1}^{n} |w_k - z_k|^{n-1})^{-2} \leqq C(\sum_{k=1}^{n} |w_k - z_k|^2)^{1-n},$$

where C is a constant with respect to w and z. Take $r_0 > 0$ and consider the set $B(w, r_0) = \{z \in \mathbf{C}^n;\ \sum_{j=1}^{n} |w_j - z_j|^2 \leqq r_0^2\}$.

Then

$$(4.2) \qquad \int_{D_1} |g_{j,w}(z)|^2 \, dv \leqq \int_{D_1 \setminus B(w,r_0)} |g_{j,w}(z)|^2 \, dv + \int_{B(w,r_0)} |g_{j,w}(z)|^2 \, dv .$$

Notice that the first integral of the right side is finite; therefore we have only to show that the second one is finite too. Indeed, we have

$$\int_{B(w,r_0)} |g_{j,w}(z)|^2 \, dv \leqq C \int_{B(w,r_0)} (\sum_{k=1}^{n} |w_k - z_k|^2)^{1-n} \, dv = C \int_{B(0,r_0)} (\sum_{k=1}^{n} |z_k|^2)^{1-n} \, dv =$$

$$= C \int_S \int_0^{r_0} r^{2-2n} r^{2n-1} \, dr \, d\omega = (C/2)\, \omega(S)\, r_0^2,$$

where $r = (\sum_{k=1}^{n} |z_k|^2)^{1/2}$ and $d\omega$ is the Lebesgue measure on $S = \{z \in \mathbf{C}^n;\ \sum_{j=1}^{n} |z_j|^2 = 1\}$ $(\subset \mathbf{C}^n = \mathbf{R}^{2n})$; therefore the right side of (4.2) is finite and $g_{j,w}$ belongs to X.

Finally, we have for all $w \in \mathbf{C}^n$, $z \in D_1$

$$\sum_{j=1}^{n} (w_j - T_j)^{n-1} g_{j,w}(z) = h(z).$$

Consequently, $0 \neq h \in \bigcap_{w \in \mathbf{C}^n} \sum_{j=1}^{n} (w_j - T_j)^{n-1} X.$

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Authors' addresses: E. J. Albrecht, Fachbereich Mathematik, Universität Trier/Kaiserslautern, Kaiserslautern, Germany and F. H. Vasilescu, Institute of Mathematics of the Academy of RSR, Bucharest, Romania.