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CIRCULANT BOOLEAN RELATION MATRICES

(A note to the foregoing paper of K. K. Hang Butler and J. R. Krabill)

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The purpose of this note is to give a new proof of Theorem 2 of the foregoing paper [1] and to modify its statement in a way which seems to be more adequate.

We briefly recall some necessary preliminaries and notations. All matrices in this note are $n \times n$ Boolean relation matrices with the usual addition and multiplication. We denote $E = \text{diag}\,[1, 1, \ldots, 1]$. Further, J denotes the $n \times n$ matrix in which all patterns are ones.

If $A = (a_{ik})$, $B = (b_{ik})$, we shall write $A \leqq B$ if $a_{ik} = 1$ implies $b_{ik} = 1$. If A is any $n \times n$ matrix it is known (see e.g. [2]) that $A^t \leqq A + A^2 + \ldots + A^n$ for any $t > 0$.

A matrix A is called irreducible if $A + A^2 + \ldots + A^n = J$. It is called primitive if there is an integer $p > 0$ such that $A^p = J$. A primitive matrix is irreducible. The converse need not be true. Nevertheless, if $E \leqq A$, then A is primitive iff A is irreducible. Indeed, $E \leqq A$ implies $E \leqq A \leqq A^2 \leqq \ldots \leqq A^n$, hence $A + A^2 + \ldots + A^n = A^n$ and $A + A^2 + \ldots + A^n = J$ iff $A^n = J$.

Let P be the $n \times n$ permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Then $P^n = E$, and every circulant (Boolean relation) matrix can be written in the form

$$C = c_0 E + c_1 P + c_2 P^2 + \ldots + c_{n-1} P^{n-1}\,.$$

Omitting those c_i which are zeros and defining $P^0 = E$, we have

$$C = P^{i_1} + P^{i_2} + \ldots + P^{i_l}\,, \tag{1}$$

where $0 \leqq i_1 < i_2 < i_3 < \ldots < i_l \leqq n - 1$. Suppose $l > 1$.

The problem treated in [1] can be formulated as follows. We have to find necessary and sufficient conditions under which C is primitive. We prove:

Theorem. *The circulant Boolean relation matrix* (1) *is primitive iff*

$$\text{g.c.d.}\ (i_2 - i_1, i_3 - i_1, \ldots, i_l - i_1, n) = 1\,.$$

Proof. Write

$$C = P^{i_1}[E + P^{i_2 - i_1} + \ldots + P^{i_l - i_1}] = P^{i_1} \,.\, T,$$

where T has the obvious meaning. We have $C^p = P^{pi_1} \,.\, T^p$. Since the permutation matrix P^{pi_1} rearranges only the rows and columns in T^p, we conclude that $C^p = J$ holds iff $T^p = J$ holds.

Since $E \leqq T$, T is primitive iff it is irreducible, i.e. iff

$$T + T^2 + \ldots + T^n = J\,. \tag{2}$$

It is advantageous to write instead of (2) $\sum_{j=1}^{N} T^j = J$ for any integer $N \geqq n$. Hence T is primitive iff for any $N \geqq n$ we have

$$\sum_{j=1}^{N} (E + P^{i_2 - i_1} + \ldots + P^{i_l - i_1})^j = J\,. \tag{3}$$

Note that $E + P + P^2 + \ldots + P^{n-1} = J$ and each summand on the left hand side is essential, i.e., omitting any P^i $(0 \leqq i \leqq n - 1)$ the sum becomes $\neq J$.

Multiply term by term the products $(E + P^{i_2 - i_1} + \ldots + P^{i_l - i_1})^j$. Using the idempotency of addition (i.e. $P^l + P^l = P^l$) and $P^n = E$, the left hand side of (3) finally becomes a sum of distinct powers of P. Now (3) holds iff the left hand side of (3) contains as a summand every power P^j for $j = 0, 1, \ldots, n - 1$. Since this expression certainly contains E, we can state that (3) holds iff to any integer $k = 1, 2, \ldots, n - 1$ there exist nonnegative integers $x_{2k}, x_{3k}, \ldots, x_{lk}$ such that

$$x_{2k}(i_2 - i_1) + x_{3k}(i_3 - i_1) + \ldots + x_{lk}(i_l - i_1) \equiv k \pmod{n}\,.$$

Now the congruence

$$x_2(i_2 - i_1) + x_3(i_3 - i_1) + \ldots + x_l(i_l - i_1) \equiv 1 \pmod{n}$$

has a solution $x_{21}, x_{31}, \ldots, x_{l1}$ iff g.c.d. $(i_2 - i_1, i_3 - i_1, \ldots, i_l - i_1, n) = 1$. On the other hand if this condition is satisfied, then for any $k = 2, 3, \ldots, n - 1$ the congruence

$$y_2(i_2 - i_1) + y_3(i_3 - i_1) + \ldots + y_l(i_l - i_1) \equiv k \pmod{n}$$

has a solution $y_{2k}, y_{3k}, \ldots, y_{lk}$. [It is sufficient to put $y_{2k} = kx_{21}, \ldots, y_{lk} = kx_{l1}$.] This proves our statement.

References

[1] *K. K. Hang Butler* and *J. R. Krabill:* Circulant Boolean relation matrices. Czech. Math. J. *24* (1974), 247—251.

[2] *Št. Schwarz:* On the semigroup of binary relations on a finite set. Czech. Math. J. *20* (1970), 632—679.

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