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SOME HIGHER ORDER OPERATIONS WITH CONNECTIONS

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Our intention being to study some higher order operations with connections, we should first remark that there exist two main kinds of higher order problems in the general theory of connections. On the one hand, one can investigate connections of the first order on some "higher order spaces", for instance on higher order prolongations of a differentiable manifold or on prolongations of a principal fibre bundle, [5]. On the other hand, there are higher order connections introduced by EHRESMANN, [3]. Moreover, in the interesting special case of vector bundles, some of these problems have specific forms and have been sometimes studied by means of special methods. Hence it is quite natural that some of our definitions are motivated by an intention to compare certain different points of view adopted independently by different authors. However, the greater part of the following concepts is of their own interest. We first define a product of higher order connections, a special case of which has been introduced recently by VIRSIK, [11]. This product represents a generalization of the prolongation of higher order connection in the sense of Ehresmann, [3]. Then we study some relations between the higher order connections on a Lie groupoid Φ and the first order connections on the prolongations of Φ. This investigation was incited by [6], where LIBERMANN has established an identification between the semi-holonomic connections of order r on the first order frame bundle of a differentiable manifold M and the first order connections on the semi-holonomic frame bundle of order r of M. After that, we define prolongations of an arbitrary first order connection C with respect to a linear connection L on the base manifold (in the special case of a connection on a vector bundle, this operation was treated by Pohl, [8]) and we find their relation to the prolongations of connections C and L in the sense of Ehresmann. In conclusion, we give a comparison of the absolute differentiation with respect to the connections related by our main operations. We hope that Propositions 3, 4, 8, 10, 12 and Corollary 3 can testify that the operations introduced here are of practical importance.

In the present paper, all connections are studied on groupoids, i.e., in the most geometric way. However, our operations have interesting forms even on principal fibre bundles and in the special case of vector bundles. We shall treat these problems as well as some other related questions in a continuation of this paper, which is under preparation. Unless otherwise specified, our considerations are in the category C^∞. The standard terminology and notation of the theory of jets is used throughout the paper, see [10]. In addition, j_r^s denotes the canonical projection of r-jets onto s-jets, $s < r$.

1. Let Φ be a Lie groupoid over B with projections a, b and let e_x denote the unit of Φ over $x \in B$, see [9]. The partial composition law in Φ will be denoted by a dot. Hence $\theta' . \theta$ is defined for every $\theta', \theta \in \Phi$ satisfying $a(\theta') = b(\theta)$ and it is $b(\theta' . \theta) = b(\theta')$, $a(\theta' . \theta) = a(\theta)$. In the sequel, we shall use frequently the prolongations of this partial composition law, [1], which will be denoted by the same symbol. If M is an arbitrary manifold, then the prolongation of the composition in Φ is defined for every pair $X', X \in \tilde{J}^r(M, \Phi)$ satisfying $aX' = bX \in \tilde{J}^r(M, B)$ and the value $X' . X$ is an element of $\tilde{J}^r(M, \Phi)$ such that $b[X' . X] = bX'$, $a[X' . X] = aX$. (We recall that e.g. aX' means the composition of mapping $a\colon \Phi \to B$ and of r-jet X' in accordance with a general agreement in the theory of jets. Using the square brackets we want to stress that e.g. $b[X' . X]$ is the composition of mapping $b\colon \Phi \to B$ and of r-jet $X' . X$ and not the value of b at $X' . X$.)

To create a convenient tool for the investigation of the higher order elements of connection, [3], we introduce the concept of a quasi-element of connection. By a non-holonomic quasi-element of connection of order r on Φ will be meant an r-jet $X \in \tilde{J}^r_y(B, \Phi)$ satisfying $aX = j^r_y\hat{x}$, $bX = j^r_y id_B$, $x, y \in B$, where $\hat{x}$ is the constant mapping $t \mapsto x$, $t \in B$. The space of all non-holonomic quasi-elements of connection of order r on Φ will be denoted by $\tilde{A}^r(\Phi)$. We have the natural projections $\beta : \tilde{A}^r(\Phi) \to \Phi$, $\bar{a} := a\beta : \tilde{A}^r(\Phi) \to B$, $\bar{b} := b\beta : \tilde{A}^r(\Phi) \to B$. Put

$$\tilde{A}^r_x(\Phi) = \{X \in \tilde{A}^r(\Phi);\ \bar{a}(X) = x\}\,, \quad {}_y\tilde{A}^r(\Phi) = \{X \in \tilde{A}^r(\Phi);\ \bar{b}(X) = y\}\,,$$

${}_y\tilde{A}^r_x(\Phi) = \tilde{A}^r_x(\Phi) \cap {}_y\tilde{A}^r(\Phi)$. Further, let $\Phi_x = a^{-1}(x) \subset \Phi$. One deduces directly from the definition, cf. [4], that $\tilde{A}^r_x(\Phi)$ coincides with the r-th non-holonomic prolongation of fibred manifold (Φ_x, b, B), i.e.

$$(1) \qquad (\tilde{A}^r_x(\Phi), \bar{b}, B) = \tilde{J}^r(\Phi_x, b, B)\,.$$

This implies that every $X \in {}_y\tilde{A}^r_x(\Phi)$ can be written in the form $X = j^1_y\sigma$, where σ is a local cross section of $(\tilde{A}^{r-1}_x(\Phi), \bar{b}, B)$. If an $X \in \tilde{A}^r(\Phi)$ satisfies $\beta X = e_x$, then it is an element of connection in the sense of Ehresmann, [3]. Hence the fibred manifold $\tilde{Q}^r(\Phi)$ of all non-holonomic elements of connection of order r on Φ is a subspace of $\tilde{A}^r(\Phi)$. In the semi-holonomic and holonomic cases, we set $\bar{A}^r(\Phi) = \tilde{A}^r(\Phi) \cap \bar{J}^r(B, \Phi)$, $A^r(\Phi) = \tilde{A}^r(\Phi) \cap J^r(B, \Phi)$.

2. Groupoid Φ acts on the right on fibred manifold ${}_y\tilde{A}^r(\Phi)$ as follows. Let $X \in {}_y\tilde{A}^r(\Phi)$, $\bar{a}(X) = x$ and let $\theta \in \Phi$, $a(\theta) = z$, $b(\theta) = x$. To simplify the notation, we shall denote by $\hat{\theta}$ the r-jet at z of the constant mapping $t \mapsto \theta$, $t \in B$. Then the prolongation of the partial composition law in Φ defines an element

$$(2) \qquad X \, . \, \hat{\theta} \in {}_y\tilde{A}^r(\Phi) \, , \quad \bar{a}(X \, . \, \hat{\theta}) = a(\theta) = z \, .$$

Obviously, if $\theta = \beta X$, then $X \, . \, \hat{\theta}^{-1}$ is an element of connection. Hence every quasi-element of connection $Y \in \tilde{A}^r(\Phi)$ can be written in the form $Y = X \, . \, \hat{\theta}$, where X is an element of connection and $\theta \in \Phi$. Formula (2) determines the mapping

$$(\tilde{A}^r(\Phi), \bar{a}, B) \oplus (\Phi, b, B) \to \tilde{A}^r(\Phi) \, , \quad (X, \theta) \mapsto X \, . \, \hat{\theta} \, .$$

(The symbol $\oplus$ denotes the fibre product over B.) In particular, the prolongation of (2) is defined for every $X \in \tilde{J}^s(M, \tilde{A}^r(\Phi))$ and $Y \in \tilde{J}^s(M, \Phi)$ satisfying $\bar{a}X = bY$. Consider now a non-holonomic connection $C : B \to \tilde{Q}^r(\Phi)$ and an element of connection $X \in \tilde{Q}^s_x(\Phi)$. Since the s-jet $j^s_x C$ satisfies $\bar{a}[j^s_x C] = j^s_x id_B = bX$, the prolongation of (2) determines an element $C * X \in \tilde{J}^s_x(B, \tilde{A}^r(\Phi))$. One finds easily $\bar{b}[C * X] = j^s_x id_B$, $\bar{a}[C * X] = j^s_x \hat{x}$ and $\beta(C * X) = e_x$. Hence

$$C * X \in \tilde{Q}^{r+s}_x(\Phi) \, .$$

If a connection $C_1 : B \to \tilde{Q}^s(\Phi)$ is given, then the mapping $x \mapsto C * C_1(x)$ is a connection of the order $r + s$ on Φ, which will be denoted by $C * C_1$ and will be called the product (or star product) of C and C_1. In particular, if C_1 is a connection of the first order on Φ, then the direct comparison with [11] shows that connection $C * C_1 : B \to \tilde{Q}^{r+1}(\Phi)$ coincides with that denoted by the same symbol by Virsik, [11]. Moreover, if one considers the connection of the first order $j^1_r C : B \to Q^1(\Phi)$ underlying to C, then $C * (j^1_r C)$ coincides with the prolongation C' of C according to Ehresmann, [3]. The star product of connections is associative, i.e., if $C_1 : B \to \tilde{Q}^r(\Phi)$, $C_2 : B \to \tilde{Q}^s(\Phi)$, $C_3 : B \to \tilde{Q}^t(\Phi)$ are three connections, then $(C_1 * C_2) * C_3 = C_1 * (C_2 * C_3)$ is the same cross section of $\tilde{Q}^{r+s+t}(\Phi)$. This follows easily from the associativity of the composition law in Φ. (We remark that we shall show in the next paper that the star product of connections has a very instructive form on principal fibre bundles.)

3. Consider the r-th non-holonomic prolongation $\tilde{\Phi}^r$ of Φ, [2]. Let $\square$ denote the partial composition law in Φ^r and let a_r or b_r or e^r_x be the source projection or the target projection or the unit of $\tilde{\Phi}^r$ over $x \in B$ respectively. We recall that an element $Z \in \tilde{\Phi}^r$, $a_r(Z) = x$, $b_r(Z) = y$ is a non-holonomic r-jet of B into Φ satisfying $\alpha Z = x$, $aZ = j^r_x id_B$, $bZ \in \tilde{\Pi}^r(B)$, $\beta(bZ) = y$ and that the composition in $\tilde{\Phi}^r$ is defined by

$$(3) \qquad Z' \square Z = (Z' bZ) \, . \, Z \, , \quad a_r(Z') = b_r(Z) \, .$$

Obviously, the mapping $\tilde{\Phi}^r \to \tilde{\Pi}^r(B)$, $Z \mapsto bZ$ is a functor. Assume now that Φ is a groupoid of operators on a fibred manifold (E, p, B), [2]. The action of Φ on E will be also denoted by a dot. Hence $\theta . z$ is defined for every $\theta \in \Phi$ and $z \in E$ satisfying $a(\theta) = p(z)$ and it is $p(\theta . z) = b(\theta)$. Let M be a manifold. The prolongation of the action Φ on E, which will be also denoted by a dot, is defined for every pair $Z \in \tilde{J}^r(M, \Phi)$, $X \in \tilde{J}^r(M, E)$ satisfying $aZ = pY \in \tilde{J}^r(M, B)$ and the value $Z . X$ is an element of $\tilde{J}^r(M, E)$ such that $p[Z . X] = bZ$. We recall that $\tilde{\Phi}^r$ is a groupoid operating on $\tilde{J}^r E$ by

$$(4) \qquad Z \,\square\, X := (Z . X)\,(bZ)^{-1}\,, \quad Z \in \tilde{\Phi}^r\,, \quad X \in \tilde{J}^r E\,, \quad a_r(Z) = \alpha X\,,$$

where the product of $Z . X$ and $(bZ)^{-1}$ is the composition of jets.

For this moment, we shall need only a special case of (4). Consider the fibred manifold (Φ_w, b, B), $w \in B$. Then Φ itself is a groupoid operating on Φ_w by the multiplication in Φ. Hence $\tilde{\Phi}^r$ is a groupoid of operators on $\tilde{J}^r(\Phi_w) = \tilde{A}^r_w(\Phi)$. In other words, if $X \in {}_x\tilde{A}^r_w(\Phi)$ and $Z \in \tilde{\Phi}^r$, $a_r(Z) = x$, $b_r(Z) = y$, then

$$(5) \qquad Z \,\square\, X = (Z . X)\,(bZ)^{-1} \in {}_y\tilde{A}^r_w(\Phi)\,,$$

where $Z . X$ is defined by the prolongation of the composition in Φ. Further, let $Z = j^1_x\zeta$, $X = j^1_x\varphi$. Then

$$(6) \qquad Z \,\square\, X = j^1_y[(\zeta(\zeta_0^{-1}(t)) . \varphi(\zeta_0^{-1}(t)))\,\lambda^{-1}(\zeta_0^{-1}(t))]\,,$$

where $\zeta_0 = b_r\zeta$, $\lambda(t) = b[\zeta(t)]$. Conversely, if $X, Y \in \tilde{A}^r_w(\Phi)$, $\bar{b}(X) = x$, $\bar{b}(Y) = y$ and $L \in \tilde{\Pi}^r(B)$, $\alpha L = x$, $\beta L = y$, then the formula

$$(7) \qquad Z = YL . X^{-1}$$

determines the unique element $Z \in \tilde{\Phi}^r$ such that $bZ = L$ and $Y = Z \,\square\, X$. Let $Y = j^1_y\psi$, $X = j^1_x\varphi$, $L = j^1_x\lambda$, $\lambda_0 = j^0_{r-1}\lambda$. Then

$$(8) \qquad Z = j^1_x[\psi(\lambda_0(t))\,\lambda(t) . \varphi^{-1}(t)]\,,$$

where, by induction, $\psi(\lambda_0(t))\,\lambda(t) . \varphi^{-1}(t)$ is an element of $\tilde{\Phi}^{r-1}$. If $r = 1$ and $L = j^1_x\mu$, then formula (8) is specialized to

$$(9) \qquad Z = j^1_x[\psi(\mu(t)) . \varphi^{-1}(t)]\,.$$

The following two assertions will be used in the proof of Proposition 8.

Lemma 1. *Let $Z_1, Z_2 \in \tilde{\Phi}^r$ be two elements of the form $Z_1 = (Y_1 . \hat{\theta})\,L_1 . X_1^{-1}$, $Z_2 = Y_2L_2 . Y_1^{-1}$. Then*

$$Z_2 \,\square\, Z_1 = (Y_2 . \hat{\theta})\,L_2L_1 . X_1^{-1}\,.$$

Proof. The assertion is trivial for $r = 1$. Assume by induction that it holds for $r - 1$. Let $Y_2 = j^1_u\chi$, $Y_1 = j^1_y\psi$, $X_1 = j^1_x\varphi$, $L_1 = j^1_x\lambda$, $L_2 = j^1_y\nu$, $j^0_{r-1}\lambda = \lambda_0$, $j^0_{r-1}\nu = \nu_0$. Then

$$Z_1 = j^1_x[(\psi(\lambda_0(t)) . \hat{\theta})\, \lambda(t) . \varphi^{-1}(t)]\,,$$
$$Z_2 = j^1_y[\chi(\nu_0(t))\, \nu(t) . \psi^{-1}(t)]$$

and we obtain

$$(10) \qquad Z_2 \Box Z_1 = j^1_x\{[\chi(\nu_0(\lambda_0(t))\, \nu(\lambda_0(t)) . \psi^{-1}(\lambda_0(t))] \Box \Box [(\psi(\lambda_0(t)) . \hat{\theta})\, \lambda(y) . \varphi^{-1}(t)]\}\,,$$

where the symbol $\Box$ on the right hand side means the multiplication in $\tilde{\Phi}^{r-1}$. Using the induction hypothesis, we simplify the right hand side of (10) to $j^1_x[(\chi(\nu_0(\lambda_0(t)) . . \hat{\theta})\, \nu(\lambda_0(t))\, \lambda(t) . \varphi^{-1}(t)] = (Y_2 . \hat{\theta})\, L_2 L_1 . X_1^{-1}$, QED.

Corollary 1. *If $Z \in \tilde{\Phi}^r$ is an element of the form $Z = (Y . \hat{\theta})\, L . X^{-1}$, its inverse element in $\tilde{\Phi}^r$ is given by $Z^{-1} = XL^{-1} . \hat{\theta}^{-1} . Y^{-1}$.*

4. We now introduce a mapping $\varkappa_{r+1} : \tilde{Q}^{r+1}(\Phi) \oplus Q^1(\tilde{\Pi}^r(B)) \to Q^1(\tilde{\Phi}^r)$ as follows. If $X = j^1_x\nu \in \tilde{Q}^{r+1}_x(\Phi)$ and $Y = j^1_x\lambda \in Q^1_x(\tilde{\Pi}^r(B))$, then we set

$$(11) \qquad \varkappa_{r+1}(X, Y) = j^1_x[\nu(y)\, \lambda(y) . \nu^{-1}(x)] \in Q^1_x(\tilde{\Phi}^r)\,.$$

We further define a mapping $k_r : Q^1(\tilde{\Phi}^r) \oplus \tilde{Q}^r(\Phi) \to \tilde{Q}^{r+1}(\Phi)$ by

$$(12) \qquad k_r(Z, T) = j^1_x(\zeta(y) \Box T) = j^1_x[(\zeta(y) . T)\, (b\, \zeta(y))^{-1}]\,,$$

provided $Z = j^1_x\zeta \in Q^1_x(\tilde{\Phi}^r)$, $T \in \tilde{Q}^r_x(\Phi)$. Moreover, the functor $\tilde{\Phi}^r \to \tilde{\Pi}^r(B)$, $X \mapsto bX$ is extended to a mapping

$$b_* : Q^1(\tilde{\Phi}^r) \to Q^1(\tilde{\Pi}^r(B))\,, \quad j^1_x\, \sigma(y) \mapsto j^1_x\, b\sigma(y)\,.$$

(In general, if Φ and Ψ are two Lie groupoids over B and $\varphi : \Phi \to \Psi$ is a base-preserving functor, then φ is extended to a mapping

$$(13) \qquad \varphi_* : Q^1(\Phi) \to Q^1(\Psi)\,, \quad j^1_x\, \sigma(y) \mapsto j^1_x\, \varphi(\sigma(y))\,.)$$

Proposition 1. *The mapping*

$$(14) \qquad (\varkappa_{r+1}, j^r_{r+1}) : \tilde{Q}^{r+1}(\Phi) \oplus Q^1(\tilde{\Pi}^r(B)) \to Q^1(\tilde{\Phi}^r) \oplus \tilde{Q}^r(\Phi)\,,$$

$(\varkappa_{r+1}, j^r_{r+1})\, (X, Y) = (\varkappa_{r+1}(X, Y),\ j^r_{r+1}X)$, is a B-isomorphism. The inverse isomorphism is

$$(15) \qquad (k_r, b_*) : Q^1(\tilde{\Phi}^r) \oplus \tilde{Q}^r(\Phi) \to \tilde{Q}^{r+1}(\Phi) \oplus Q^1(\tilde{\Pi}^r(B))\,,$$

$(k_r, b_*)\, (Z, T) = (k_r(Z, T), b_*(Z))$.

Proof. It suffices to establish

$$\varkappa_{r+1}(k_r(Z, T), b_*(Z)) = Z \,, \tag{16}$$

$$k_r(\varkappa_{r+1}(X, Y), j^r_{r+1}X) = X \,. \tag{17}$$

In the above notation, we have $\varkappa_{r+1}(j^1_x[(\zeta(y) \,.\, T)\,(b\,\zeta(y))^{-1}],\ j^1_x b\,\zeta(y)) = j^1_x[\zeta(y) \,.\, T \,.\, T^{-1}] = Z$. On the other hand, it is $k_r(j^1_x[\nu(y)\,\lambda(y) \,.\, \nu^{-1}(x)], \nu(x)) = j^1_x[(\nu(y)\,\lambda(y) \,.\, \nu^{-1}(x) \,.\, \nu(x))\,\lambda^{-1}(y)] = X$, QED.

Thus, the pair (14), (15) gives an identification

$$\tilde{Q}^{r+1}(\Phi) \oplus Q^1(\tilde{\Pi}^r(B)) \approx Q^1(\tilde{\Phi}^r) \oplus \tilde{Q}^r(\Phi)\,.$$

5. Denote by $\bar{\bar{Q}}^r(\Phi) \subset \tilde{Q}^r(\Phi)$ the subspace of all elements of connection of the form $j^1_x\varrho$ where ϱ is a local cross section of $\bar{A}^{r-1}_x(\Phi)$. Obviously, it holds $\varkappa_{r+1}(\bar{\bar{Q}}^{r+1}(\Phi) \oplus Q^1(\bar{\Pi}^r(B))) \subset Q^1(\bar{\Phi}^r)$ and $k_r(Q^1(\bar{\Phi}^r) \oplus \bar{\bar{Q}}^r(\Phi)) \subset \bar{\bar{Q}}^{r+1}(\Phi)$. Thus, in the semi-holonomic case we have an identification

$$\bar{\bar{Q}}^{r+1}(\Phi) \oplus Q^1(\bar{\Pi}^r(B)) \approx Q^1(\bar{\Phi}^r) \oplus \bar{\bar{Q}}^r(\Phi)\,.$$

Let $Z \in Q^1_x(\bar{\Phi}^r)$ and $T \in \bar{\bar{Q}}^r_x(\Phi)$. We shall find a necessary and sufficient condition for $k_r(Z, T)$ to lie in $\bar{\bar{Q}}^{r+1}(\Phi)$. Introduce a mapping $\sigma_r : Q^1(\bar{\Phi}^r) \to \bar{\bar{Q}}^{r+1}(\Phi)$ by the following induction. For $r = 0$, σ_0 is the identity of $Q^1(\Phi)$. Assume by induction that we have defined the mapping $\sigma_{r-1} : Q^1(\bar{\Phi}^{r-1}) \to \bar{\bar{Q}}^r(\Phi)$. Then we set

$$\sigma_r(Z) = k_r(Z, \sigma_{r-1}(j^{r-1}_{r*}(Z)))\,, \quad Z \in Q^1(\bar{\Phi}^r)\,, \tag{18}$$

where $j^{r-1}_{r*}(Z) \in Q^1(\bar{\Phi}^{r-1})$ is the image of Z by the functor $j^{r-1}_r : \bar{\Phi}^r \to \bar{\Phi}^{r-1}$ in the sense of (13).

Lemma 2. *The values of σ_r lie in $\bar{\bar{Q}}^{r+1}(\Phi)$.*

Proof. Let $Z = j^1_x\zeta$, $\zeta_1 = j^{r-1}_r\zeta$, $Z_1 = j^1_x\zeta_1 = j^{r-1}_{r*}(Z)$. By definition, it is $\sigma_r(Z) = j^1_x[\zeta(y) \,\square\, \sigma_{r-1}(Z_1)]$. This implies

$$j^r_{r+1}\, \sigma_r(Z) = \sigma_{r-1}(j^{r-1}_{r*}(Z))\,. \tag{19}$$

By induction, (19) is also true for $r - 1$, i.e. $j^{r-1}_r\, \sigma_{r-1}(Z_1) = \sigma_{r-2}(j^{r-2}_{r-1*}(Z_1))$. Then we find $j^1_x[j^{r-1}_r(\zeta(y) \,\square\, \sigma_{r-1}(Z_1))] = j^1_x[\zeta_1(y) \,\square\, \sigma_{r-2}(j^{r-2}_{r-1*}(Z_1))] = \sigma_{r-1}(Z_1)$. Hence $\sigma_r(Z) \in \bar{\bar{Q}}^{r+1}(\Phi)$, QED.

Proposition 2. *Let $Z \in Q^1_x(\bar{\Phi}^r)$ and $T \in \bar{\bar{Q}}^r_x(\Phi)$. Then $k_r(Z, T) \in \bar{\bar{Q}}^{r+1}_x(\Phi)$ if and only if $T = \sigma_{r-1}(j^{r-1}_{r*}(Z))$.*

Proof. The "if part" coincides with Lemma 2. To prove the converse assertion, we shall proceed by induction. The assertion is trivial for $r = 0$. Assume that it holds for $r - 1$. Let $Z = j^1_x\zeta$, $\zeta_1 = j^{r-1}_r\zeta$, $Z_1 = j^1_x\zeta_1$, $T_1 = j^{r-1}_r T$. Since $k_r(Z, T) = j^1_x(\zeta(y) \square T) \in \bar{Q}^{r+1}_x(\Phi)$, it is $T = j^1_x(\zeta_1(y) \square T_1)$. By the induction hypothesis, $T_1 \in \bar{Q}^{r-1}_x(\Phi)$ implies $T_1 = \sigma_{r-2}(j^{r-2}_{r-1*}(Z_1))$. Hence $T = j^1_x[\zeta_1(y) \square \sigma_{r-1}(j^{r-2}_{r-1*}(Z_1))]$, which is $\sigma_{r-1}(Z_1)$ by definition, QED.

6. We shall now treat a special case $\Phi = \Pi^1(B)$. In this case, the target projection b of groupoid $\Pi^1(B)$ is the target jet projection $j^0_1 : \Pi^1(B) \to B$. One finds easily that the r-th non-holonomic prolongation $\widetilde{\Pi^1(B)}^r$ of $\Pi^1(B)$ coincides with $\tilde{\Pi}^{r+1}(B)$. Further, $\bar{\Pi}^{r+1}(B)$ is a subgroupoid of the r-th semi-holonomic prolongation $\overline{\Pi^1(B)}^r$ of $\Pi^1(B)$. On the one hand, we have the mapping $\overline{\Pi^1(B)}^r \to \bar{\Pi}^r(B)$, $X \mapsto bX$. On the other hand, we have the target projection $j^r_{r+1} : \overline{\Pi^1(B)}^r \to \overline{\Pi^1(B)}^{r-1}$.

Lemma 3. *An element $X \in \overline{\Pi^1(B)}^r$ belongs to $\bar{\Pi}^{r+1}(B)$ if and only if*

$$j^r_{r+1}X = bX \in \bar{\Pi}^r(B)\,. \tag{20}$$

Proof. We shall use the lateral projections of non-holonomic jets introduced in [7]. Since $X \in \overline{\Pi^1(B)}^r$, Proposition 4 of [7] yields

$$\begin{aligned} j^r_{r+1}X &= {}^1l^r_{r+1}X = \ldots = {}^{r-2}l^r_{r+1}X = {}^{r-1}l^r_{r+1}X\,,\\ j^{r-1}_{r+1}X &= {}^1l^{r-1}_{r+1}X = \ldots = {}^{r-2}l^{r-1}_{r+1}X\,,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ j^2_{r+1}X &= {}^1l^2_{r+1}X\,. \end{aligned} \tag{21}$$

We first deduce $bX = {}^rl^r_{r+1}X$. If $r = 1$ and $X = j^1_x\varphi$, then $b[j^1_x\,\varphi(y)] = j^1_x[j^0_1\,\varphi(y)] = {}^1l^1_2X$. Assume by induction that every $Y \in \overline{\Pi^1(B)}^{r-1}$ satisfies $bY = {}^{r-1}l^{r-1}_r Y$. Let $X \in \overline{\Pi^1(B)}^r$, $X = j^1_x\varphi$. Then $bX = b[j^1_x\,\varphi(y)] = j^1_x b\,\varphi(y) = j^1_x[{}^{r-1}l^{r-1}_r\,\varphi(y)] = {}^rl^r_{r+1}X$. Hence (20) can be written as

$$j^r_{r+1}X = {}^rl^r_{r+1}X\,. \tag{22}$$

Using Proposition 2 of [7], we derive from (21) and (22) the relations

$$j^{r-1}_{r+1}X = {}^{r-1}l^{r-1}_{r+1}X, \ldots, j^2_{r+1}X = {}^2l^2_{r+1}X\,,\quad j^1_{r+1}X = {}^1l^1_{r+1}X\,. \tag{23}$$

However, (21), (22) and (23) imply $X \in \bar{\Pi}^{r+1}(B)$ by Corollary 1 of [7]. The converse assertion is obvious, QED.

We introduce a mapping $\varrho_{r+1} : \bar{Q}^{r+1}(\Pi^1(B)) \to Q^1(\bar{\Pi}^{r+1}(B))$ by the following induction. For $r = 0$, ϱ_1 is the identity of $Q^1(\Pi^1(B))$. Assume by induction that we have defined the mapping

$$\varrho_r : \bar{Q}^r(\Pi^1(B)) \to Q^1(\overline{\Pi}^r(B)) . \tag{24}$$

Then we set

$$\varrho_{r+1}(X) = \varkappa_{r+1}(X, \varrho_r(j^r_{r+1}X)) , \quad X \in \bar{Q}^{r+1}(\Pi^1(B)) . \tag{25}$$

Lemma 4. *The values of* ϱ_{r+1} *lie in* $Q^1(\overline{\Pi}^{r+1}(B))$.

Proof. For technical reasons, we shall need a little stronger induction hypothesis than (24). Assume by induction that, for every $Y \in \bar{Q}^r(\Pi^1(B))$ and every local cross section ψ satisfying $Y = j^1_x\psi$, one can express $\varrho_{r-1}(j^{r-1}_r Y)$ as $j^1_x\varphi$ in such a way that $\psi(y)\,\varphi(y) \,.\, \psi^{-1}(x) \in \overline{\Pi}^r(B)$. (For $r = 1$, this can be directly verified.) Let $X \in \bar{Q}^{r+1}(\Pi^1(B))$, $X = j^1_x v$. Since X is semi-holonomic, it holds $v(x) = j^1_x v_1$, $v_1 = j^{r-1}_r v$. By the induction hypothesis, there is a local cross section λ satisfying $\varrho_{r-1}(v_1(x)) = j^1_x\lambda$ and $v_1(y)\,\lambda(y) \,.\, v_1^{-1}(x) \in \overline{\Pi}^r(B)$. Since $b[v_1(y)\,\lambda(y) \,.\, v_1^{-1}(x)] = \lambda(y)$, Lemma 3 gives

$$j^{r-1}_r(v_1(y)\,\lambda(y) \,.\, v_1^{-1}(x)) = \lambda(y) . \tag{26}$$

By definition, it is $\varrho_{r+1}(X) = j^1_x[v(y)\,(v_1(y)\,\lambda(y) \,.\, v_1^{-1}(x)) \,.\, v^{-1}(x)] := j^1_x\, H(y)$. Then we have $b\,H(y) = v_1(y)\,\lambda(y) \,.\, v_1^{-1}(x)$, while (26) implies $j^r_{r+1}\,H(y) = v_1(y)\,\lambda(y) \,.\, v_1^{-1}(x)$. Hence $H(y) \in \overline{\Pi}^{r+1}(B)$ by Lemma 3, $\varrho_{r+1}(X) \in Q^1(\overline{\Pi}^{r+1}(B))$ and the stronger induction hypothesis is also proved, QED.

Lemma 5. *The following diagram commutes*:

$$\begin{array}{ccc}
\bar{Q}^{r+1}(\Pi^1(B)) & \xrightarrow{\ \varrho_{r+1}\ } & Q^1(\overline{\Pi}^{r+1}(B)) \\
{\scriptstyle j^r_{r+1}}\downarrow & & \downarrow{\scriptstyle j^r_{r+1*}} \\
\bar{Q}^r(\Pi^1(B)) & \xrightarrow{\ \varrho_r\ } & Q^1(\overline{\Pi}^r(B))
\end{array}$$

Proof. Using the notation of the proof of Lemma 4, we obtain $j^r_{r+1*}(\varrho_{r+1}(X)) = j^1_x[v_1(y)\,\lambda(y) \,.\, v_1^{-1}(x)] = \varrho_r(j^r_{r+1}X)$, QED.

We shall further show that

$$\sigma_r(\varrho_{r+1}(X)) = X , \quad \varrho_{r+1}(\sigma_r(Z)) = Z \tag{27}$$

for every $X \in \bar{Q}^{r+1}(\Pi^1(B))$ and every $Z \in Q^1(\overline{\Pi}^{r+1}(B))$, provided $\overline{\Pi}^{r+1}(B)$ is considered as a subgroupoid of $\overline{\Pi^1(B)}^r$. We shall proceed by induction. For $r = 0$, both σ_0 and ϱ_1 coincide with the identity of $Q^1(\Pi^1(B))$. Assume that (27) holds for $r-1$. According to (18), it is

$$(28)\qquad \sigma_r(\varrho_{r+1}(X)) = k_r(\varrho_{r+1}(X), \sigma_{r-1}(j^r_{r+1*}(\varrho_{r+1}(X)))) .$$

We have $j^r_{r+1*}(\varrho_{r+1}(X)) = \varrho_r(j^r_{r+1}X)$ by Lemma 5 and $\sigma_{r-1}(\varrho_r(j^r_{r+1}X)) = j^r_{r+1}X$ by the induction hypothesis. Hence the right hand side of (28) is equal to $k_r(\varkappa_{r+1}(X, \varrho_r(j^r_{r+1}X)), j^r_{r+1}X) = X$. On the other hand, (25) gives

$$(29)\qquad \varrho_{r+1}(\sigma_r(Z)) = \varkappa_{r+1}(\sigma_r(Z), \varrho_r(j^r_{r+1}\, \sigma_r(Z))) .$$

We have $j^r_{r+1}\, \sigma_r(Z) = \sigma_{r-1}(j^{r-1}_{r*}(Z))$ by (19) and $\varrho_r(\sigma_{r-1}(j^{r-1}_{r*}(Z))) = j^{r-1}_{r*}(Z)$ by the induction hypothesis. Since $Z \in Q^1(\bar{\Pi}^{r+1}(B))$, Lemma 3 implies $j^{r-1}_{r*}(Z) = b_*(Z)$. Hence the right hand side of (29) is equal to $\varkappa_{r+1}(k_r(Z, \sigma_{r-1}(j^{r-1}_{r*}(Z))), b_*(Z)) = Z$. Thus, we have deduced

Proposition 3. *The restriction of* σ_r *to* $Q^1(\bar{\Pi}^{r+1}(B))$ *is a B-isomorphism* $Q^1(\bar{\Pi}^{r+1}(B)) \to \bar{Q}^{r+1}(\Pi^1(B))$. *The inverse isomorphism is* ϱ_{r+1}.

Remark. In the foregoing consideration, we have been interested in the semi-holonomic case only. However, it is remarkable that one can establish an identification of $\tilde{Q}^r(\Pi^1(B))$ and $Q^1(\tilde{\Pi}^r(B))$ in a quite analogous manner. Let us define a B-morphism $\tilde{\varrho}_r$: $\tilde{Q}^r(\Pi^1(B)) \to Q^1(\tilde{\Pi}^r(B))$ by the following induction

a) $\tilde{\varrho}_1$ is the identity of $Q^1(\Pi^1(B))$,
b) $\tilde{\varrho}_r(X) = \varkappa_r(X, \tilde{\varrho}_{r-1}(j^{r-1}_r X))$, $X \in \tilde{Q}^r(\Pi^1(B))$.

Obviously, the above mapping ϱ_r is the restriction of $\tilde{\varrho}_r$ to $\bar{Q}^r(\Pi^1(B))$. We assert that $\tilde{\varrho}_r$ is a B-isomorphism. To prove it, we introduce a mapping $\lambda_r : Q^1(\tilde{\Pi}^r(B)) \to$ $\to \tilde{Q}^r(\Pi^1(B))$ by the induction

a) λ_1 is the identity of $Q^1(\Pi^1(B))$,
b) $\lambda_r(Z) = k_{r-1}(Z, \lambda_{r-1}(b_*(Z)))$.

It suffices to deduce that $\lambda_r \circ \tilde{\varrho}_r$ is the identity of $\tilde{Q}^r(\Pi^1(B))$ and $\tilde{\varrho}_r \circ \lambda_r$ is the identity of $Q^1(\tilde{\Pi}^r(B))$. We shall proceed by induction. For $r = 1$, the assertion is trivial. Assume that it holds for $r-1$. If $X \in \tilde{Q}^r(\Pi^1(B))$, then $\lambda_r(\tilde{\varrho}_r(X)) =$ $= \lambda_r(\varkappa_r(X, \tilde{\varrho}_{r-1}(j^{r-1}_r X))) = k_{r-1}(\varkappa_r(X, \tilde{\varrho}_{r-1}(j^{r-1}_r X)), \lambda_{r-1}(\tilde{\varrho}_{r-1}(j^{r-1}_r X))) = X$ by (17) and by the induction hypothesis. Conversely, if $Z \in Q^1(\tilde{\Pi}^r(B))$, then $\tilde{\varrho}_r(\lambda_r(Z)) =$ $= \tilde{\varrho}_r(k_{r-1}(Z, \lambda_{r-1}(b_*(Z)))) = \varkappa_r(k_{r-1}(Z, \lambda_{r-1}(b_*(Z))), \tilde{\varrho}_{r-1}(\lambda_{r-1}(b_*(Z)))) = Z$ by (16) and by the induction hypothesis.

7. Combining the left action of $\tilde{\Phi}^r$ on $\tilde{A}^r_x(\Phi)$ and the right action of Φ on ${}_y\tilde{A}^r(\Phi)$ in the way shown below, we obtain a left action of $\tilde{\Phi}^r$ on $\tilde{Q}^r(\Phi)$. (In the holonomic

case, this action was treated by Que, [9].) Let $Z \in \tilde{\Phi}^r$, $a_r(Z) = x$, $b_r(Z) = y$, $\beta Z = \theta$ and let $X \in \tilde{Q}^r_x(\Phi)$. Then

$$(30) \qquad (Z \,\square\, X) \,.\, \hat{\theta}^{-1} = (Z \,.\, X)\,(bZ)^{-1} \,.\, \hat{\theta}^{-1} \in \tilde{Q}^r_y(\Phi)\,,$$

which establishes $\tilde{\Phi}^r$ as a groupoid of operators on $\tilde{Q}^r(\Phi)$. Similarly to [9], the set of all elements of $\tilde{\Phi}^r$ transforming a connection $C : B \to \tilde{Q}^r(\Phi)$ into itself is a subgroupoid $R(C) \subset \tilde{\Phi}^r$ isomorphic to $\Phi \times \tilde{\Pi}^r(B)$. More precisely, if we consider the natural projection $(\beta, b) : \tilde{\Phi}^r \to \Phi \times \tilde{\Pi}^r(B)$, then there exists a unique functor $\gamma : \Phi \times \tilde{\Pi}^r(B) \to \tilde{\Phi}^r$ such that $\gamma(\Phi \times \tilde{\Pi}^r(B)) = R(C)$ and $(\beta, b) \circ \gamma$ is the identity of $\Phi \times \tilde{\Pi}^r(B)$. If $\theta \in \Phi$, $X \in \tilde{\Pi}^r(B)$, $a(\theta) = \alpha X = x$, $b(\theta) = \beta X = y$, then $\gamma(\theta, X)$ is given by

$$(31) \qquad \gamma(\theta, X) = (C(y) \,.\, \hat{\theta})\, X \,.\, C^{-1}(x)\,.$$

For the semi-holonomic case, we take the following agreement up. If $C : B \to \bar{Q}^r(\Phi)$, then $R(C)$ will denote the set of all elements of $\bar{\Phi}^r$ transforming C into itself. Consequently, in the semi-holonomic case γ will mean the restriction of functor (31) to $\Phi \times \bar{\Pi}^r(B)$, so that $\gamma(\Phi \times \bar{\Pi}^r(B)) = R(C) \subset \bar{\Phi}^r$.

Proposition 4. *Assume that three connections $C : B \to \tilde{Q}^r(\Phi)$, $C_0 : B \to Q^1(\Phi)$ and $L : B \to Q^1(\tilde{\Pi}^r(B))$ are given. Consider the connection $C_0 \times L$ on $\Phi \times \tilde{\Pi}^r(B)$ and the functor $\gamma : \Phi \times \tilde{\Pi}^r(B) \to \tilde{\Phi}^r$. Then*

$$\varkappa_{r+1}(C * C_0, L) = \gamma_*(C_0 \times L)\,.$$

Proof. Let $C_0(x) = j^1_x\sigma$, $L(x) = j^1_x\lambda$. According to item 2, it is $C * C_0(x) = j^1_x[C(y) \,.\, \hat{\sigma}(y)]$. By (11) and (8), we deduce $\varkappa_{r+1}(C * C_0(x), L(x)) = j^1_x[(C(y) \,.\, \hat{\sigma}(y))\, \lambda(y) \,.\, C^{-1}(x)]$. On the other hand, (31) gives

$$\gamma_*(j^1_x\sigma, j^1_x\lambda) = j^1_x[(C(y) \,.\, \hat{\sigma}(y))\, \lambda(y) \,.\, C^{-1}(x)],\ \text{QED.}$$

8. In particular, if a connection $C : B \to Q^1(\Phi)$ and a linear connection on the base manifold $L : B \to Q^1(\Pi^1(B))$ are given, then we define the prolongation $p(C, L)$ of C with respect to L by

$$(32) \qquad p(C, L) := \varkappa_2(C * C, L) = \varkappa_2(C', L) : B \to Q^1(\Phi^1)\,.$$

Let $C(x) = j^1_x\varphi$, $L(x) = j^1_x\lambda$. Using (11), we find

$$(33) \qquad p(C, L)\,(x) = j^1_x[(C(y) \,.\, \hat{\varphi}(y))\, \lambda(y) \,.\, C^{-1}(x)]\,.$$

If we further denote $C(y) = j^1_x\psi$, $\lambda(y) = j^1_x\mu$, then (9) implies

$$(34) \qquad p(C, L)\,(x) = j^1_x\{j^1_x[\psi(\mu(t)) \,.\, \varphi(y) \,.\, \varphi^{-1}(t)]\}\,.$$

By Proposition 4, we also have

$$p(C, L) = \gamma_*(C \times L)\,, \tag{35}$$

provided $\gamma : \Phi \times \Pi^1(B) \to \Phi^1$ is the functor determined by C. Considering the functor $j_1^0 : \Phi^1 \to \Phi$, we derive directly from the definition

$$j^0_{1*}(p(C, L)) = C\,. \tag{36}$$

Proposition 5. *Consider a connection $C : B \to Q^1(\tilde{\Phi}^r)$ and a connection $L : B \to \to Q^1(\Pi^1(B))$, so that $p(C, L)$ is a connection on $(\tilde{\Phi}^r)^1 = \tilde{\Phi}^{r+1}$ and $b_*(C)$ or $b_*(p(C, L))$ is a connection on $\tilde{\Pi}^r(B)$ or $\tilde{\Pi}^{r+1}(B)$ respectively. Then*

$$b_*(p(C, L)) = p(b_*(C), L)\,.$$

Proof. This is verified directly by applying projection b to (34).

The r-th prolongation $p^r(C, L)$ of C with respect to L is defined by the iteration $p^r(C, L) = p(p^{r-1}(C, L), L)$, $p^0(C, L) = C$. This is a connection on $(\tilde{\Phi}^{r-1})^1 = \tilde{\Phi}^r$, which is semi-holonomic in the following sense.

Proposition 6. *The values of $p^r(C, L)$ lie in $Q^1(\overline{\Phi}^r) \subset Q^1(\tilde{\Phi}^r)$.*

Proof. The assertion is trivial for $r = 1$. Assume by induction that it holds for $r - 1$. Put $M = p^{r-1}(C, L)$, $M_1 = p^{r-2}(C, L)$, $M_1(x) = j^1_x\varphi$, $M_1(y) = j^1_y\psi$, $L(x) = = j^1_x\lambda$, $L(y) = j^1_y\nu$, $\lambda(y) = j^1_x\mu$. Hence it holds $p^r(C, L) = p(M, L)$, $M = p(M_1, L)$, $M_1 = j^{r-2}_{r-1*}(M)$. According to (33), it is

$$M(x) = j^1_x[(M_1(y) \,.\, \hat{\varphi}(y))\, \lambda(y) \,.\, M_1^{-1}(x)] := j^1_x\, \varrho(y)\,,$$
$$M(y) = j^1_y[(M_1(z) \,.\, \hat{\psi}(z))\, \nu(z) \,.\, M_1^{-1}(y)] := j^1_y\, \sigma(z)\,,$$

so that $j^{r-2}_{r-1}\varrho = \varphi$, $j^{r-2}_{r-1}\sigma = \psi$. Applying (9), we further obtain $\varrho(y) = j^1_x[\psi(\mu(t)) \,\square\, \square\, \varphi(y) \,\square\, \varphi^{-1}(t)]$. On the other hand, (34) gives

$$p(M, L)\,(x) = j^1_x\{j^1_x[\varrho(\mu(t)) \,\square\, \sigma(y) \,\square\, \sigma^{-1}(t)]\} := j^1_x\, \chi(y)\,.$$

We have to prove $\chi(y) \in \overline{\Phi}^r$. By the induction hypothesis, ϱ and σ are local cross sections of $\overline{\Phi}^{r-1}$. Further we find $j^{r-1}_r\, \chi(x) = e^{r-1}_y \,\square\, \varrho(y) \,\square\, e^{r-1}_x = \varrho(y)$ and $j^1_x[j^{r-2}_{r-1}(\sigma(\mu(t)) \,\square\, \varrho(y) \,\square\, \varrho^{-1}(t))] = j^1_x[\psi(\mu(t)) \,\square\, \varphi(y) \,\square\, \varphi^{-1}(t)] = \varrho(y)$, QED.

In particular, if the original connection is linear, i.e. $C = L$, then we set $p^r(L, L) = = p^r(L)$. If necessary, this connection will be called the r-th p-prolongation of L to be distinguished from the r-th prolongation $L^{(r)} : B \to \bar{Q}^{r+1}(\Pi^1(B))$ of L in the sense of Ehresmann. By Proposition 6, $p^r(L)$ is a connection on $\overline{\Pi^1(B)}^r$, but it is also a special one.

Proposition 7. *The values of* $p^r(L)$ *lie in* $Q^1(\overline{\Pi}^{r+1}(B))$.

Proof. We shall first deduce $b_*(p^r(L)) = p^{r-1}(L)$. For $r = 1$, one finds easily $b_*(p(L)) = L = p^0(L)$. Assume by induction that $b_*(p^{r-1}(L)) = p^{r-2}(L)$. Using Proposition 5, we obtain $b_*(p^r(L)) = b_*(p(p^{r-1}(L), L)) = p(b_*(p^{r-1}(L)), L) = p(p^{r-2}(L), L) = p^{r-1}(L)$. Further, $p^{r+1}(L)$ is a connection on $\overline{\Pi^1(B)}^{r+1}$ by construction, so that $b_*(p^{r+1}(L)) = p^r(L)$ is a connection on $\overline{\Pi}^{r+1}(B)$, QED.

9. The prolongations of C and L in the sense of Ehresmann and the prolongations of C with respect to L are in the following relation.

Proposition 8. *Consider two connections* $C : B \to Q^1(\Phi)$, $L : B \to Q^1(\Pi^1(B))$ *and their prolongations* $C^{(r)} : B \to \bar{Q}^{r+1}(\Phi)$, $L^{(r-1)} : B \to \bar{Q}^r(\Pi^1(B))$ *and* $p^r(C, L) : B \to Q^1(\bar{\Phi}^r)$. *Then it holds*

$$\varkappa_{r+1}(C^{(r)}, \varrho_r(L^{(r-1)})) \doteq p^r(C, L)\,. \tag{37}$$

In particular, for $C = L$ *we have*

$$\varrho_{r+1}(L^{(r)}) = p^r(L)\,. \tag{38}$$

Proof. For $r = 1$, it is $\varkappa_2(C', L) = p(C, L)$ by definition. Assume by induction that (37) and (38) hold for $r - 1$. Let $p^{r-1}(C, L) = M$, $M(x) = j^1_x\varphi$, $M(y) = j^1_y\psi$, $L(x) = j^1_x\lambda$, $\lambda(y) = j^1_x\mu$. Then

$$p^r(C, L)\,(x) = p(M, L)\,(x) = j^1_x\{j^1_x[\psi(\mu(t)) \,\Box\, \varphi(y) \,\Box\, \varphi^{-1}(t)]\} := j^1_x\, H(y)\,.$$

Put $N = \varrho_r(L^{(r-1)})$, $N(x) = j^1_x\nu$, $N(y) = j^1_y\eta$, $\nu_1 = j^{r-1}_r\nu$, $\eta_1 = j^{r-1}_r\eta$, $N_1 = j^{r-1}_{r*}(N)$, $C(x) = j^1_x\varrho$, $C(y) = j^1_y\sigma$. By Lemma 5, it is $\varrho_{r-1}(L^{(r-2)}) = N_1$. By the induction hypothesis (37), i.e. $M = \varkappa_r(C^{(r-1)}, \varrho_{r-1}(L^{(r-2)}))$, we obtain

$$\begin{aligned} \varphi(y) &= (C^{(r-2)}(y)\,.\,\hat{\sigma}(y))\,\nu_1(y)\,.\,[C^{(r-2)}(x)]^{-1}\,, \\ \psi(z) &= (C^{(r-2)}(z)\,.\,\hat{\varrho}(z))\,\eta_1(z)\,.\,[C^{(r-2)}(y)]^{-1}\,. \end{aligned} \tag{39}$$

On the other hand, we have

$$\begin{aligned} &\varkappa_{r+1}(C^{(r)}(x), \varrho_r(L^{(r-1)}(x))) = \\ &= j^1_x\{(C^{(r-1)}(y)\,.\,\hat{\sigma}(y))\,\nu(y)\,.\,[C^{(r-1)}(x)]^{-1}\} := j^1_x\,K(y)\,. \end{aligned}$$

By the induction hypothesis (38), i.e. $N = p(N_1, L)$, we find

$$\nu(y) = j^1_x[\eta_1(\mu(t))\,\nu_1(y)\,\nu_1^{-1}(t)]\,.$$

Applying (9), we further deduce

$$\begin{aligned} K(y) = j^1_x\{(C^{(r-2)}(\mu(t))\,.\,\hat{\varrho}(\mu(t))\,.\,\hat{\sigma}(y))\,\eta_1(\mu(t))\,\nu_1(y)\,\nu_1^{-1}(t)\,. \\ .\,\hat{\sigma}^{-1}(t)\,.\,[C^{(r-2)}(t)]^{-1}\} := j^1_x\,V(t)\,. \end{aligned}$$

Finally, using (39), Lemma 1 and Corollary 1, we obtain $\psi(\mu(t)) \square \varphi(y) \square \varphi^{-1}(t) = V(t)$. This implies $H(y) = K(y)$, QED.

10. We shall need another construction related with a connection $L : B \to Q^1(\bar{\Pi}^r(B))$. Such a connection determines a functor $f(L) : \Pi^1(B) \to \bar{\Pi}^{r+1}(B)$ by the following induction. For $r = 1$, consider a connection $L_0 : B \to Q^1(\Pi^1(B))$ and the corresponding functor $\lambda_0 : \Pi^1(B) \times \Pi^1(B) \to (\Pi^1(B))^1 = \tilde{\Pi}^2(B)$ in the sense of (31). Then we set $f(L_0)(X) = \lambda_0(X, X)$, $X \in \Pi^1(B)$. By (31), it is $f(L_0)(X) = (L_0(y) . \hat{X}) X . L_0^{-1}(x)$, $x = \alpha X$, $y = \beta X$, which implies $b[f(L_0)(X)] = X$, $j_2^1(f(L_0)(X)) = X$. Hence $f(L_0)(X) \in \bar{\Pi}^2(B)$ by Lemma 3. Set $L_1 = j_{r*}^{r-1}(L) : B \to Q^1(\bar{\Pi}^{r-1}(B))$. Assume by induction that we have defined the functor $f(L_1) : \Pi^1(B) \to \bar{\Pi}^r(B)$. In the sense of (31), L determines a functor $\lambda : \bar{\Pi}^r(B) \times \Pi^1(B) \to (\bar{\Pi}^r(B))^1$. Then we define

(40) $$f(L)(X) = \lambda(f(L_1)(X), X) = (L(y) . \hat{f}(L_1)(X)) X . L^{-1}(x) .$$

Proposition 9. *The values of $f(L)$ lie in $\bar{\Pi}^{r+1}(B) \subset (\bar{\Pi}^r(B))^1$.*

Proof. Let $L(x) = j_x^1\varphi$, $L(y) = j_y^1\psi$, $\varphi_1 = j_r^{r-1}\varphi$, $\psi_1 = j_r^{r-1}\psi$, $L_2 = j_{r-1*}^{r-2}(L_1)$, $X = j_x^1\mu$. By virtue of (9), we can write

$$f(L)(X) = j_x^1[\psi(\mu(t)) (f(L_1)(X)) \varphi^{-1}(t)] := j_x^1 \sigma(y) .$$

It is $\sigma(x) = f(L_1)(X)$, while

(41) $$j_x^1[j_r^{r-1} \sigma(y)] = j_x^1[\psi_1(\mu(t)) (f(L_2)(X)) \varphi_1^{-1}(t)] .$$

By the induction hypothesis, the right hand side of (41) is equal to $f(L_1)(X)$, QED.

Let $L_0 : B \to Q^1(\Pi^1(B))$ be another connection. Using the functor $f(L) : \Pi^1(B) \to \bar{\Pi}^{r+1}(B)$, one can construct an induced connection $f(L)_* (L_0) : B \to Q^1(\bar{\Pi}^{r+1}(B))$.

Proposition 10. *For a linear connection $L : B \to Q^1(\Pi^1(B))$, we have*

(42) $$p^r(L) = f(p^{r-1}(L))_* (L) .$$

Proof. For $r = 0$, we define $f(p^{-1}(L))$ to be the identity of $\Pi^1(B)$, so that (42) is trivial. Assume by induction that it holds for $r - 1$. Set $M = p^{r-1}(L)$, $L(x) = j_x^1\lambda$, $M(x) = j_x^1\varphi$, $M_1 = j_{r-1*}^{r-2}(M)$. Then $f(M)_* (L(x)) = j_x^1[(M(y) . \hat{f}(M_1)(\lambda(y))) \lambda(y) . M^{-1}(x)]$. On the other hand, it is

(43) $$p(M, L)(x) = j_x^1[(M(y) . \hat{\varphi}(y)) \lambda(y) . M^{-1}(x)] .$$

By the induction hypothesis, $M(x) = f(M_1)_* (L(x)) = j_x^1[f(M_1)(\lambda(y))]$. Hence $\varphi(y) = f(M_1)(\lambda(y))$. Substituting it into (43), we obtain our assertion.

Remark. Dealing with the corresponding principal fibre bundles, we shall deduce in the next paper by Proposition 10 that the r-th p-prolongation $p^r(L)$ of a linear connection L coincides with that studied by YUEN, [12].

11. Assume now that Φ is a groupoid of operators on a fibred manifold (E, p, B). In particular, every $\theta \in \Phi$, $a(\theta) = x$, $b(\theta) = y$ determines a mapping

$$(44) \qquad \tilde{\theta} : E_x \to E_y , \quad z \mapsto \theta . z .$$

The general definition of the absolute differential of a jet of a manifold M into E with respect to an element of connection on Φ introduced by Ehresmann, [3], can be immediately extended to quasi-elements of connection on Φ. Let $Y \in \tilde{J}^r(M, E)$, $p(\beta Y) = y$, and let $X \in {}_y\tilde{A}_x(\Phi)$. Then $X^{-1}pY \in \tilde{J}^r(M, \Phi)$ satisfies $a[X^{-1}pY] = pY$, so that the prolongation of the action of Φ on E determines an element

$$(45) \qquad X^{-1}(Y) := (X^{-1}pY) . Y \in \tilde{J}^r(M, E_x) ,$$

which will be called the absolute differential of Y with respect to X. By definition, one finds easily

$$(46) \qquad (X . \hat{\theta})^{-1} (Y) = \tilde{\theta}^{-1}[X^{-1}(Y)] .$$

In the sequel, we shall use (45) in the special case $Y \in \tilde{J}^r E$ only. In this case, it is $X^{-1}(Y) = X^{-1} . Y$. Let $X = j_y^1 \sigma(z)$ and $Y = j_y^1 \varrho(z)$. Then

$$(47) \qquad X^{-1}(Y) = X^{-1} . Y = j_y^1[\sigma^{-1}(z) . \varrho(z)] \in \tilde{J}_y^r(B, E_x) .$$

Consider the identification of Proposition 1. Let $X \in \tilde{Q}_x^{r+1}(\Phi)$, $Y \in Q_x^1(\tilde{\Pi}^r(B))$, $Z = \varkappa_{r+1}(X, Y) \in Q_x^1(\tilde{\Phi}^r)$ and $X_1 = j_{r+1}^r X \in \tilde{Q}_x^r(\Phi)$. We are going to compare the absolute differentiation with respect to these elements. In the sense of Ehresmann, we have the mappings $X^{-1} : \tilde{J}_x^{r+1}E \to \tilde{J}_x^{r+1}(B, E_x)$, $X_1^{-1} : \tilde{J}_x^r E \to \tilde{J}_x^r(B, E_x)$ and $Z^{-1} : \tilde{J}_x^{r+1}E \to J_x^1(B, \tilde{J}_x^r E)$. Further, let M be a manifold and let $Y = j_x^1 \lambda$. Then Y determines a diffeomorphism

$$(48) \qquad \mu(Y) : \tilde{J}_x^{r+1}(B, M) \to J_x^1(B, \tilde{J}_x^r(B, M)) , \quad j_x^1 \varphi(y) \mapsto j_x^1[\varphi(y) \lambda(y)] .$$

In particular, we have a mapping $\mu(Y) : \tilde{J}_x^{r+1}(B, E_x) \to J_x^1(B, \tilde{J}_x^r(B, E_x))$. Moreover, for any two manifolds M_1, M_2, every mapping $f : M_1 \to M_2$ is extended to a mapping

$$(49) \qquad f_* : J_x^1(B, M_1) \to J_x^1(B, M_2) , \quad j_x^1 \varphi(y) \mapsto j_x^1 f(\varphi(y)) .$$

In particular, for $X_1^{-1} : \tilde{J}_x^r E \to \tilde{J}_x^r(B, E_x)$ we have $X_{1*}^{-1} : J_x^1(B, \tilde{J}_x^r E) \to J_x^1(B, \tilde{J}_x^r(B, E_x))$.

Proposition 11. *The following diagram commutes:*

$$(50)\qquad \begin{array}{ccc} \tilde{J}_x^{r+1}E & \xrightarrow{[\varkappa_{r+1}(X,Y)]^{-1}} & J_x^1(B, \tilde{J}_x^r E) \\ \downarrow X^{-1} & & \downarrow X_{1*}^{-1} \\ \tilde{J}_x^{r+1}(B, E_x) & \xrightarrow{\mu(Y)} & J_x^1(B, \tilde{J}_x^r(B, E_x)) \end{array}$$

Proof. Let $\varkappa_{r+1}(X, Y) = Z = j_x^1\,\zeta(y)$, $b\,\zeta(y) = \lambda(y)$ and let $U \in \tilde{J}_x^{r+1}E$, $U = j_x^1\eta$. Hence we have $X = j_x^1[(\zeta(y) \,.\, X_1)\,\lambda^{-1}(y)]$, $Y = j_x^1\lambda$. Then we find $X^{-1}(U) = j_x^1[(X_1^{-1} \,.\, \zeta^{-1}(y))\,\lambda^{-1}(y) \,.\, \eta(y)]$ and $\mu(Y)\,(X^{-1}(U)) = j_x^1[X_1^{-1} \,.\, \zeta^{-1}(y) \,.\, \eta(y)\,\lambda(y)]$. On the other hand, the action of $\tilde{\Phi}^r$ on $\tilde{J}^r E$ is given by (4), so that $Z^{-1}(U) = j_x^1[(\zeta^{-1}(y)\,\lambda^{-1}(y) \,.\, \eta(y))\,\lambda(y)]$ and $X_{1*}^{-1}(Z^{-1}(U)) = j_x^1[X_1^{-1} \,.\, \zeta^{-1}(y) \,.\, \eta(y)\,\lambda(y)]$, QED.

We define by induction

$$N_x^1(B, M) = J_x^1(B, M)\,, \quad N_x^r(B, M) = J_x^1(B, N_x^{r-1}(B, M))\,.$$

Every $Y \in Q_x^1(\tilde{\Pi}^r(B))$ determines a total map $\tau(Y) : \tilde{J}_x^{r+1}(B, M) \to N_x^{r+1}(B, M)$ as follows. For $r = 0$, we have the trivial groupoid $\Pi^0(B) = B \times B$, for which $Q_x^1(\Pi^0(B))$ contains a single element $I_x = j_x^1(x \mapsto (x, t))$, $t \in B$. Then we set $\tau(I_x) := \mu(I_x) =$ the identity of $J_x^1(B, M)$. Assume by induction that we have introduced a mapping $\tau(V) : \tilde{J}_x^r(B, M) \to N_x^r(B, M)$ for every $V \in Q_x^1(\tilde{\Pi}^{r-1}(B))$. Let $Y \in Q_x^1(\tilde{\Pi}^r(B))$, $Y_1 = j_{r*}^{r-1}(Y) \in Q_x^1(\tilde{\Pi}^{r-1}(B))$. According to (49), $\tau(Y_1)_*$ is a mapping $J_x^1(B, \tilde{J}_x^r(B, M)) \to J_x^1(B, N_x^r(B, M)) = N_x^{r+1}(B, M)$. Then we define

$$(51)\qquad \tau(Y) = \tau(Y_1)_* \circ \mu(Y) : \tilde{J}_x^{r+1}(B, M) \to N_x^{r+1}(B, M)\,.$$

Explicitly, if $Y = j_x^1\lambda$ and $U = j_x^1\eta \in \tilde{J}_x^{r+1}(B, M)$, then

$$(52)\qquad \tau(Y)\,(U) = j_x^1[\tau(Y_1)\,(\eta(y)\,\lambda(y))]\,.$$

Obviously, it holds

$$(53)\qquad \beta(\tau(Y)\,(U)) = \tau(Y_1)\,(j_{r+1}^r U)\,.$$

Similarly, every element of connection $Z \in Q_x^1(\tilde{\Phi}^r)$ determines a mapping $t(Z^{-1}) : \tilde{J}_x^{r+1}E \to N_x^{r+1}(B, E_x)$ as follows. For $r = 0$, we put $t(Z^{-1}) = Z^{-1} : J_x^1 E \to J_x^1(B, E_x)$. Assume by induction that we have defined a mapping $t(V^{-1}) : \tilde{J}_x^r E \to N_x^r(B, E_x)$ for every $V \in Q_x^1(\tilde{\Phi}^{r-1})$. Let $Z \in Q_x^1(\tilde{\Phi}^r)$, $Z_1 = j_{r*}^{r-1}(Z) \in Q_x^1(\tilde{\Phi}^{r-1})$. According to (49), $t(Z_1^{-1})$ is a mapping $J_x^1(B, \tilde{J}_x^r E) \to J_x^1(B, N_x^r(B, E_x))$. Then we set

$$(54)\qquad t(Z^{-1}) = t(Z_1^{-1})_* \circ Z^{-1} : \tilde{J}_x^{r+1}E \to N_x^{r+1}(B, E_x)\,.$$

Explicitly, if $Z = j^1_x\zeta$ and $U = j^1_x\eta \in \bar{J}^{r+1}_x E$, then

(55) $$t(Z^{-1})(U) = j^1_x[t(Z_1^{-1})(\zeta^{-1}(y) \,\square\, \eta(y))]\,.$$

The mapping $t(Z^{-1})$ will be said to be the full absolute differential with respect to Z.

Corollary 2. *The following diagram commutes*:

(56) $$\begin{array}{ccc} \bar{J}^{r+1}_x E & & \\ {\scriptstyle X^{-1}}\downarrow & \searrow^{t([\varkappa_{r+1}(X,\,Y)]^{-1})} & \\ \bar{J}^{r+1}_x(B, E_x) & \xrightarrow{\ \tau(Y)\ } & N^{r+1}_x(B, E_x) \end{array}$$

Proof. The assertion is trivial for $r = 0$. Assume by induction that it holds for $r - 1$. Let $X_1 = j^r_{r+1}X$, $Y_1 = j^{r-1}_{r*}(Y)$. From the induction hypothesis we derive directly $t([\varkappa_r(X_1, Y_1)]^{-1})_* = \tau(Y_1)_* \circ X^{-1}_{1*}$. Adding such a triangle to (50), we prove Corollary 2.

12. Quite analogously to the concept of a semi-holonomic jet, we define a subspace $S^r_x(B, M) \subset N^r_x(B, M)$ by the following induction:

a) $S^1_x(B, M) = N^1_x(B, M) = J^1_x(B, M)$,

b) an element $X \in N^r_x(B, M)$ belongs to $S^r_x(B, M)$ if it is of the form $X = j^1_x\sigma$, where σ is a local mapping of B into $S^{r-1}_x(B, M)$ satisfying $\sigma(x) = j^1_x[\beta\,\sigma(y)]$.

In particular, if M is a vector space V, then we have a canonical identification

(57) $$S^r_x(B, V) = V \oplus V \otimes T^*_x(B) \oplus \ldots \oplus V \otimes \overset{r}{\otimes}\, T^*_x(B)\,.$$

Lemma 6. *If* $U \in \bar{J}^{r+1}_x(B, M)$ *and* $Y \in Q^1_x(\Pi^r(B))$, *then*

$$\tau(Y)(U) \in S^{r+1}_x(B, M)\,.$$

Proof. The assertion is trivial for $r = 0$. Assume by induction that it holds for $r - 1$. Let $U = j^1_x v$, $Y = j^1_x\lambda$, $Y_1 = j^{r-1}_{r*}(Y)$, $v_1 = j^{r-1}_r v$, $\lambda_1 = j^{r-1}_r\lambda$. By definition, it is $\tau(Y)(U) = j^1_x[\tau(Y_1)(v(y)\,\lambda(y))] := j^1_x\,\sigma(y)$. By the induction hypothesis we find that $\sigma(y)$ belongs to $S^r_x(B, M)$. Further, as U is semi-holonomic, it holds $v(x) = j^1_x v_1$. Hence $\sigma(x) = \tau(Y_1)(v(x)) = j^1_x[\tau(j^{r-2}_{r-1*}(Y_1))(v_1(y)\,\lambda_1(y))]$ by the definition of $\tau(Y_1)$. On the other hand, $\beta\,\sigma(y) = \tau(j^{r-2}_{r-1*}(Y_1))(v_1(y)\,\lambda_1(y))$ by (53), which implies $\tau(Y)(U) \in S^{r+1}_x(B, M)$, QED.

Lemma 7. *If* $Z \in Q^1_x(\Phi^r)$ *and* $U \in \bar{J}^{r+1}_x E$, *then*

$$t(Z^{-1})(U) \in S^{r+1}_x(B, E_x)\,.$$

Proof is a simple replica of the proof of Lemma 6.

Introduce

$$S^r(E) = \bigcup_{x \in B} S^r_x(B, E_x), \quad \bar{F}^r(E) = \bigcup_{x \in B} \bar{J}^r_x(B, E_x).$$

Consider two connections $C : B \to Q^1(\Phi)$, $L : B \to Q^1(\Pi^1(B))$, their prolongations $C^{(r)} : B \to \bar{Q}^{r+1}(\Phi)$, $L^{(r-1)} : B \to \bar{Q}^r(\Pi^1(B))$ in the sense of Ehresmann and the r-th prolongation $p^r(C, L)$ of C with respect to L. By (56) and Proposition 8, the following diagram commutes:

$$\tag{58} \begin{array}{ccc} J^{r+1}E & & \\ {\scriptstyle [C^{(r)}]^{-1}} \big\downarrow & \searrow^{t([p^r(C, L)]^{-1})} & \\ \bar{F}^{r+1}(E) & \xrightarrow{\tau(\varrho_r(L^{(r-1)}))} & S^{r+1}(E) \end{array}$$

This diagram compares the absolute differentiation by means of $C^{(r)}$ in the sense of Ehresmann with the full absolute differentiation by means of $p^r(C, L)$.

13. There is a natural action of groupoid $\Phi \times \Pi^1(B)$ on $S^r(E)$ defined by the following induction. Let $\theta \in \Phi$, $X \in \Pi^1(B)$, $a(\theta) = \alpha X = x$, $b(\theta) = \beta X = y$, $X = j^1_x\mu$. If $r = 1$ and $Y \in J^1_x(B, E_x)$, $Y = j^1_x\eta$, then we set

$$\tag{59} (\theta, X) \bullet Y = j^1_y[\theta \,.\, \eta(\mu^{-1}(t))] \in J^1_y(B, E_y).$$

Assume by induction that we have introduced an action $((\theta, X), U) \mapsto (\theta, X) \bullet U$ of $\Phi \times \Pi^1(B)$ on $S^{r-1}(E)$. If $Y \in S^r_x(B, E_x)$, $Y = j^1_x\eta$, then we define

$$\tag{60} (\theta, X) \bullet Y = j^1_y[(\theta, X) \bullet \eta(\mu^{-1}(t))] \in S^r_y(B, E_y).$$

Let $\bar{C} : B \to Q^1(\bar{\Phi}^r)$ be a connection on $\bar{\Phi}^r$ and let $\sigma : B \to E$ be a cross section. Then $j^{r+1}\sigma$ is a cross section of $J^{r+1}E$ and the cross section

$$\tag{61} t(\bar{C}^{-1})(\sigma) : B \to S^{r+1}(E), \quad x \mapsto t(\bar{C}^{-1}(x))(j^{r+1}_x\sigma)$$

will be said to be the full absolute differential of σ with respect to $\bar{C}$. Consider further two connections $C : B \to Q^1(\Phi)$, $L : B \to Q^1(\Pi^1(B))$ and the r-th prolongation $p^r(C, L)$ of C with respect to L. Using the action (60), we shall describe a simple step by step construction of the full absolute differential of σ with respect to $p^r(C, L)$ by means of C and L only.

Proposition 12. *The full absolute differential of a cross section $\sigma : B \to E$ with respect to $p^r(C, L)$ coincides with the absolute differential with respect to $C \times L$ of the full absolute differential of σ with respect to $p^{r-1}(C, L)$, i.e.*

$$(C \times L)^{-1}\,(t([p^{r-1}(C, L)]^{-1})(\sigma)) = t([p^r(C, L)]^{-1})(\sigma).$$

Corollary 3. *Let E be a vector bundle associated with Φ, let $C : B \to Q^1(\Phi)$, $L : B \to Q^1(\Pi^1(B))$ be two connections and let $\sigma : B \to E$ be a cross section. Then the full absolute differential $t([p^{r-1}(C, L)]^{-1})(\sigma) : B \to S^r(E) = E \oplus E \otimes \otimes T^*(B) \oplus \ldots \oplus E \otimes \overset{r}{\otimes} T^*(B)$ of σ with respect to $p^{r-1}(C, L)$ is of the form $(\sigma, \nabla\sigma, \ldots, \nabla^r\sigma)$.*

For $r = 2$, Corollary 3 shows that in the vector bundle case our prolongation of a connection with respect to a linear connection on the base manifold coincides with the operation treated by Pohl, [8].

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NEWS and NOTICES

SIXTIETH ANNIVERSARY OF THE BIRTHDAY OF ACADEMICIAN ŠTEFAN SCHWARZ

JÁN JAKUBÍK, Košice, MILAN KOLIBIAR, Bratislava

An outstanding czechoslovak mathematician, Academician ŠTEFAN SCHWARZ, reaches sixty years of age on May 18, 1974.

He was born at Nové Mesto nad Váhom where he attended the secondary school. Already then he showed his deep interest in mathematics. In the years 1932–1936 he studied at the Faculty of Sciences of the Charles University at Prague. Immediately after completing his studies he became assistent at the Faculty, remaining member of its staff until 1939. Then he came to the just established Slovak Technical University at Bratislava where he has worked until now. In the year 1946 he became reader and in 1947 he was appointed professor.

In the year 1952, Š. Schwarz was elected corresponding member of the Czechoslovak Academy of Sciences. When the Slovak Academy of Sciences was founded in 1953, he was appointed its regular member, and in 1960 he was elected regular member of the Czechoslovak Academy of Sciences. In both Czechoslovak and Slovak Academies he acted in a number of important offices. In the years 1965–70 he was President of the Slovak Academy of Sciences and Vice-President of the Czechoslovak Academy of Sciences. Since 1964 he has been director of the Institute of Mathematics of the Slovak Academy of Sciences.

The scientific activity of Academician Štefan Schwarz concentrates on algebra and theory of numbers. It was Schwarz's teacher, an outstanding mathematician KAREL PETR, professor of the Charles University, who encouraged Š. Schwarz to direct his work to algebraic problems connected with the theory of numbers.

In his first studies Š. Schwarz dealt with the problems concerning irreducibility of polynomials over an integral domain. His further papers are devoted to the decomposability of polynomials over a finite field into a product of irreducible polynomials. One of the main tasks advanced by Š. Schwarz was to find explicit formulae for the number σ_k of mutually different irreducible factors of degree k for a given polynomial over a finite field. He first improved some older results and finally found an essentially new solution of a, so to say, "algorithmic" character, which turned out to be suitable for machine computation. In other papers which obtained also considerable

Corollary 3. *Let E be a vector bundle associated with Φ, let $C : B \to Q^1(\Phi)$, $L : B \to Q^1(\Pi^1(B))$ be two connections and let $\sigma : B \to E$ be a cross section. Then the full absolute differential* $t([p^{r-1}(C, L)]^{-1})(\sigma) : B \to S^r(E) = E \oplus E \otimes \otimes T^*(B) \oplus \ldots \oplus E \otimes \overset{r}{\otimes} T^*(B)$ *of σ with respect to $p^{r-1}(C, L)$ is of the form* $(\sigma, \nabla\sigma, \ldots, \nabla^r\sigma)$.

For $r = 2$, Corollary 3 shows that in the vector bundle case our prolongation of a connection with respect to a linear connection on the base manifold coincides with the operation treated by Pohl, [8].

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