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# ON A GROUP OF HOLOMORPHIC TRANSFORMATIONS IN $\mathscr{C}^2$

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**0.** Consider the space $\mathscr{C}^2$ with the complex coordinates $(x, y)$ and let $V$ be the layer of real hypersurfaces given by

$$(0.1) \qquad i(y - \bar{y}) + (x - \bar{x})^2 = r\,, \quad r \in \mathscr{R}\,.$$

Each hypersurface of $V$ has a non-degenerate Levi form at each its point. The Lie group

$$(0.2) \qquad X = \alpha x + \beta\,, \quad Y = 2i\alpha(\beta - \bar{\beta})\,x + i\alpha(\alpha - \bar{\alpha})\,x^2 + \alpha\bar{\alpha}y + \gamma\,;$$
$$\alpha, \beta, \gamma \in \mathscr{C}\,;$$

of the biholomorphic mappings of $\mathscr{C}^2$ preserves $V$, the hypersurface (0.1) with the parameter $r$ being transformed into the hypersurface (0.1) with the parameter

$$(0.3) \qquad r' = \frac{1}{\alpha\bar{\alpha}}\,\{r + i(\bar{\gamma} - \gamma) - (\beta - \bar{\beta})^2\}\,.$$

Obviously, $\dim_{\mathscr{R}} G = 6$. We are going to prove the following

**Theorem.** *Let $V$ be a layer of real hypersurfaces in $\mathscr{C}^2$ such that each hypersurface of $V$ has a non-degenerate Levi form. Let $G$ be a Lie group of biholomorphic transformations of $\mathscr{C}^2$ which is transitive on $\mathscr{C}^2$ and preserves the layer $V$. Then* $4 \leqq \dim G \leqq 6$. *In the case* $\dim G = 6$ *there are, in $\mathscr{C}^2$, holomorphic coordinates $(x, y)$ such that $G$ is given by* (0.2) *and $V$ by* (0.1).

**1.** Be given a differentiable manifold $M^{2n}$ and an almost complex structure $J$ over it; all manifolds and maps are supposed to be of class $C^\infty$. The torsion of $J$ is defined as the vector 2-form $[J, J]$ given by

$$(1.1) \qquad \tfrac{1}{2}[J, J]\,(u, v) = [Ju, Jv] - J[Ju, v] - J[u, Jv] - [u, v]\,.$$

On $M^{2n}$, let us choose vector fields $v_\alpha, v_{n+\alpha}; \alpha = 1, \ldots, n$; such that

(1.2) $$Jv_\alpha = v_{n+\alpha}, \quad Jv_{n+\alpha} = -v_\alpha; \quad \alpha = 1, \ldots, n;$$

and write

(1.3) $$\begin{aligned} [v_\alpha, v_\beta] &= a^\gamma_{\alpha\beta} v_\gamma + a^{n+\gamma}_{\alpha\beta} v_{n+\gamma}, \\ [v_\alpha, v_{n+\beta}] &= a^\gamma_{\alpha,n+\beta} v_\gamma + a^{n+\gamma}_{\alpha,n+\beta} v_{n+\gamma}, \\ [v_{n+\alpha}, v_{n+\beta}] &= a^\gamma_{n+\alpha,n+\beta} v_\gamma + a^{n+\gamma}_{n+\alpha,n+\beta} v_{n+\gamma}. \end{aligned}$$

For

(1.4) $$u = x^\alpha v_\alpha - x^{n+\alpha} v_{n+\alpha}, \quad v = y^\alpha v_\alpha - y^{n+\alpha} v_{n+\alpha},$$

we obtain

(1.5) $$\begin{aligned} &\tfrac{1}{2}[J, J](u, v) = \\ &= \left(a^\gamma_{n+\alpha,n+\beta} - a^\gamma_{\alpha\beta} + a^{n+\gamma}_{\alpha,n+\beta} - a^{n+\gamma}_{\beta,n+\alpha}\right) . \\ &. \left\{\left(x^\alpha y^\beta - x^{n+\alpha} y^{n+\beta}\right) v_\gamma + \left(x^\alpha y^{n+\beta} + x^{n+\alpha} y^\beta\right) v_{n+\gamma}\right\} + \\ &+ \left(a^{n+\gamma}_{\alpha\beta} - a^{n+\gamma}_{n+\alpha,n+\beta} + a^\gamma_{\alpha,n+\beta} - a^\gamma_{\beta,n+\alpha}\right) . \\ &. \left\{\left(x^\alpha y^{n+\beta} + x^{n+\alpha} y^\beta\right) v_\gamma - \left(x^\alpha y^\beta - x^{n+\alpha} y^{n+\beta}\right) v_{n+\gamma}\right\} . \end{aligned}$$

The condition $[J, J] = 0$ is thus equivalent to

(1.6) $$\begin{aligned} a^\gamma_{n+\alpha,n+\beta} - a^\gamma_{\alpha\beta} + a^{n+\gamma}_{\alpha,n+\beta} - a^{n+\gamma}_{\beta,n+\alpha} = 0, \\ a^{n+\gamma}_{\alpha\beta} - a^{n+\gamma}_{n+\alpha,n+\beta} + a^\gamma_{\alpha,n+\beta} - a^\gamma_{\beta,n+\alpha} = 0; \quad \alpha, \beta, \gamma = 1, \ldots, n. \end{aligned}$$

The following result is classic: Be given a manifold $M^{2n}$, the almost complex structure $J$ over $M^{2n}$ be given by means of the vector fields $v_\alpha, v_{n+\alpha}$ and (1.2); the structure $J$ is complex if and only if (1.6).

**2.** Consider a manifold $M^4$, a complex structure $J$ over $M^4$, and let $V$ be a layer of hypersurfaces in $M^4$. At each point $m \in M^4$, let us choose vectors $v_1, \ldots, v_4 \in T_m(M^4)$ such that: (i) $v_1, v_2, v_3$ are tangent to the hypersurface of $V$ going through $m$, (ii) $Jv_1 = v_3$, $Jv_2 = v_4$. (iii) the vector fields $v_1, \ldots, v_4$ are of class $C^\infty$. The vector fields $w_1, \ldots, w_4$ satisfying (i)–(iii) as well, there are real-valued functions $\alpha, \beta, \gamma, \varphi, \delta$ on $M^4$ such that

(2.1) $$\begin{aligned} &v_1 = \alpha w_1 - \beta w_3, \quad v_2 = \gamma w_1 + \varphi w_2 - \delta w_3, \\ &v_3 = \beta w_1 + \alpha w_3, \quad v_4 = \delta w_1 + \gamma w_3 + \varphi w_4; \quad (\alpha^2 + \beta^2)\, \varphi \neq 0. \end{aligned}$$

The complex structure $J$ together with the layer $V$ induce a $G$-structure $B_G$ on $M^4$, the group $G$ being the set of non-singular matrices of the type

(2.2)
$$\begin{pmatrix} \alpha & 0 & -\beta & 0 \\ \gamma & \varphi & -\delta & 0 \\ \beta & 0 & \alpha & 0 \\ \delta & 0 & \gamma & \varphi \end{pmatrix}.$$

Let us write

(2.3)
$$\begin{aligned}
[v_1, v_2] &= a_1 v_1 + a_2 v_2 + a_3 v_3 , \\
[v_1, v_3] &= b_1 v_1 + b_2 v_2 + b_3 v_3 , \\
[v_1, v_4] &= c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 , \\
[v_2, v_3] &= d_1 v_1 + d_2 v_2 + d_3 v_3 , \\
[v_2, v_4] &= e_1 v_1 + e_2 v_2 + e_3 v_3 + e_4 v_4 , \\
[v_3, v_4] &= f_1 v_1 + f_2 v_2 + f_3 v_3 + f_4 v_4 ;
\end{aligned}$$

the conditions (1.6) reduce to

(2.4)
$$f_1 - a_1 + c_3 - d_3 = 0 , \quad f_2 - a_2 + c_4 = 0 , \quad a_3 - f_3 + c_1 - d_1 = 0 ,$$
$$f_4 - c_2 + d_2 = 0 .$$

Analoguously, let us write

(2.5)
$$\begin{aligned}
[w_1, w_2] &= A_1 w_1 + A_2 w_2 + A_3 w_3 , \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
[w_3, w_4] &= F_1 w_1 + F_2 w_2 + F_3 w_3 + F_4 w_4 ,
\end{aligned}$$

(2.6)
$$F_1 - A_1 + C_3 - D_3 = 0 , \quad F_2 - A_2 + C_4 = 0 ,$$
$$A_3 - F_3 + C_1 - D_1 = 0 , \quad F_4 - C_2 + D_2 = 0 .$$

From $(2.3_2)$, $(2.5_2)$ and (2.1), we get

$$[v_1, v_3] = [\alpha w_1 - \beta w_3, \beta w_1 + \alpha w_3] = (\cdot) w_1 + (\cdot) w_3 + (\alpha^2 + \beta^2) B_2 w_2 =$$
$$= (\cdot) w_1 + (\cdot) w_3 + b_2 \varphi w_2 ,$$

i.e.,

(2.7)
$$(\alpha^2 + \beta^2) B_2 = \varphi b_2 .$$

It is easy to see that $b_2 \neq 0$ because of the non-degeneracy of the Levi forms of the hypersurfaces of $V$. Thus we are in the position to choose the frames $(v_1, \dots, v_4)$ of the $G$-structure $B_G$ in such a way that

(2.8)
$$b_2 = 1 ;$$

from $B_2 = b_2 = 1$, we get

(2.9) $$\varphi = \alpha^2 + \beta^2 .$$

Further,

$$[v_1, v_4] = [\alpha w_1 - \beta w_3, \delta w_1 + \gamma w_3 + \varphi w_4] =$$
$$= (\cdot)w_1 + (\cdot)w_3 + (\cdot)w_4 + (\alpha\gamma + \beta\delta + \alpha\varphi C_2 - \beta\varphi F_2)\, w_2 =$$
$$= (\cdot)w_1 + (\cdot)w_3 + (\cdot)w_4 + c_2\varphi w_2$$
$$[v_3, v_4] = [\beta w_1 + \alpha w_3, \delta w_1 + \gamma w_3 + \varphi w_4] =$$
$$= (\cdot)w_1 + (\cdot)w_3 + (\cdot)w_4 + (\beta\gamma - \alpha\delta + \beta\varphi C_2 + \alpha\varphi F_2)\, w_2 =$$
$$= (\cdot)w_1 + (\cdot)w_3 + (\cdot)w_4 + f_2\varphi w_2 ,$$

i.e.,

(2.10) $$\alpha\gamma + \beta\delta + \alpha\varphi C_2 - \beta\varphi F_2 = \varphi c_2 ,$$
$$\beta\gamma - \alpha\delta + \beta\varphi C_2 + \alpha\varphi F_2 = \varphi f_2 .$$

The frames of $B_G$ may be chosen in such a way that (2.8) and

(2.11) $$c_2 = f_2 = 0 ,$$

and we get

(2.12) $$\gamma = \delta = 0 .$$

Further,

$$[v_2, v_4] = [\varphi w_2, \varphi w_4] =$$
$$= \varphi w_2\varphi \,.\, w_4 - \varphi w_4\varphi \,.\, w_2 + \varphi^2(E_1w_1 + E_2w_2 + E_3w_3 + E_4w_4) =$$
$$= e_1(\alpha w_1 - \beta w_3) + e_2\varphi w_2 + e_3(\beta w_1 + \alpha w_3) + e_4\varphi w_4 ,$$

i.e.,

(2.13) $$\varphi^2 E_1 = \alpha e_1 + \beta e_3 , \ \varphi^2 E_3 = -\beta e_1 + \alpha e_3$$

and

(2.14) $$\varphi^2(E_1w_1 + E_3w_3) = e_1v_1 + e_3v_3 .$$

The direction of the vector $e_1v_1 + e_3v_3$ is thus invariant. Suppose that $e_1v_1 + e_3v_3 \neq 0$. The frames $(v_1, \ldots, v_4) \in B_G$ may be chosen in such a way that

(2.15) $$e_1 = 1 , \quad e_3 = 0 .$$

This means

(2.16) $$\alpha = 1 , \quad \beta = 0 , \quad \varphi = 1 ,$$

and the induced structure $B_G$ is reduced to the $\{e\}$-structure $B_{\{e\}}$. Denote by $G(V)$ the group of biholomorphic transformations of $\mathscr{C}^2$ preserving $V$. Obviously, $G(V)$ preserves the induced $G$-structure $B_G$ and the reduced structure $B_{\{e\}}$. In our case $\dim G(V) \leqq 4$.

If dim $G(V) > 4$, we should have

(2.17) $$e_1 = e_3 = 0,$$

i.e.,

(2.18)
$$\begin{aligned}
[v_1, v_2] &= a_1 v_1 + a_2 v_2 + a_3 v_3, \\
[v_1, v_3] &= b_1 v_1 + v_2 + b_3 v_3, \\
[v_1, v_4] &= c_1 v_1 + c_3 v_3 + a_2 v_4, \\
[v_2, v_3] &= d_1 v_1 + d_2 v_2 + d_3 v_3, \\
[v_2, v_4] &= e_2 v_2 + e_4 v_4, \\
[v_3, v_4] &= f_1 v_1 + f_3 v_3 - d_2 v_4,
\end{aligned}$$

(2.19) $$f_1 - a_1 + c_3 - d_3 = 0, \quad a_3 - f_3 + c_1 - d_1 = 0;$$

the admissible transformations of the frames are given by

(2.20)
$$\begin{aligned}
&v_1 = \alpha w_1 - \beta w_3, \quad v_2 = \varphi w_2, \\
&v_3 = \beta w_1 + \alpha w_3, \quad v_4 = \varphi w_4; \quad \varphi = \alpha^2 + \beta^2.
\end{aligned}$$

From the Jacobi identities

(2.21) $$[v_i, [v_j, v_k]] + [v_j, [v_k, v_i]] + [v_k, [v_i, v_j]] = 0; \quad i, j, k = 1, 2, 3, 4;$$

it follows

(2.22)
$$\begin{aligned}
&v_1 d_1 - v_2 b_1 + v_3 a_1 + (d_2 + b_1) a_1 + (d_3 - a_1) b_1 - (b_3 + a_2) d_1 = 0, \\
&v_1 d_2 + v_3 a_2 + (d_2 + b_1) a_2 + d_3 - a_1 - (b_3 + a_2) d_2 = 0, \\
&v_1 d_3 - v_2 b_3 + v_3 a_3 + (d_2 + b_1) a_3 + (d_3 - a_1) b_3 - (b_3 + a_2) d_3 = 0, \\
&-v_2 c_1 + v_4 a_1 + (e_2 + c_1) a_1 + (e_4 - a_1) c_1 - c_3 d_1 - a_3 f_1 = 0, \\
&v_1 e_2 + v_4 a_2 + (e_2 + c_1) a_2 - c_3 d_2 - 2 a_2 e_2 = 0, \\
&-v_2 c_3 + v_4 a_3 + (e_2 + c_1) a_3 + (e_4 - a_1) c_3 - c_3 d_3 - a_3 f_3 = 0, \\
&v_1 e_4 - v_4 a_2 + (e_4 - a_1) a_2 - 2 a_2 e_4 + a_3 d_2 = 0, \\
&v_1 f_1 - v_3 c_1 + v_4 a_1 + (f_3 + c_1) b_1 - (d_2 + a_1) c_1 - (a_2 + a_3) f_1 = 0, \\
&v_4 a_2 + f_3 + c_1 - a_2 e_2 = 0, \\
&v_1 f_3 - v_3 c_3 + v_4 a_3 + (f_3 + c_1) b_3 - (d_2 + a_1) c_3 - (a_2 + a_3) f_3 = 0, \\
&-v_1 d_2 - v_3 a_2 - (d_2 + a_1) a_2 + (a_2 + a_3) d_2 - a_2 e_4 = 0, \\
&v_2 f_1 + v_4 d_1 - f_1 a_1 + (f_3 + e_2) d_1 - (e_4 + d_3) f_1 - d_1 c_1 = 0, \\
&-v_3 e_2 + v_4 d_2 - f_1 a_2 + (f_3 + e_2) d_2 - 2 d_2 e_2 = 0, \\
&v_2 f_3 + v_4 d_3 - f_1 a_3 + (f_3 + e_2) d_3 - (e_4 + d_3) f_3 - d_1 c_3 = 0, \\
&-v_2 d_2 - v_3 e_4 - 2 d_2 e_4 + (e_4 + d_3) d_2 - d_1 a_2 = 0.
\end{aligned}$$

From (2.18), the analoguous equations for $[w_i, w_j]$ and from (2.20), we get

$$\begin{aligned} -\varphi w_2\alpha + \alpha\varphi A_1 + \beta\varphi D_1 &= \alpha a_1 + \beta a_3\,, \\ \varphi w_2\alpha - \beta\varphi A_3 + \alpha\varphi D_3 &= -\beta d_1 + \alpha d_3\,, \\ \varphi w_2\beta + \alpha\varphi A_3 + \beta\varphi D_3 &= -\beta a_1 + \alpha a_3\,, \\ \varphi w_2\beta - \beta\varphi A_1 + \alpha\varphi D_1 &= \alpha d_1 + \beta d_3\,, \\ -\varphi w_4\alpha + \alpha\varphi C_1 - \beta\varphi F_1 &= \alpha c_1 + \beta c_3\,, \\ -\varphi w_4\alpha + \beta\varphi C_3 + \alpha\varphi F_3 &= -\beta f_1 + \alpha f_3\,, \\ \varphi w_4\beta + \alpha\varphi C_3 - \beta\varphi F_3 &= -\beta c_1 + \alpha c_3\,, \\ -\varphi w_4\beta + \beta\varphi C_1 + \alpha\varphi F_1 &= \alpha f_1 + \beta f_3\,, \end{aligned} \tag{2.23}$$

$$\begin{aligned} \alpha w_1\varphi - \beta w_3\varphi + \alpha\varphi A_2 + \beta\varphi D_2 &= \varphi a_2\,, \\ -\beta w_1\varphi - \alpha w_3\varphi - \beta\varphi A_2 + \alpha\varphi D_2 &= \varphi d_2\,, \end{aligned} \tag{2.24}$$

$$w_2\varphi + \varphi E_4 = e_4\,, \quad -w_4\varphi + \varphi E_2 = e_2\,, \tag{2.25}$$

$$\begin{aligned} \alpha w_1\beta - \beta w_3\beta - \beta w_1\alpha - \alpha w_3\alpha + \varphi B_1 &= \alpha b_1 + \beta b_3\,, \\ \alpha w_1\alpha - \beta w_3\alpha + \beta w_1\beta + \alpha w_3\beta + \varphi B_3 &= -\beta b_1 + \alpha b_3\,. \end{aligned} \tag{2.26}$$

From $(2.23_{1,2}) + (2.23_{3,4})$ and $(2.23_{5,6}) + (2.23_{7,8})$, we get

$$\begin{aligned} \alpha\varphi(A_1 + D_3) + \beta\varphi(D_1 - A_3) &= \alpha(a_1 + d_3) - \beta(d_1 - a_3)\,, \\ \alpha\varphi(D_1 - A_3) - \beta\varphi(A_1 + D_3) &= \alpha(d_1 - a_3) + \beta(a_1 + d_3)\,, \end{aligned} \tag{2.27}$$

$$\begin{aligned} \alpha\varphi(C_1 - F_3) - \beta\varphi(F_1 + C_3) &= \alpha(c_1 - f_3) + \beta(f_1 + c_3)\,, \\ \alpha\varphi(F_1 + C_3) + \beta\varphi(C_1 - F_3) &= \alpha(f_1 + c_3) - \beta(c_1 - f_3)\,. \end{aligned} \tag{2.28}$$

The equations (2.28) are the consequence of (2.27) because of (2.19). From (2.27),

$$\varphi\{(A_1 + D_3)^2 + (D_1 - A_3)^2\} = (a_1 + d_3)^2 + (d_1 - a_3)^2\,. \tag{2.29}$$

Suppose

$$(a_1 + d_3)^2 + (d_1 - a_3)^2 \neq 0\,. \tag{2.30}$$

Thus we may choose the frames of $B_G$ in such a way that

$$(a_1 + d_3)^2 + (d_1 - a_3)^2 = 1\,, \tag{2.31}$$

i.e.,

$$\varphi = \alpha^2 + \beta^2 = 1\,. \tag{2.32}$$

We have dim $G(V) \leqq 5$ because the system (2.23)–(2.26) is, in the best case, completely integrable.

Suppose

$$(2.33) \qquad d_3 = -a_1 , \quad d_1 = a_3 ;$$

from (2.19), we obtain

$$(2.34) \qquad f_1 = -c_3 , \quad f_3 = c_1 .$$

From $(2.23_{1,3}) + (2.25_1)$ and $(2.23_{5,7}) + (2.25_2)$,

$$(2.35) \qquad \varphi(2A_1 + E_4) = 2a_1 + e_4 , \quad \varphi(E_2 - 2C_1) = e_2 - 2c_1 .$$

Again, $2a_1 + e_4 \neq 0$ or $e_2 - 2c_1 \neq 0$ implies $\dim G(V) \leqq 5$.

**3.** Finally, suppose (2.18) with

$$(3.1) \qquad d_3 = -a_1 , \quad d_1 = a_3 , \quad f_1 = -c_3 , \quad f_3 = c_1 , \quad e_4 = -2a_1 , \quad e_2 = 2c_1 .$$

The equations (2.25) reduce to

$$(3.2) \qquad w_2\varphi = 2\varphi A_1 - 2a_1 , \quad w_4\varphi = 2\varphi C_1 - 2c_1 .$$

Consider the system

$$(3.3) \qquad w_2\varphi = -2a_1 , \quad w_4\varphi = -2c_1 .$$

Then $w_4 w_2 \varphi = -2w_4 a_1$, $w_2 w_4 \varphi = -2w_2 c_1$, and we get

$$[w_2, w_4]\, \varphi = -2w_2 c_1 + 2w_4 a_1 = 2c_1 w_2 \varphi - 2a_1 w_4 \varphi = -4c_1 a_1 + 4a_1 c_1 = 0$$

by means of $(2.18_5)$. The integrability condition of the system (3.3) is $w_2 c_1 - w_4 a_1 = 0$, i.e., $v_2 c_1 - v_4 a_1 = 0$. This equation being satisfied because of $(2.22_{14})$, the system (3.3) is completely integrable. It follows the possibility to choose the frames of $B_G$ such that $A_1 = C_1 = 0$. Let us suppose

$$(3.4) \qquad a_1 = c_1 = 0$$

and

$$(3.5) \qquad w_2\varphi = w_4\varphi = 0 .$$

From $(2.23_{3,7})$,

$$(3.6) \qquad \varphi w_2 \beta + \alpha\varphi A_3 = \alpha a_3 , \quad \varphi w_4 \beta + \alpha\varphi C_3 = \alpha c_3 .$$

Consider the system

$$(3.7) \qquad v_2\beta = \alpha a_3 , \quad v_4\beta = \alpha c_3 .$$

From (3.5) and (3.7), $v_2\alpha = -\beta a_3$, $v_4\alpha = -\beta c_3$, from (3.7) and ($2.18_5$), $v_2 v_4 \beta = -\beta a_3 c_3 + \alpha v_2 c_3$, $v_4 v_2 \beta = -\beta c_3 a_3 + \alpha v_4 a_3$. The integrability condition of (3.7) $0 = [v_2, v_4]\, \beta = \alpha(v_2 c_3 - v_4 a_3)$ is now satisfied because of ($2.22_6$). The system (3.7) being integrable, we are in the position to choose the frames in such a way that $A_3 = C_3 = 0$. Suppose

(3.8) $$a_3 = c_3 = 0$$

and, consequently,

(3.9) $$w_2\alpha = w_2\beta = w_4\alpha = w_4\beta = 0\,.$$

The condition $\dim G(V) > 5$ for a layer $V$ implies the existence of sections $(v_1, \ldots, v_4)$ of $B_G$ such that

(3.10) $$\begin{aligned} &[v_1, v_2] = a_2 v_2\,, && [v_2, v_3] = d_2 v_2\,,\\ &[v_1, v_3] = b_1 v_1 + v_2 + b_3 v_3\,, && [v_2, v_4] = 0\,,\\ &[v_1, v_4] = a_2 v_4\,; && [v_3, v_4] = d_2 v_4\,. \end{aligned}$$

The equations (2.22) reduce to

(3.11) $$\begin{aligned} &v_2 b_1 = v_2 b_3 = 0\,, \quad v_2 a_2 = v_4 a_2 = 0\,, \quad v_2 d_2 = v_4 d_2 = 0\,,\\ &v_1 d_2 + v_3 a_2 = 0\,, \quad a_2 d_2 + a_2 b_1 - b_3 d_2 = 0\,. \end{aligned}$$

The equations (2.24) may be written as

(3.12) $$v_1\varphi + \alpha\varphi A_2 + \beta\varphi D_2 = \varphi a_2\,, \quad -v_3\varphi - \beta\varphi A_2 + \alpha\varphi D_2 = \varphi d_2\,,$$

the equation (2.26) as

(3.13) $$v_1\beta - v_3\alpha + \varphi B_1 = \alpha b_1 + \beta b_3\,, \quad v_1\alpha + v_3\beta + \varphi B_3 = -\beta b_1 + \alpha b_3\,.$$

The integrability condition of (3.12) is

(3.14) $$a_2 d_2 = \varphi A_2 D_2\,.$$

The condition $a_2 d_2 \neq 0$ implies $\dim G(V) \leqq 5$. Suppose

(3.15) $$a_2 = 0\,,$$

the case $d_2 = 0$ being symmetric. Because of (3.11), the system $v_1\varphi = 0$, $v_3\varphi = -\varphi d_2$ is integrable, and we may choose the frames of $B_G$ in such a way that

(3.16) $$a_2 = d_2 = 0$$

which implies

(3.17) $$v_1\varphi = v_3\varphi = 0\,.$$

Then $\alpha v_1\alpha + \beta v_1\beta = \alpha v_3\alpha + \beta v_3\beta = 0$, and we get

$$v_1\alpha = \alpha\beta B_1 - \beta^2 B_3 - \beta b_1, \quad v_3\alpha = \beta^2 B_1 + \alpha\beta B_3 - \beta b_3 \tag{3.18}$$

from (3.13). The integrability condition of (3.18) is

$$v_1 b_3 - v_3 b_1 - b_1^2 - b_3^2 = \varphi(w_1 B_3 - w_3 B_1 - B_1^2 - B_3^2). \tag{3.19}$$

The condition $v_1 b_3 - v_3 b_1 - b_1^2 - b_3^2 \neq 0$ implies $\dim G(V) \leqq 5$. Let us suppose

$$v_1 b_3 - v_3 b_1 - b_1^2 - b_3^2 = 0. \tag{3.20}$$

The system $v_1\alpha = -\beta b_1$, $v_3\alpha = -\beta b_3$ being integrable, there are sections $(v_1, \ldots, v_4)$ satisfying

$$b_1 = b_3 = 0, \tag{3.21}$$

and we have

$$v_1\alpha = v_1\beta = v_3\alpha = v_3\beta = 0. \tag{3.22}$$

**4.** The condition $\dim G(V) > 5$ implies $\dim G(V) = 6$ and the existence of a section $(v_1, \ldots, v_4)$ of $B_G$ such that

$$[v_1, v_3] = v_2, \quad [v_1, v_2] = [v_1, v_4] = [v_2, v_3] = [v_2, v_4] = [v_3, v_4] = 0. \tag{4.1}$$

Consider the layer $V(0.1)$. It is easy to check that the real vector fields

$$\begin{aligned} v_1 &= i\frac{\partial}{\partial x} + 2(\bar{x} - x)\frac{\partial}{\partial y} - i\frac{\partial}{\partial \bar{x}} + 2(x - \bar{x})\frac{\partial}{\partial \bar{y}}, \quad v_2 = 4\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial \bar{y}}\right), \\ v_3 &= -\frac{\partial}{\partial x} + 2i(\bar{x} - x)\frac{\partial}{\partial y} - \frac{\partial}{\partial \bar{x}} - 2i(x - \bar{x})\frac{\partial}{\partial \bar{y}}, \quad v_4 = 4i\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial \bar{y}}\right) \end{aligned} \tag{4.2}$$

over $\mathscr{C}^2$ satisfy the following conditions: (i) $v_3 = Jv_1$, $v_4 = Jv_2$, (ii) at each point $z \in \mathscr{C}^2$, $v_1, v_2, v_3$ are tangent to the hypersurface of $V$ going through $z$, (iii) $v_1, \ldots, v_4$ satisfy (4.1). The Lie group (0.2) preserving $V$, we have obtained an example of a layer satisfying the conditions of our Theorem. It remains to show that any two layers satisfying these conditions are biholomorphically equivalent. Consider the complex manifold $M^4$ and its layer $V$ such that in its corresponding structure $B_G$ there is a section $(v_1, \ldots, v_4)$ satisfying (4.1). Let $N^4$ be another complex manifold with a layer $W$ of hypersurfaces such that in the associated structure $B'_G$ there is a section $(w_1, \ldots, w_4)$ such that

$$\begin{gathered} [w_1, w_3] = w_2, \\ [w_1, w_2] = [w_1, w_4] = [w_2, w_3] = [w_2, w_4] = [w_3, w_4] = 0. \end{gathered} \tag{4.3}$$

On $M^4 \times N^4$, consider the vector fields $v_i^*$, $w_i^*$ defined by the relations $d\pi_1(v_i^*) = v_i$, $d\pi_2(v_i^*) = 0$, $d\pi_1(w_i^*) = 0$, $d\pi_2(w_i^*) = w_i$; $\pi_1 : M^4 \times N^4 \to M^4$, $\pi_2 : M^4 \times N^4 \to N^4$ being the natural projections. Let $\alpha, \beta \in \mathcal{R}$, $\varphi = \alpha^2 + \beta^2 \neq 0$. On $M^4 \times N^4$, consider the distribution $D$ such that its space $D^4_{(m,n)} \subset T_{(m,n)}(M^4 \times N^4)$ is spanned by the vectors

$$V_1 = v_1^* + \alpha w_1^* - \beta w_3^*, \quad V_2 = v_2^* + \varphi w_2^*, \quad V_3 = v_3^* + \beta w_1^* + \alpha w_3^*,$$
$$V_4 = v_4^* + \varphi w_4^*.$$

Because of

$$[V_1, V_3] = V_2, \quad [V_1, V_2] = [V_1, V_4] = [V_2, V_3] = [V_2, V_4] = [V_3, V_4] = 0,$$

the distribution $D$ is integrable and its integral manifold represents a (local) biholomorphic map $M^4 \to N^4$ transforming $V$ into $W$.

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