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REMARKS ON TOPOLOGIES UNIQUELY DETERMINED BY THEIR CONTINUOUS SELF MAPS

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In this note, some remarks to Warndof's paper [7] are given. Let a pair $(A, \mathscr{A})$ denote a topological space, where A is the set of points in the space and $\mathscr{A}$ is the collection of all open sets of the space. Let $C(\mathscr{A}, \mathscr{B})$ denote the collection of all continuous mappings from $(A, \mathscr{A})$ into $(B, \mathscr{B})$, $C(\mathscr{A}) = C(\mathscr{A}, \mathscr{A})$. Warndof have introduced the concept of the generated space and the special space. A T_1-space $(A, \mathscr{A})$ is generated if for every T_1-topology $\mathscr{B}$ on A such that $C(\mathscr{A}) \subseteq C(\mathscr{B})$, $\mathscr{A} \subseteq \mathscr{B}$ holds. A T_1-space $(A, \mathscr{A})$ is generated if and only if $\{f^{-1}(x)/f \in C(\mathscr{A}), x \in A\}$ forms a subbasis for closed sets of $(A, \mathscr{A})$. A (T_1-) space $(A, \mathscr{A})$ is (T_1-) special if the only (T_1-) topology $\mathscr{B}$ on A such that $C(\mathscr{A}) = C(\mathscr{B})$ is the topology $\mathscr{B} = \mathscr{A}$.

In [5] it is shown that any non-discrete T_1-special space is special. Further, any space $(A, \mathscr{A})$ containing a two-point set $C \subseteq A$ with $\{\emptyset, C\} \neq \mathscr{A}/C \neq \exp C$ is special. If for any two-point set $C \subseteq A$ it holds either $\mathscr{A}/C = \{\emptyset, C\}$ or $\mathscr{A}/C = \exp C$, then $(A, \mathscr{A})$ is special if and only if the T_0-reflexion of $(A, \mathscr{A})$, which is a T_1-space, is special. This facts imply that to find all special spaces it is sufficient to deal with T_1-special spaces. Therefore, by a space (topology) we shall always understand a T_1-space (T_1-topology).

Let $\bar{\mathscr{A}}$ be the system of all closed sets of a space $(A, \mathscr{A})$ and $\mathscr{A}^0$ the system of all clopen sets. It is shown in [5] that $\mathscr{A}^0 = \mathscr{B}^0$ must hold in the case $C(\mathscr{A}) = C(\mathscr{B})$. The closure of $X \subseteq A$ in $(A, \mathscr{A})$ is denoted by $Cl_{\mathscr{A}}(X)$.

1. Warndof has proved in [7] that the space $(A, \mathscr{A})$, $\mathscr{A} = \{X/\text{card}\,(A - X) < \mathfrak{m}\} \cup \{\emptyset\}$ is either special or discrete for any infinite cardinal number $\mathfrak{m}$. His result can be generalized.

Theorem 1. *Let $(A, \mathscr{A})$ be a non-discrete space containing a point p any neighbourhood of which is open. Then $(A, \mathscr{A})$ is both generated and special.*

Proof. Let $X \in \overline{\mathscr{A}}$. Define $f_X : A \to A$ as follows: $f_X(x) = p$ for $x \in X$ and $f_X(x) = x$ otherwise. Let $V \in \mathscr{A}$. If $p \notin V$, one gets $f_X^{-1}(V) = V - X \in \mathscr{A}$. If $p \in V$, it holds $f_X^{-1}(V) = V \cup X \in \mathscr{A}$ since $V \cup X$ is a neighbourhood of p. Therefore $f_X \in C(\mathscr{A})$.

We have proved that $\mathscr{A}$ is generated. Let $C(\mathscr{A}) = C(\mathscr{B})$ for a topology $\mathscr{B}$ on A. Let U be a neighbourhood of p in $\mathscr{B}$. Then we can find $V \in \mathscr{B}$ with $p \in V \subseteq U$. It is $V \cup \{x\} = f_{\{x\}}^{-1}(V) \in \mathscr{B}$ for every $x \in U - V$. Therefore $U \in \mathscr{B}$, i.e., any neighbourhood of p in $\mathscr{B}$ is open in $\mathscr{B}$. Hence $\mathscr{B}$ is generated and $\mathscr{A}$ is special.

Corollary 1. *Let $\mathscr{F}$ be a proper free filter on a set A. Then the topology $\mathscr{F} \cup \{\emptyset\}$ is both generated and special.*

Corollary 2. *Let $\mathscr{F}$ be a proper free filter on a set A and $x \in A$. Then the topology $\mathscr{S}(\mathscr{F}, x) = \mathscr{F} \cup \exp(E - \{x\})$ is both generated and special.*

If $\mathscr{F}$ is a free ultrafilter, then $\mathscr{S}(\mathscr{F}, x)$ is called a free ultraspace. Free ultraspaces are precisely the dual atoms of the lattice of all topologies. Magill has shown in [4] that any completely regular space is a subspace of a generated space.

Corollary 3. *Any space $(A, \mathscr{A})$ is a subspace of a generated special space.*

Proof. Let $a \notin A$, $B = A \cup \{a\}$. Define $\mathscr{B} = \mathscr{A} \cup \{A - X / X \text{ finite}\}$. $(A, \mathscr{A})$ is a subspace of $(B, \mathscr{B})$ and $(B, \mathscr{B})$ is generated and special because any neighbourhood of a is open.

Theorem 2. *The topological sum of special spaces is a special space.*

Proof: Let $(A_i, \mathscr{A}_i)$ be special for any $i \in I$ and $(A, \mathscr{A}) = \sum_{i \in I} (A_i, \mathscr{A}_i)$. Let $C(\mathscr{A}) = C(\mathscr{B})$ for a topology $\mathscr{B}$ on A. $C(\mathscr{A}_i) \subseteq C(\mathscr{B}/A_i)$ because any $f \in C(\mathscr{A}_i)$ is a restriction of a suitable $g \in C(\mathscr{A})$. Since $C(\mathscr{A}) = C(\mathscr{B})$ and A_i is clopen in $\mathscr{A}$, A_i is clopen in $\mathscr{B}$ (see [5]). Therefore $\mathscr{B} = \sum_{i \in I} \mathscr{B}/A_i$. Hence $C(\mathscr{B}/A_i) \subseteq C(\mathscr{A}_i)$. As $\mathscr{A}_i$ is special, we have $\mathscr{A}_i = \mathscr{B}/A_i$, i.e. $\mathscr{A} = \mathscr{B}$.

Corollary 4. *Any topological space is a quotient of a generated special space.*

Proof. Since any space is an intersection of free ultraspaces, one gets that any space is a quotient of a topological sum of free ultraspaces. This follows, for instance, from the characterizations of coreflective subcategories of the category of topological spaces (see [2]). By Corollary 1 any space is a quotient of a topological sum of generated special spaces. It remains to prove that the topological sum of generated spaces is generated. However, this follows from [7], Th. 1.5, because a topological sum of a family of spaces is a subspace of the product of spaces of this family and a discrete space (see [2]).

2. Theorem 3. *Any space of ordinals is either special or discrete.*

Proof. Let $\alpha > \omega_0$, $A = W(\alpha) = \{\beta / \beta < \alpha\}$ and let $\mathscr{A}$ be the interval topology on A. Let $\mathscr{B}$ be a topology on A with $C(\mathscr{A}) = C(\mathscr{B})$. $\mathscr{A}$ is zero-dimensional and thus $\mathscr{A} \subseteq \mathscr{B}$ (see [4]). Suppose that there exists $X \in \overline{\mathscr{B}} - \overline{\mathscr{A}}$. Let β be the least element of the set $Cl_{\mathscr{A}}(X) - X$. β is limit.

Suppose that there exists $\gamma < \beta$ such that any $\xi \in (\gamma, \beta) \cap X$ is isolated. Put $V = W(\gamma + 1) \cup [(\gamma, \beta) \cap X] \cup (\beta, \alpha)$. Clearly $V \in \mathscr{A}$. Further $A - V = (\gamma, \beta + 1) - X \in \mathscr{B}$. V is clopen in $\mathscr{B}$ and therefore V is clopen in $\mathscr{A}$. This contradicts to $\beta \in Cl_{\mathscr{A}}(X) - X$. Thus there exists a limit $\xi \in (\gamma, \beta) \cap X$ for every $\gamma < \beta$.

Define $f : A \to A$ as follows: $f(\gamma)$ is the least element of the set $\{\zeta / \gamma \leqq \zeta, \zeta \text{ limit}, \zeta \in Cl_{\mathscr{A}}(X)\}$ for $\gamma \leqq \beta$, $f(\gamma) = \gamma$ otherwise. Let $\gamma_1 < \gamma_2 \leqq \beta + 1$. Clearly $f^{-1}(\gamma_1, \gamma_2)$ is convex. Let δ be the least element of $f^{-1}(\gamma_1, \gamma_2)$. To verify that $f^{-1}(\gamma_1, \gamma_2) \in \mathscr{A}$ it is sufficient to show that δ is isolated. Suppose that δ is limit. If $\delta \in Cl_{\mathscr{A}}(X)$, so $\delta = f(\delta) \in (\gamma_1, \gamma_2)$. Hence $\gamma_1 + 1 \in f^{-1}(\gamma_1, \gamma_2)$, i.e. $\delta = \gamma_1 + 1$, a contradiction. Therefore $\delta \notin Cl_{\mathscr{A}}(X)$. We can find $\gamma < \delta$ such that $(\gamma, \delta + 1) \cap Cl_{\mathscr{A}}(X) = \emptyset$. Hence $f(\gamma) = f(\delta)$, a contradiction. If $\beta \leqq \gamma_1 \leqq \gamma_2$, it holds $f^{-1}(\gamma_1, \gamma_2) = (\gamma_1, \gamma_2) \in \mathscr{A}$. In the case $\gamma_1 < \beta < \gamma_2$ we have $f^{-1}(\gamma_1, \gamma_2) = f^{-1}(\gamma_1, \beta + 1) \cup f^{-1}(\beta, \gamma_2) \in \mathscr{A}$. We have proved that $f \in C(\mathscr{A})$.

Let $\gamma < \beta$. Since there exists a limit $\xi \in (\gamma, \beta) \cap X$, one gets that $f(\gamma) < \beta$. Since $f(\gamma) \in Cl_{\mathscr{A}}(X)$, it follows from the definition of β that $f(\gamma) \in X$. Therefore $f^{-1}(X) = X \cup W(\beta)$, i.e. $A - (X \cup W(\beta)) \in \mathscr{B}$. It is $\{\beta\} = [A - (X \cup W(\beta)] \cap W(\beta + 1) \in \mathscr{B}$. Hence $\{\beta\}$ is clopen in $\mathscr{B}$, i.e. in $\mathscr{A}$, a contradiction.

Theorem 4. *Any ordered space is generated.*

Proof. Let A be a chain, $\mathscr{A}$ the interval topology on A. Let $a \in A$. Define $f(x) = x \vee a$ for every $x \in A$. Let $[b, c]$ be a closed interval. If $a \notin [b, c]$, it holds $f^{-1}[b, c] = [b, c]$ or $\emptyset$. If $a \in [b, c]$, we have $f^{-1}[b, c] = (c]$. Therefore f is continuous. It is $(a] = f^{-1}(a)$. Analogously it may be proved that $[a)$ is a preimage of a in some $f \in C(\mathscr{A})$. Thus $\mathscr{A}$ is generated.

Theorem 5. *Any infinitely distributive complete lattice A is generated in its interval topology.*

Proof. Let $a \in A$. Define $f(x) = x \vee a$ for every $x \in A$. Let $t \in A$. $f^{-1}(t] = \emptyset$ for $a \nleqq t$ and $f^{-1}(t] = (t]$ otherwise. Since $(\bigwedge_{x \vee a \geqq t} x) \vee a = \bigwedge_{x \vee a \geqq t} (x \vee a) \geqq t$, it holds $f^{-1}[t) = [\bigwedge_{x \vee a \geqq t} x)$. Therefore f is continuous. It is $f^{-1}(a) = (a]$. Dually we can show that $[a)$ is a preimage of a in some $f \in C(\mathscr{A})$. Therefore $\mathscr{A}$ is generated.

3. A space $(A, \mathscr{A})$ is called upper special if the only topology $\mathscr{B}$ on A such that $\mathscr{A} \subseteq \mathscr{B}$ and $C(\mathscr{A}) = C(\mathscr{B})$ is the topology $\mathscr{B} = \mathscr{A}$. $(A, \mathscr{A})$ is called full if it has no isolated points, and the only topology $\mathscr{B}$ on A without isolated points such that $\mathscr{A} \subseteq \mathscr{B}$ and $C(\mathscr{A}) \subseteq C(\mathscr{B})$ is the topology $\mathscr{B} = \mathscr{A}$ (see [7]). We define a space $(A, \mathscr{A})$ to be limited if the only topology $\mathscr{B}$ on A such that $\mathscr{A} \subseteq \mathscr{B}$, $\mathscr{A}^0 = \mathscr{B}^0$ and $C(\mathscr{A}) \subseteq C(\mathscr{B})$ is the topology $\mathscr{B} = \mathscr{A}$.

Lemma 1. *Any full space is limited and any limited space is upper special.*

In [3] a space $(A, \mathscr{A})$ is defined to be a V-space if for any points $p, q, x, y \in A$, where $p \neq q$, there exists $f \in C(\mathscr{A})$ such that $f(p) = x$ and $f(q) = y$. $(A, \mathscr{A})$ is a weak V-space if for any two different points $p, q \in A$ and any non-empty set $P \subsetneqq A$, there exists $f \in C(\mathscr{A})$ such that $f(p) \in P$ and $f(q) \notin P$. Šneperman has proved in [6] that if $(A, \mathscr{A})$ is a completely regular space containing an arc $(B, \mathscr{A}/B)$ and $\mathscr{A}'$ a topology on A with $C(\mathscr{A}') = C(\mathscr{A})$, then $\mathscr{A}'/B$ is an arc. This result can be generalized. A space is called $\mathscr{S}$-regular if it is a subspace of a product of copies of $(S, \mathscr{S})$ (see [1] and [2]).

Theorem 6. *Let $(S, \mathscr{S})$ be a connected, generated, limited and weak* V-*space. Let $(A, \mathscr{A})$ be an $\mathscr{S}$-regular space, $S \subseteq A$ and $\mathscr{A}/S = \mathscr{S}$. Let $\mathscr{A}'$ be a topology on A such that $C(\mathscr{A}) = C(\mathscr{A}')$. Then $\mathscr{A}'/S = \mathscr{S}$.*

Proof. Since $\mathscr{S}$ is generated and $\mathscr{A}$ $\mathscr{S}$-regular, one easily gets that $\mathscr{A}$ is generated, too. Therefore $\mathscr{A} \subseteq \mathscr{A}'$, i.e. $\mathscr{S} \subseteq \mathscr{A}'/S$. Let $Z \in \mathscr{A}'/S$. Let $\mathscr{B}$ be a topology on S with the subbasis $\{f^{-1}(Z)/f \in C(\mathscr{S})\} \cup \mathscr{S}$. It holds $\mathscr{S} \subseteq \mathscr{B}$ and $C(\mathscr{S}) \subseteq C(\mathscr{B})$. Assume first that $\mathscr{B}$ is connected. Then $\mathscr{S} = \mathscr{B}$ because $\mathscr{S}$ is limited and therefore $Z \in \mathscr{S}$, which completes the proof.

Suppose that $\mathscr{B}$ is not connected. Then there exists a clopen set $X \in \mathscr{B}^0$ with $\emptyset \neq X \neq S$. Let $x \in X$ and $y \in S - X$. Since $\mathscr{A}$ is $\mathscr{S}$-regular, there exists $h_1 \in C(\mathscr{A}, \mathscr{S})$ with $h_1(x) \neq h_1(y)$. Since $\mathscr{S}$ is a weak V-space, there exists $h_2 \in C(\mathscr{S})$ with $h_2 h_1(x) \in X$ and $h_2 h_1(y) \notin X$. Put $h = h_2 h_1$. It is $h \in C(\mathscr{A}, \mathscr{S}) \subseteq C(\mathscr{A}) = C(\mathscr{A}')$. Let $Y \in \mathscr{B}$. It is $Y = \bigcup_{i \in I} [Y_i \cap \bigcap_{k=1}^{n} f_{i,k}^{-1}(Z)]$, where $Y_i \in S$ and $f_{i,k} \in C(\mathscr{S})$ for $i \in I$ and $k = 1, 2, \ldots, n$. Thus $h^{-1}(Y) = \bigcup_{i \in I} [h^{-1}(Y_i) \cap \bigcap_{k=1}^{n} h^{-1} f_{i,k}^{-1}(Z)]. f_{i,k} h \in C(\mathscr{A}) = C(\mathscr{A}')$ and therefore $h^{-1}(Y) \in \mathscr{A}'$. Thus $h \in C(\mathscr{A}', \mathscr{B})$. Hence $h^{-1}(X) \in (\mathscr{A}')^0 = \mathscr{A}^0$, i.e. $h^{-1}(X) \cap S \in \mathscr{S}^0$. Since $\mathscr{S}$ is connected, $x \in h^{-1}(X)$ and $y \notin h^{-1}(X)$, we get a contradiction.

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