Vladimír Müller; Jan Pelant

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ON STRONGLY HOMOGENEOUS TOURNAMENTS

VLADIMÍR MÜLLER, JAN PELANT, Praha

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INTRODUCTION

Let $\mathcal{T} = \langle T, t \rangle$ be a finite tournament. If $\mathcal{T}$ has constant degrees (i.e., $|\{z \mid \langle x, z \rangle \in t\}|$ is a constant independent on x) we call $\mathcal{T}$ homogeneous. It is known that homogeneous tournaments have certain "regular" properties, e.g. it holds that the number of 3-cycles containing a given vertex is constant and maximal in every homogeneous tournament. Here we consider more "regular" tournaments, calling them strongly homogeneous ones (briefly S + H tournaments): we say that a tournament $\mathcal{T} = \langle T, t \rangle$ is an *S + H tournament* if $|\{z \mid \langle x, z \rangle \in t \text{ and } \langle y, z \rangle \in t\}| = \text{const.}$ for any two distinct vertices x, y of $\mathcal{T}$. In this paper we give three different structural characterizations of S + H tournaments and we show that any further sharpening of homogeneity makes no sense (§ 2). We give several constructions of S + H tournaments and as a consequence we prove that S + H tournaments form a universal class of tournaments. In § 2 and § 3 we prove theorems on extensions of tournaments related to the extensions of Fraisé. As S + H tournaments are strongly connected with Hadamard block designs the general question if there is an S + H tournament on every set $4k + 3$ remains unsolved. We discuss this relationship in § 4.

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This paper is closely related to the paper [3].

1. K-HOMOGENEOUS TOURNAMENTS. BASIC PROPERTIES OF STRONGLY HOMOGENEOUS TOURNAMENTS.

Definition 1. A tournament $\mathcal{T}$ is a couple $\langle T, t \rangle$, where T is a finite set and t is a subset of T^2 such that the following holds:

1) $x \in T \Rightarrow \langle x, x \rangle \in t$,
2) $x, y \in T,\ x \neq y \Rightarrow (\langle x, y \rangle \in t \Leftrightarrow \langle y, x \rangle \notin t)$.

We put: $v(x) = \{z \mid z \neq x \text{ and } \langle x, z \rangle \in t\}, f(x) = \{z \mid z \neq x \text{ and } \langle z, x \rangle \in t\}$, $v_S(x) = S \cap v(x)$, $\mathcal{T}|_S = \langle S, S^2 \cap t \rangle$, for an $x \in T$, $S \subset T$.

Definition 2. A tournament $\mathcal{T} = \langle T, t\rangle$ is said to be *k-homogeneous* if there exists an integer $m \geqq 1$ such that $|\bigcap\{v(x_i) \mid i = 1, \ldots, k\}| = m$ for each k-distinct vertices $x_1, \ldots, x_k$ of $\mathcal{T}$.

This notion is a natural strengthening of the tournaments with constant score (i.e. 1-homogeneous tournaments) as it is shown by the following:

Proposition 1. *Let $\mathcal{T} = \langle T, t\rangle$ be $(k + 1)$-homogeneous, $k \geqq 1$. Then $\mathcal{T}$ is k-homogeneous.*

Proof. Let $a_1, \ldots, a_k$ be k distinct vertices of $\mathcal{T}$, put $\bigcap\{v(a_i) \mid i = 1, \ldots, k\} = T'$. By the assumption, we have $|v_{T'}(b)| = |v(b) \cap T'| = |v(b) \cap \{v(a_i) \mid i = 1, \ldots, k\}| = = m$ for every $b \in T'$, where m is the constant belonging to $(k + 1)$-homogeneitly of $\mathcal{T}$. As $\binom{|T'|}{2} = |T'|.\, m$ holds for the tournament $\mathcal{T}|_{T'}$ we have $|T'| = 2m + 1$. Since we have chosen the vertices $a_1, \ldots, a_k$ arbitrarily, $\mathcal{T}$ is k-homogeneous with the constant $m' = 2m + 1$.

Proposition 2. *Let $\mathcal{T} = \langle T, t\rangle$ be $(k + 1)$-homogeneous, $k \geqq 1$, $a \in T$. Then $\mathcal{T}|_{v(a)}$ is k-homogeneous.*

Proof. It is $|\bigcap\{v_{T'}(a_i) \mid i = 1, \ldots, k\}| = |v(a) \cap \bigcap\{v(a_i) \mid i = 1, \ldots, k\}| = m$ for each k distinct points $a_1, \ldots, a_k \in v(a) = T'$.

Proposition 3. *Let $\mathcal{T} = \langle T, t\rangle$ be a k-homogeneous tournament, m the corresponding constant. Then $|T| = 2^k . m + 2^k - 1$.*

Proof will be done by induction on k. In the case $k = 1$ we have $|v(a)| = m$ for every $a \in T$, hence $\binom{|T|}{2} = T.\, m$ and consequently $|T| = 2m + 1$.

Let $\mathcal{T} = \langle T, t\rangle$ be $(k + 1)$-homogeneous tournament. Then $\mathcal{T}$ is also k-homogeneous and $|T| = 2^k . m' + 2^k - 1$ by induction hypothesis. (Here $m' = = |\bigcap\{v(a_i) \mid i = 1, \ldots, k\}|$. According to Proposition 1 we have $m' = 2m + 1$ and hence $|T| = 2^{k+1} . m + 2^{k+1} - 1$.

Theorem 1. *There exist no 3-homogeneous tournaments.*

Proof. Let $\mathcal{T} = \langle T, t\rangle$, $|T| = n = 8m + 7$ be a 3-homogeneous tournament. It is $\Sigma\{|v(a_1) \cap v(a_2) \cap v(a_3)| \mid a_1, a_2, a_3 \in T, a_i \text{ pairwise distinct}\} = \Sigma\left\{\binom{|f(a)|}{3} \,\middle|\, a \in T\right\}$ (as both expressions denote the number of 3-claws in $\mathcal{T}$). From this equation we can successively derive:

$$\binom{n}{3} . m = n . \binom{\frac{1}{2}(n - 1)}{3},$$

$$(n - 2) . 8m = (n - 3)(n - 5), \quad (n - 2)(n - 7) = (n - 3)(n - 5)$$

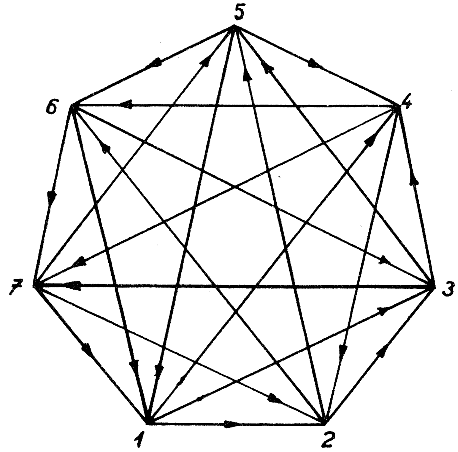
which is a contradiction.

Proposition 1 implies:

Corollary. *There exists no k-homogeneous tournament for any integer* $k \geqq 3$.

Notation. A 1-homogeneous tournament is called *homogeneous*, a 2-homogeneous tournament is called strongly *homogeneous* (shortly S + H tournament*)).

By Proposition 3 every S + H tournament has $4k + 3$ vertices. Thus the smallest example of an S + H tournament has 7 vertices: see Theorem 9.



Other examples of S + H tournaments are given in § 4.

Theorem 2. (First characterization of S + H tournaments.) *$\mathcal{T} = \langle T, t \rangle$ is strongly homogeneous iff $\mathcal{T}$ is a homogeneous tournament and $\mathcal{T}|_{v(a)}$ is a homogeneous tournament for every $a \in T$.*

Proof. The necessity of the above conditions follows by Propositions 1 and 2. Let a, b be distinct vertices of $\mathcal{T}$, $\langle a, b \rangle \in t$, $v(a) = T_a$. Then

$$|v(a) \cap v(b)| = |v_{T_a}(b)| = \frac{|v(a)| - 1}{2} = \frac{|T| - 3}{4}.$$

Proposition 4. *Let $\mathcal{T} = \langle T, t \rangle$ be an S + H-tournament, $|T| = 4k + 3$, a, b two distinct vertices, $\langle a, b \rangle \in t$. Then*

i) $|v(a) \cap v(b)| = k$,
ii) $|v(a) \cap f(b)| = k$,
iii) $|f(a) \cap v(b)| = k + 1$,
iv) $|f(a) \cap f(b)| = k$.

*) By the courtesy of Spejbl & Hurvínek Puppet Theatre of Prague.

Proof. i) Follows by the definition.

ii) It is $|v(a)| = 2k + 1$, $|v(a) \cap v(b)| = k$ and $v(a) = [v(a) \cap v(b)] \cup [v(a) \cap \cap f(b)] \cup \{b\}$ (here the righthand side is disjoint union). Hence it follows $|v(a) \cap \cap f(b)| = k$.

iii) and iv) can be proved similarly.

Proposition 5. *Let $\mathscr{T} = \langle T, t\rangle$ be an S + H-tournament, $|T| = 4k + 3$. Then every two distinct vertices of $\mathscr{T}$ belong to exactly $k + 1$ 3-cycles.*

Proof follows from Proposition 4 iii).

Theorem 3. *Every two distinct vertices of an S + H tournament with $4k + 3$ vertices belong to exactly $6 . \binom{k+1}{2}$ 4-cycles.*

Proof. Let a, b be distinct vertices of an S + H tournament $\mathscr{T} = \langle T, t\rangle$, $|T| = 4k + 3$, $\langle a, b\rangle \in t$. Determine the number of sets $\{x, y\}$ for which $\mathscr{T}|_{\{a,b,x,y\}}$ is a cyclic tournament. There are exactly 10 possibilities for vertices x, y:

1) $x, y \in v(a) \cap v(b)$,
2) $x, y \in f(a) \cap f(b)$,
3) $x, y \in v(a) \cap f(b)$,
4) $x, y \in f(a) \cap v(b)$,
5) $x \in v(a) \cap v(b)$, $y \in v(a) \cap f(b)$,
6) $x \in v(a) \cap f(b)$, $y \in f(a) \cap f(b)$,
7) $x \in f(a) \cap v(b)$, $y \in v(a) \cap f(b)$,
8) $x \in v(a) \cap v(b)$, $y \in f(a) \cap f(b)$,
9) $x \in f(a) \cap v(b)$, $y \in f(a) \cap f(b)$,
10) $x \in v(a) \cap v(b)$, $y \in f(a) \cap v(b)$.

Define only for the purpose of this proof: a set M consisting of two vertices of $\mathscr{T}$ is said to be admissible if $\mathscr{T}|_{\{a,b\}\cup M}$ is the cyclic tournament on 4 vertices.

In the cases 1), 2), 3), 5), 6) none of the sets $\{x, y\}$ is admissible. According to Proposition 4, there are exactly $\binom{k+1}{2}$ admissible sets $\{x, y\}$ in the case 4).

Obviously the number of admissible sets $\{x, y\}$ is $k . (k + 1)$ in the case 7). Denote the number of admissible sets $\{x, y\}$ $A(B, C)$ in the case 8) (9), 10), respectively).

Since $\binom{k+1}{2} + k(k + 1) = 3 . \binom{k+1}{2}$, it is enough to show $A + B + C = 3 . \binom{k+1}{2}$.

We have:

$$A = \Sigma\{|v(z) \cap f(a) \cap f(b)| \mid z \in v(a) \cap v(b)\} =$$
$$= \Sigma\{|f(z) \cap v(a) \cap v(b)| \mid z \in f(a) \cap f(b)\},$$
$$B = \Sigma\{|v(z) \cap f(a) \cap f(b)| \mid z \in f(a) \cap v(b)\} =$$
$$= \Sigma\{|f(z) \cap f(a) \cap (b)| \mid z \in f(a) \cap f(b)\},$$
$$C = \Sigma\{|v(z) \cap f(a) \cap v(b)| \mid z \in v(a) \cap v(b)\} =$$
$$= \Sigma\{|f(z) \cap v(a) \cap v(b)| \mid z \in f(a) \cap v(b)\}.$$

We shall prove $A + B = C + B = A + C = k(k + 1)$, which will complete the proof.

i) It is $|f(a) \cap v(z)| = k + 1$ and $f(a) \cap f(z) = [f(z) \cap f(a) \cap f(b)] \cup \cup [f(z) \cap f(a) \cap v(b)]$ for every $z \in v(a) \cap v(b)$. Hence $A + C = |f(a) \cap v(z)| \,.\, . |v(a) \cap v(b)| = (k + 1) \,.\, k$.

ii) $A + B = (k + 1) \,.\, k$ can be obtained similarily as i).

iii) As $\mathscr{T}|_{f(a)}, \mathscr{T}|_{v(b)}$ are homogeneous tournaments it is $|v(z) \cap f(a) \cap v(b)| + + |v(z) \cap f(a) \cap f(b)| = k$ and $(k - |v(z) \cap f(a) \cap v(b)|) + |f(z) \cap v(a) \cap v(b)| = = k$ for every $z \in f(a) \cap v(b)$. Hence $|v(z) \cap f(a) \cap v(b)| + |f(z) \cap v(a) \cap v(b)| = = k$ and $B + C = k \,.\, (k + 1)$.

Corollary. *In every $S + H$ tournament with $4k + 3$ vertices there are exactly $\binom{4k+3}{2}\binom{k+1}{2}$ 4-cycles. This is the maximal number of 4-cycles in a tournament with $4k + 3$ vertices.*

2. EXTENSIONS OF STRONGLY HOMOGENEOUS TOURNAMENTS

We shall describe two recursive constructions of S + H tournaments and prove as a consequence that S + H tournaments allows universal object.

We shall give another characterization of S + H tournaments.

Proposition 6. *Let $\mathscr{T} = \langle T, t\rangle$ be an $S + H$ tournament on $n = 4k + 3$ vertices. Then there exists an $S + H$ tournament on $2n + 1$ vertices such that $\mathscr{T}$ is its subtournament.*

Proof. The construction: Let T' be a set, $|T| = |T'|$, $T \cap T' = \emptyset$, $0 \notin T \cup T'$. Let $x \to x'$ be a bijection of T onto T'. Define the tournament $\mathscr{U} = \langle U, u\rangle$ by $U = T \cup T' \cup \{0\}$; we have six kinds of u:

1) $\langle x, y\rangle \in u \Leftrightarrow \langle x, y\rangle \in t$ for $x, y \in T$,

2) $\langle x', y'\rangle \in u \Leftrightarrow \langle y, x\rangle \in t$ for $x', y' \in T'$,

3) $\langle 0, x\rangle \in u$ for $x \in T$,

4) $\langle x', 0\rangle \in u$ for $x' \in T'$,

5) $\langle x, x'\rangle \in u$ for $x \in T$, $\langle x, x\rangle \in u$ for $x \in U$,

6) $\langle x, y'\rangle \in u$ and $\langle x', y\rangle \in u$ for every $x \neq y$, $\langle x, y\rangle \in t$.

It is a matter of routine to check $|v(a) \cap v(b)| = 2k + 1$ any distinct vertices a, b of $\mathscr{U}$.

Theorem 4. *Let $\mathscr{T} = \langle T, t\rangle$ and $\mathscr{U} = \langle U, u\rangle$ be $S + H$ tournaments, let $\mathscr{T}$ be subtournament of $\mathscr{U}$, $|T| \neq |U|$. Then $|U| \geqq 2|T| + 1$. (Thus the construction given in Proposition 6 is the minimal one.)*

First we prove an easy lemma to make the paper selfcontained:

Lemma. *Let N be a set with n elements. Let $B_1, B_2, \ldots, B_r$ be subsets of N such that*

i) *$|B_i| = k$ for $i = 1, \ldots, r$,*

ii) *$i \neq j$, $B_i \cap B_j = \lambda$,*

where k, λ are positive integers, $k \neq \lambda$. Then $r \leqq n$.

Proof. Let $N = \{1, \ldots, n\}$, let A be the incidence matrix of the above set-system (i.e. an (n, r)-matrix (a_{ij}) with $a_{ij} = 1$ if $x_i \in B_j$, otherwise $a_{ij} = 0$). The statement easily follows if we consider the matrix $A^T A$.

Proof of Theorem 4: We can suppose $T \subset U$. Put $T' = U - T$. It is $|T'| = 4m$ for an integer $m > 0$. Denote $B_x = \{y \in T' \mid \langle x, y\rangle \in u\}$ for every $x \in T$. Let $|T| = 4k + 3$. Now it is $|B_x| = |v_{\mathscr{U}}(x) - v_{\mathscr{T}}(x)| = (2k + 2m + 1) - (2k + 1) = 2m$ and $|B_x \cap B_y| = |v_{\mathscr{U}}(x) \cap v_{\mathscr{U}}(y)| - |v_{\mathscr{T}}(x) \cap v_{\mathscr{T}}(y)| = k + m - k = m$ for every pair of distinct vertices x, y of $\mathscr{T}$. It is $4k + 3 \leqq 4m$ by the above lemma and hence $|U| \geqq 2|T|$. As $|U| = 3 \pmod 4$, it is $|U| \geqq 2|T| + 1$.

Definition 3. A tournament $\mathscr{T} = \langle T, t\rangle$ is called *invertible* if there exists a bijection $\varphi : T \to T$ such that $\langle x, y\rangle \in t$ iff $\langle \varphi(x), \varphi(y)\rangle \notin t$. φ is called an *inversion*.

Proposition 7. *Let $\mathscr{T} = \langle T, t\rangle$ be an invertible tournament. Then there exists an inversion φ such that $\varphi \circ \varphi = 1_T$* (the identity mapping on T).

Proof. Denote by $A(\mathscr{T})$ the set of all automorphisms of $\mathscr{T}$. Put $B(\mathscr{T}) = \{\varphi : T \to T, \varphi \text{ is an inversion}\} \cup A(\mathscr{T})$. Then $B(\mathscr{T})$ with the operation of composition of mappings is a group and $A(\mathscr{T})$ is its subgroup. Furthermore, $A(\mathscr{T})$ are precisely the elements of an odd order in $B(\mathscr{T})$. From this it is easy to conclude the statement.

Proposition 8. *Let $\mathscr{T} = \langle T, t\rangle$ be an $S + H$ tournament, let $\mathscr{U} = \langle U, u\rangle$ be an invertible $S + H$ tournament. Then there exists an $S + H$ tournament on $(|T| + 1) \,.\, (|U| + 1) - 1$ points.*

Proof. We can assume that the sets T, U and $T \times U$ are disjoint. Put $V = T \cup \cup U \cup T \times U$. Let φ be an inversion of $\mathscr{U}$ with $\varphi \circ \varphi = 1_U$. Define the relation v on V:

1) $t \subset v,\ u \subset v,\ x \in V \Rightarrow \langle x, x \rangle \in V,$
2) $x \in T,\ y \in U \Rightarrow \langle y, x \rangle \in v,\ \langle x, \langle x, y \rangle \rangle \in v,$
3) $x_1 \neq x_2,\ \langle x_1, x_2 \rangle \in t,\ y \in U \Rightarrow \langle x_1, \langle x_2, y \rangle \rangle \in v,\ \langle \langle x_1, y \rangle, x_2 \rangle \in v,$
4) $x \in T,\ \langle y_1, y_2 \rangle \in u \Rightarrow \langle \langle x, y_1 \rangle, \langle x, x_2 \rangle \rangle \in v,$
5) $x \in T,\ y \in U \Rightarrow \langle \langle x, y \rangle, \varphi(y) \rangle \in v,$
6) $y_1 \neq \varphi(y_2),\ \langle y_1, \varphi(y_2) \rangle \in u,\ x \in T \Rightarrow \langle \langle x, y_1 \rangle, y_2 \rangle \in v,\ \langle \varphi(y_1), \langle x, \varphi(y_2) \rangle \rangle \in v,$
7) $x_1 \neq x_2,\ \langle x_1, x_2) \in t,\ y \in U \Rightarrow \langle \langle x_1, y \rangle, \langle x_2, \varphi(y) \rangle \rangle \in v,$
8) $x_1 \neq x_2,\ \langle x_1, x_2 \rangle \in t,\ y_1 \neq \varphi(y_2),\ \langle y_1, \varphi(y_2) \rangle \in u \Rightarrow \langle \langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \rangle \in v,$ $\langle \langle x_2, \varphi(y_1) \rangle, \langle x_1, \varphi(y_2) \rangle \rangle \in v.$

By checking all possibilities one can prove that $\mathscr{V} = \langle V, v \rangle$ is an S + H tournament.

Now we can prove:

Theorem 5. *Let k be a positive integer. Then there exists an S + H tournament $\mathscr{T}_k$ which contains every tournament on k vertices as its subtournament.*

Proof. The statement being for $k = 1$ trivial, we prove the theorem by induction on k.

Let $\mathscr{T}_{k+1}$ be the tournament which we get by applying the construction described in Proposition 6 to the tournament $\mathscr{T}_k$. Let $\mathscr{T} = \langle T, t \rangle$ be a tournament, $|T| = k + 1$. Let $x_0 \in T$ be an arbitrary fixed vertex. Put $\mathscr{T}' = \langle T - \{x_0\}, [t - (f(x_0))^2] \cup [(f(x_0))^2 \cap t^{-1}] \rangle$. By the induction hypothesis there exists a one-to one homomorphism $g : T - \{x_0\} \to T_k$. We define an embedding $\tilde{g} : \mathscr{T} \to \mathscr{T}_{k+1} : \tilde{g}(x_0) = 0$, $\tilde{g}(x) = g(x)$ for all $x \in v(x_0)$, $\tilde{g}(x) = g(x)'$ for all $x \in f(x_0)$. (We use the notation from the construction in Proposition 6). It is clear that $\tilde{g}$ is a homomorphism from $\mathscr{T}$ into $\mathscr{T}_{k+1}$.

Remark. Consequently, $\mathscr{T}_k$ is a k-universal tournament which is the notion studied e.g. in [4]. In [4] it is introduced $T(k)$ = the smallest cardinality of a k-universal tournament. Denote by S + H(k) the smallest cardinality of k-universal tournament which is strongly homogeneous.

Clearly $T(k) \leqq$ S + H(k). From Theorem 5 it follows that S + H(k) is asymptotically bounded by $c \,.\, 2^k$. Furthermore S + H(1) = S + H(2) = S + H(3) = 7, S + H(4) = 11, S + H(5) = 15. This can be very probably continued.

Definition 4. We say that the tournament $\mathscr{T} = \langle T, t \rangle$ and $\mathscr{U} = \langle U, u \rangle$ are *degree equivalent* if there exists a bijection $\varphi : T \to U$ such that $|v_{\mathscr{T}}(x)| = |v_{\mathscr{U}}(\varphi(x))|$ for every $x \in T$. We denote this by $\mathscr{T} \sim \mathscr{U}$.

Convention. In the following theorem we shall use the following notation: Let $\mathscr{T} = \langle T, t \rangle$ be a tournament, $i \in N$. Put $T_i = \{x \in T \mid |v(x)| = i\}$, $q_i = |T_i|$, $\mathscr{T}_x = \mathscr{T}|_{T - \{x\}}$, $T_i^x = \{y \in T - \{x\} \mid |v_{T-\{x\}}(y)| = i\}$, $q_i^x = |T_i^x|$.

Proposition 9. *Let $\mathcal{T} = \langle T, t \rangle$ be a tournament. It is $\mathcal{T}_x \sim \mathcal{T}_y$ for every two distinct vertices of $\mathcal{T}$ iff there are $p, q \in N$ such that $q_0 = q_1 = \ldots = q_{p-1} = q_{n-p} = \ldots = q_{n-1} = 0$ and for every i, j, $p \leqq i \leqq j \leqq n - p - 1$ and for every $x \in T_i$, $y \in T_j$ it holds $|v(x) \cap T_j = \frac{1}{2}(q - 1) = |f(y) \cap T_i|$ where $q = q_p = q_{p+1} = \ldots = q_{n-p-1}$.*

Proof of the sufficiency of the conditions: Let us choose $x \in T$, $i \in N$ $(i < |T|)$. Then $T_i^x = [T_i \cap v(x)] \cup [T_{i+1} \cap f(x)]$.

a) Either $i < p - 1$ or $i \geqq n - p$. Then obviously $T_i^x = \emptyset$, $q_i^x = 0$.

b) $i = n - p - 1$. Then $T_{n-p-1}^x = T_{n-p-1} \cap v(x)$, hence $q_{n-p-1}^x = \frac{1}{2}(q - 1)$.

c) $i = p - 1$. Then $T_p^x = T_p \cap f(x)$, hence $q_{p-1}^x = \frac{1}{2}(q - 1)$,

d) $p \leqq i < n - p - 1$, $x \in T$. Then $|T_i^x| = |T_i \cap v(x)| + |T_{i+1} \cap f(x)| = \frac{1}{2}(q + 1) + \frac{1}{2}(q - 1) = q$ if $j > i$ and similarily $|T_i^x| = q$ if $j \leqq i$.

Proof of the necessity of the conditions: Claim A: Let $T_j \neq \emptyset$, $i \geqq j$, $x \in T_i$. Then $|T_j \cap f(x)| = \frac{1}{2}(q_j - 1)$. Proof of the Claim will be done by induction on j. Let p be the smallest number with $T_p \neq \emptyset$. Then given an $x \in T_p$, it is $q_{p-1}^x = f_{T_p}(x)$, hence $\mathcal{T}|_{T_p}$ is a homogeneous tournament and $|v_{T_p}(x)| = \frac{1}{2}(q_p - 1)$. Let $y \in T$, then $q_{p-1}^y = |T_p \cap f(y)| = \frac{1}{2}(q_p - 1)$. This proves the Claim for $j = p$. Assume the statement holds for every $k \leqq j - 1$. Supposing $q_j \neq 0$ it is for every $x \in T_j$ either $q_{j-1}^x = |f_{T_j}(x)|$ or $q_{j-1}^x = |f_{T_j}(x)| + T_{j-1} \cap v(x)| = |f_{T_j}(x)| + \frac{1}{2}(q_{j-1} + 1)$. Thus $\mathcal{T}|_{T_j}$ is homogeneous and $|f_{T_j}(x)| = \frac{1}{2}(q_j - 1)$. If $i \geqq j$, $y \in T_i$ then $q_{j-1} = |f_{T_j}(y)| + |v_{T_{j-1}}(y)| = |f_{T_j}(y)| + \frac{1}{2}(q_{j-1} + 1)$ and hence $|f_{T_j}(y)| = \frac{1}{2}q_{j-1}$.
Quite analogously we can prove:

Claim B: Let $T_j \neq \emptyset$, $i \leqq j$, $x \in T_i$. Then $|T_j \cap v(x)| = \frac{1}{2}(q_j - 1)$.

Claim C: Let $p < k < r$, $T_r \neq \emptyset$. Then $T_k \neq \emptyset$.

Suppose in the way of contradiction $T_k = \emptyset$. Choose r such that $T_r \neq \emptyset$, $T_{r-1} = \emptyset$. Then $\mathcal{T}|_{T_r}$ is homogeneous as $q_{r-1}^x = |f_{T_r}(x)|$ for every $x \in T_r$, consequently $q_{r-1}^x = \frac{1}{2}(q_{r-1})$. By Claim A $\Sigma\{|f_{T_r}(y)| \mid y \in T_p\} \, \Sigma\{|v_{t_p}(y)| \mid y \in T_r\} = \frac{1}{2}q_r q_{p+1}$. It is $|f_{T_r}(x)| = q_{r-1}^x = \frac{1}{2}(q_r - 1)$ for every $x \in T_p$ and hence $q_r + q_p = 0$, which is a contradiction.

Claim D: It is $q_i = q_j$ for every $T_i \neq \emptyset \neq T_j$.

Suppose e.g. $i > j$, then according to A and B it is

$$q_j \frac{q_i - 1}{2} = \Sigma\{|v_{T_i}(x)| \, x \in T_j\} = \Sigma\{|f_{T_j}(x)| \mid x \in T_i\} = q_i \frac{q_j - 1}{2}$$

and hence $q_i = q_j$.

This completes the proof of Proposition 9.

Theorem 6. *Let $\mathscr{T} = \langle T, t\rangle$ be a tournament and let t not be an ordering of T. Then*

a) *there is no tournament $\mathscr{U} = \langle U, u\rangle$ such that $\mathscr{T}|_{T-\{x_1,\ldots,x_k\}} \sim \mathscr{U}$ for each set of k distinct vertices of $\mathscr{T}$ and for every $k \geqq 3$.*

b) *If there exists a tournament $\mathscr{U} = \langle U, u\rangle$ such that $\mathscr{T}|_{T-\{x,y\}} \sim \mathscr{U}$ for every set of two distinct vertices of $\mathscr{T}$, then $|U| = 4m + 1$ for an $m \in N$ and $|U_0| = |U_1| = \ldots = |U_{2m-2}| = |U_{2m+2}| = \ldots = |U_{n-1}| = 0, |U_{2m-1}| = |U_{2m+1}| = m, |U_{2m}| = 2m + 1$.*

In this case $\mathscr{T}$ is a homogeneous tournament.

Proof follows by the above Proposition and by the fact that the degree sequence is determined by the maximal proper suntournaments.

Theorem 7. (Second characterization of strongly homogeneous tournaments). *Let $\mathscr{T}\langle = T, t\rangle$ be a tournament, let t not be an ordering of T. Then the following two statements are equivalent*:

a) *$\mathscr{T}$ is an $S + H$ tournament*

b) *$\mathscr{T}|_{T-\{x,y\}} \sim \mathscr{T}|_{T-\{u,v\}}$ for every pair of distinct vertices x, y (u, v respectively) of $\mathscr{T}$.*

Proof. $a \Rightarrow b$ follows by Proposition 4.

If b) is satisfied, then by the above Theorem 6 $\mathscr{T}$ is homogeneous and $|T| = 4k + 3$. Let x, y be two distinct vertices of $\mathscr{T}$. Then obviously $|v(x) \cap v(y)| = |\{z \in T - \{x, y\}| \, |v(z) \cap T - \{x, y\}| = 2k + 1\}|$ and thus by Theorem 6 $|v(x) \cap v(y)| = k$.

Remark. Theorem 7 is related to the notion (Fraisé) k-extension, see [2] and the literature quoted there. We can formulate the above results as follows:

A tournament $\mathscr{T} = \langle T, t\rangle$ has a $|T|$-extension to $|T| + k$ points for a $k \geqq 3$ iff t is an ordering of T.

3. SIMPLE TOURNAMENTS

Definition 5. We call a tournament *simple* if every its non-constant endomorphism is automorphism.

Proposition 10. *A tournament $\mathscr{T} = \langle T, t\rangle$ is simple iff there exists no at least two-point set $K \subset T$, $K \neq T$ such that either $K \subset v(a)$ or $K \subset f(a)$ for every $a \in T - K$.*

Proof. Let $\mathscr{T}$ not be simple. Then there exists a non-constant endomorphism φ such that $\varphi(a) = \varphi(b)$ for two distinct vertices of $\mathscr{T}$. As φ is non-constant it is

$\varphi^{-1}(a) \neq T$ and we can put $K = \varphi^{-1}(a)$. If there exists a set K with the above properties then the mapping φ defined by $\varphi(x) = x$ for all $x \in T - K$ and $\varphi(x) = k$ where k is an arbitrarily fixed point of K for all $x \in K$ is a non-constant endomorphism of $\mathcal{T}$.

The following definition gives a measure for simplicity:

Definition 6. Let $\mathcal{T} = \langle T, t\rangle$ be a tournament. The *arrow-simplicity* $s(\mathcal{T})$ is the least number $|M|$ where $M \subset t$ is a set of arrows for which the tournament $\mathcal{T}' = \langle T, (t - M) \cup M^{-1}\rangle$ is not simple. (i.e. $s(\mathcal{T})$ is the least number of arrows chasing of which yields a non-simple tournament).

Theorem 8. (Third characterization of strongly homogenous tournaments.) *A tournament is strongly homogeneous iff* $s(\mathcal{T}) = \frac{1}{2}(|T| - 1)$.

Proof of the sufficiency of the condition: Let $s(\mathcal{T}) = \frac{1}{2}(|T| - 1)$. If there exists $x \in T$ such that $v(x) < \frac{1}{2}(|T| - 1)$, then let us denote $M = \{\langle x, y\rangle \in T^2 \mid y \in v(x)\}$. Clearly $|M| < \frac{1}{2}(|T| - 1)$, and the tournament $\langle T, (t - M) \cup M^{-1}\rangle$ is not simple which is a contradiction. Hence $\mathcal{T}$ is homogeneous. Let x, y be distinct vertices of $\mathcal{T}$. It follows from the homogeneity of $\mathcal{T}$ that $|v(x) \cap v(y)| = |f(x) \cap f(y)|$ and from the simplicity number $s(\mathcal{T})$ it follows that $|f(x) \cap v(y)| + |v(x) \cap f(y)| \geqq \frac{1}{2}(|T| - 1)$ and consequently

$$(1) \qquad |v(x) \cap v(y)| \leqq \frac{|T| - 3}{4}.$$

On the other side

$$\Sigma\{v(x) \cap v(y)| \mid x, y \in T,\ x \neq y\} = \Sigma\left\{\binom{f(x)}{2} \,\middle|\, x \in T\right\} = \binom{|T|}{2}\frac{|T| - 3}{4}$$

and consequently

$$(2) \qquad \Sigma\{|v(x) \cap v(y)| \mid x, y \in T,\ x \neq y\} = \binom{|T|}{2}\frac{|T| - 3}{4}.$$

Combining (1) a (2) we have $|v(x) \cap v(y)| = \frac{1}{4}(|T| - 3)$ for each couple of distinct vertices x and y.

Proof of the necessity of the condition: Let $\mathcal{T} = \langle T, t\rangle$ be an S + H tournament, put $|T| = 4k + 3$. Suppose in the way of contradiction $s(\mathcal{T}) \leqq 2k$, hence there exists a set $M \subset t$, $|M| \leqq 2k$ such that the tournament $\mathcal{T}' = \langle T, (t - M) \cup M^{-1}\rangle$ is not simple. Then according to Proposition 10 there exists a set C, $|C| > 1$, $C \subset T$, $C \neq T$ such either $v_{\mathcal{T}'}(x) \supset C$ or $f_{\mathcal{T}'}(x) \supset C$ for every vertex $x \in T - C$. Denote

$$V(C) = \{x \mid x \in T - C,\ f_{\mathcal{T}'}(x) \supset C\}, \quad F(C) = \{x \mid x \in T - C,\ v_{\mathcal{T}'}(x) \supset C\}.$$

Put $z(x) = \{y \mid \langle y, x\rangle \in M \text{ vel } \langle x, y\rangle \in M\}$ for every $x \in T$. Then obviously

$\Sigma\{|z(x)| \mid x \in T\} \leqq 4k$ and we can assume without loss of generality that $\Sigma\{z(x)| \mid x \in \in C\} = \Sigma\{|z(x)| \mid x \in T - C\} \leqq 2k$.

Let us prove a few simple formulas:

a) $(|C| - (2k + 1)) . (|V(C)| + |F(C)|) \leqq 2k$.

$\mathcal{T}$ is homogeneous and that is why the inequality holds for every point

$$x \in T - C = V(C) \cup F(C) : |z(x)| \geqq |C| - (2k + 1) .$$

Then we have:

$$(|C| - (2k + 1)) . (|V(C)| + |F(C) \leqq \Sigma\{|z(x)| \mid x \in T - C\} \leqq 2k .$$

b) $|C| (|V(C) + |F(C)| - 2k) \leqq 4k$.

This can beseen as follows: it is $|v_{\mathcal{T}}(c) \cap v_{\mathcal{T}}(c')| = |f_{\mathcal{T}}(c) \cap f_{\mathcal{T}}(c')| = k$ for any two distinct vertices $c, c' \in C$. Hence

$$|z(c)| + |z(c')| \geqq V(C)| - k + |F(C)| - k$$

and

$$(|C| - 1) \Sigma\{|z(x)| |x \in C\} \geqq \binom{|C|}{2} (|V(C)| + |F(C)| - 2k) ,$$

which is b).

c) $(|C| - (k + 1)) . (|V(C)| + |F(C)|) \leqq 4k$.

For any two points $x, y \in T - C$, $x \neq y$ it is $|v_{\mathcal{T}}(x) \cap v_{\mathcal{T}}(y)| = |f_{\mathcal{T}}(x) \cap f_{\mathcal{T}}(y)| = k$ and $k \leqq |v_{\mathcal{T}}(x) \cap f_{\mathcal{T}}(y)| \leqq k + 1$, thus we have:

$$|z(x)| + |z(y)| \geqq |C| - (k + 1) ,$$

$$\{(|C| - (k + 1)\} . \binom{V(C) + F(C)}{2} \leqq$$

$$\leqq (|V(C)| + |F(C)| - 1) . \Sigma\{z(x) \mid x \in T - C\} \leqq (|V(C)| + |F(C)| - 1) . 2k .$$

Now we show that the assumption $|C| \geqq 2$ leads to a contradiction. We divide the proof into nine cases according to the cardinality of C:

i) $|C| = 2$. By b) it is $2(|V(C)| + |F(C)| - 2k) \leqq 4k$. As $|V(C)| + |F(C)| - 2k = = 2k + 1$ we have a contradiction.

ii) The case $|C| = 3$ can be handled similarly as i).

iii) $4 \leqq C \leqq k$. We get from b): $k + 3 \leqq |V(C)| + |F(C)| - 2k \leqq 4k/|C| \leqq k$, a contradiction.

iv) $k + 1 \leqq |C| \leqq 2k - 1$. By b) $|V(C)| + |F(C)| - 2k \leqq 4k/|C|$ hence $|V(C)| + + |F(C)| \leqq 2k + 3$ which is a contradiction.

v) $C = 2k$. By b) $2k . 3 \leqq 4k$, which is a contradiction.

vi) $2k + 1 \leqq |C| \leqq 4k - 2$. By c) $k(|V(C)| + |F(C)|) \leqq 4k$, while $|V(C)| + |F(C)| \geqq 5$.

vii) $C = 4k - 1$. For $k = 1$ it is by b) $3.2 \leqq 4$, a contradiction. For $k \geqq 2$ it is by a): $(2k - 2) . 4 \leqq 2k$ and consequently $k \leqq \frac{4}{3}$, which is a contradiction.

viii) $|C| = 4k$. By c) $(3k - 1)(|V(C)| + |F(C)|) \leqq 4k$ and consequently $|V(C)| + |F(C)| \leqq 2$, a contradiction with $|C|$.

ix) $4k + 1 \leqq |C| \leqq 4k + 2$, $C \neq T$. By a): $2k(|V(C)| + |F(C)|) \leqq 2k$ and as $|V(C)| + |F(C)| > 0$, it is $|V(C)| + |F(C)| = 1$. If $|V(C)| = 1$ we get a contradiction with the homogeneity of $\mathcal{T}$ as $|C| \leqq 4k + 2$ and $\Sigma\{|z(x)| \mid x \in T - C\} \leqq 2k$. Similarly if $|F(C)| = 1$.

This completes the proof of the theorem.

Remark. Obviously $s(\mathcal{T}) \leqq \frac{1}{2}(|T| - 1)$ for every tournament $\mathcal{T} = \langle T, t \rangle$. Hence we can reformulate Theorem 8.

Strongly homogeneous tournaments are precisely the tournaments with maximal simplicity.

4. STRONGLY HOMOGENEOUS TOURNAMENTS AND BLOCK DESIGNS

Definition 7. Let M be a set, $\mathcal{B} \subset \exp M$ (= the power set of M). Define the mappings b_1, h_1, h_2:

$b_1 : \mathcal{B} \to N$ by $b_1(B) = |B|$,

$h_1 : M \to N$ by $h_1(x) = |\{B \mid B \in \mathcal{B} \text{ and } x \in B\}|$,

$h_2 : P_2(M) \to N$ by $(\{x, y\}) = |\{B \mid B \in \mathcal{B} \text{ and } \{x, y\} \subset B\}|$, where $P_2(M) = \{\{x, y\} \mid x, y \in M \text{ and } x \neq y\}$.

In the case that the mappings b_1, h_1, h_2 are constant we say that $\langle M, \mathcal{B}, b_1, h_1, h_2 \rangle$ is a *block design* of the type $\langle M, \mathcal{B}, b_1, h_1, h_2 \rangle$ (we denote the corresponding constants by the same symbol as the mappings themselves).

Every S + H-tournament determines a block-design in the following way: Let $\mathcal{T} = \langle T, t \rangle$ be an S + H-tournament. Put $\mathcal{B} = \{v(x) \mid x \in T\}$. Then $b_1 = \frac{1}{2}(|T| - 1)$ $\mathcal{T} = \langle T, t \rangle$ be an S + H-tournament. Put $\mathcal{B} = \{v(x) \mid x \in T\}$. Then $b_1 = \frac{1}{2}(|T| - 1) = h_1$ and $h_2 = \frac{1}{4}(|T| - 3)$ as can be seen easily. It is also clear that every such block design satisfies all the properties of Hadamard block designs.

There is anothe way to derive a block design from an S + H-tournament; namely by putting $\mathcal{B}' = \{v(x) \cup \{x\} \mid x \in T\}$. Obviously in this case $b_1 = h_1 = 2k + 2$ and $h_2 = k + 1$.

Let $\mathcal{T}$ be an S + H-tournament, denote by $B_1(\mathcal{T})$ the block design given by the former construction and by $B_2(\mathcal{T})$ the block design given by the latter one.

It is not clear whether for every block design of Hadamard type or of the type $\langle 4k + 3, 4k + 3, 2k + 2, 2k + 2, k + 1\rangle$ there exists an S + H-tournament which determines the given block by one of the above constructions. Here we give a partial answer to this question.

Proposition 11. *Let $\langle M, \mathscr{B}, b_1, h_1, h_2\rangle$ be a Hadamard block design. Then there exists a tournament $\mathscr{T} = \langle M, t\rangle$ such that $B_1(\mathscr{T}) = \langle M, \mathscr{B}, b_1, h_1, h_2\rangle$ iff there exists a bijection $q : M \to \mathscr{B}$ such that $\langle M, \mathscr{B}', b'_1, h'_1, h'_2\rangle$ is a block design of the type $\langle 4k + 3, 4k + 3, 2k + 2, 2k + 2, k + 1\rangle$ where $\mathscr{B}' = \{q(x) \cup \{x\} \mid x \in M\}$ and b'_1, h'_1, h'_2, are the corresponding mappings.*

Proof. The necessity of the condition is obvious. Let there exist a mapping q with the above properties. Define the relation $t \subset M \times M$ by $\langle x, y\rangle \in t$ iff $y \in q(x)$. First we shall prove that $\langle M, t\rangle$ is a tournament. As $h'_2(\{x, y\} - h_2(\{x, y\}) = 1$, either $x \in q(y)$ or $y \in q(x)$, hence either $\langle x, y\rangle \in t$ or $\langle y, x\rangle \in t$. On the other hand $x \in q(y)$ and $y \in q(x)$ is impossible because $h'_2(\{x, y\}) - h_2(\{x, y\}) < 2$, and we get $\langle x, y\rangle \notin t \Leftrightarrow \langle y, x\rangle \in t$. The rest of the statement is clear as $\langle M, \mathscr{B}, b_1, h_1, h_2\rangle$ is a Hadamard block design. Now we shall show that to a Hadamard block design constructed in a a very special way there exists the corresponding S + H-tournament.

Definition 8. Let $\langle G, +\rangle$ be an Abelian group, $D \subset G$. We say that D is a (v, p, s)-*difference set* if $|G| = v$, $|D| = q$ and if for every $a \in G$, $a \neq 0$ there exist exactly s couples $x, y \in D$ such that $a = x - y$.

Proposition 12. *Let $\langle G, +\rangle$ be an Abelian group, let $D \subset G$ be a (v, p, s)-difference set. Then $\langle G, \mathscr{B}, b_1, h_1, h_2\rangle$ is a block design of the type $\langle v, v, p, p, s\rangle$ where $\mathscr{B} = \{i + D \mid i \in G\}$.*

See [1] for the proof.

Proposition 13. *Let $\langle T, +, .\rangle$ be a finite field and let $T = q^m = 4k + 3$ for q prime, k positive integer. Then $D = \{a^2 \mid a \in T,\ a \neq 0\}$ is a $(4k + 3, 2k + 1, k + 1)$-difference set.*

This can be proved by a standard method via quadratic residues. See e.g. [1].

Theorem 9. *Let $\langle T, +, .\rangle$ be a finite field of the order $p^m = 4k + 3$ for a p prime, k positive integer. Let $D = \{a^2 \mid a \in T,\ a \neq 0\}$. Define the relation $t \subset T \times T$ by $\langle i, j\rangle \in t$ iff $j - i \in D$. Then $\langle T, t\rangle$ is an $S + H$-tournamen twhich is invertible.*

Lemma. *Let $\langle T, +, .\rangle$ be a finite field, $|T| = 4k + 3$, $a \in T$, $b \in T$. Then $a^2 + b^2 \neq 0$.*

Proof. Supposing the contrary we derive easily $a^{4k+2} = -b^{4k+2}$ and consequently $a = -b$ which is a contradiction (T is not even).

Proof of Theorem 9. It is $D \cap -D = \emptyset$ and $-D = T - (\{0\} \cup D)$ by the above lemma. The strong homogeneity of $\mathscr{T}$ follows by Proposition 12.

Thus we have an invertible S + H-tournament with prime power of vertices (and of course with $4k + 3$ vertices). Hence starting from any one or two of these tournaments we can apply the constructions given in Propositions 6 and 8.

5. CONCLUDING REMARKS

1. As the general question whether there exists a Hadamard block design on very set of $4k + 3$ points is not yet settled it is not surprising that it is not known whether there is an S + H-tournament on every set $4k + 3$. Theorem 9 gives the affirmative answer on every set of prime power cardinality and the constructions given by Propositions 6 and 8 provide another partial affirmative answer.

2. The structure of S + H-tournaments being so restricted it is not clear what is the behaviour of the groups of their autormorphisms. It is not even known if there is an S + H-tournament which has the identity mapping as its only automorphisms.

APPENDIX

While finishing this paper, we have got acquainted with a paper by E. SZEKERES: *Tournaments and Hadamard matrices* which contains some of the results given here: It contains the definition of a $T_{k,m}$ tournament (which is a tournament with the property $|\bigcap v(x_i)| \geqq m$ for any k distinct vertices $x_1, \ldots, x_k$). It is proved that a $T_{k,m}$-tournament contains at least $2^k(m + 1) - 1$ vertices.

$T_{k,m}$ with $2^k(m + 1) - 1$ vertices are studied for $k = 2$. It turns out that they are precisely S + H-tournaments. Szekeres proved our Proposition 6 and gave the connection with Hadamard block designs. Theorems on existence of $T_{2,m}$ tournaments with prime power vertices are proved via complementary difference sets. The method is similar to that employed here. Szekeres also exhibited an example of a Hadamard block design which cannot be constructed from an S + H-tournament (using the construction B_1). Nevertheless, it seems that Proposition 8 and the fact that there are invertible S + H-tournaments with prime power vertices yield a few Hadamard block designs. (Particularly it follows from Theorem 1 that there are no extremal $T_{k,m}$-tournaments for any $k \geqq 3$.)

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Author's address: 186 00 Praha 8, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta KU).