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Marko Švec

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ASYMPTOTIC RELATIONSHIP BETWEEN SOLUTIONS OF TWO SYSTEMS OF DIFFERENTIAL EQUATIONS

MARKO ŠVEC, Bratislava

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Let x be an n-dimensional vector, A an $n \times n$ matrix, $f(t, x)$ n-dimensional vector-function continuous in (t, x) for $t \geqq 0$, $|x| < \infty$. $|.|$ denotes any convenient matrix (vector) norm. We will consider two systems,

$$x' = Ax + f(t, x) \tag{1}$$

and

$$y' = Ay\,. \tag{2}$$

The problem we will deal with in this paper is the asymptotic equivalence of the two systems (1) and (2). We will say that the two systems (1) and (2) are *asymptotically equivalent* iff to each solution $x(t)$ of (1) there exists a solution $y(t)$ of (2) such that

$$\lim_{t\to\infty} |x(t) - y(t)| = 0\,, \tag{3}$$

and conversely, to each solution $y(t)$ of (2) there exists a solution $x(t)$ of (1) such that (3) holds.

We will speak of *restricted asymptotic equivalence* between (1) and (2) if the relation (3) is satisfied only between some subsets of solutions of (1) and (2), e.g. between the bounded solutions.

To the problem of (restricted) asymptotic equivalence are dedicated many papers by various authors, as WEYL [1], LEVINSON [2], [3], WINTNER [4], [5], JAKUBOVIČ [6], BRAUER [7], [8], BRAUER and WONG [9], [10], ONUCHIC [11]–[13], and others.

In what follows we will denote by A an $n \times n$ constant matrix, by $A(t)$ an $n \times n$ matrix-function continuous on $[0, \infty)$.

We start our considerations with the systems

$$x' = A(t)\,x + f(t)\,, \tag{1'}$$

and

$$(2') \qquad y' = A(t)\, y\,.$$

Using the property that each solution of (1) can be represented in the form

$$x(t) = y(t) + x_0(t)$$

where $x_0(t)$ is some given solution of (1′) and $y(t)$ a suitable solution of (2′), it is easy to prove the following theorem:

Theorem 1. *The systems* (1′) *and* (2′) *are asymptotically equivalent iff there exists* (*at least*) *one solution* $x_0(t)$ *of* (1′) *such that* $\lim x_0(t) = 0$ *as* $t \to \infty$.

Therefore, our next problem is to find sufficient conditions for the existence of the solution $x_0(t)$ of (1′) which has the property that $\lim x_0(t) = 0$ as $t \to \infty$.

Let us suppose that A has the Jordan form. Let $\mu_1 < \mu_2 < \ldots < \mu_s = \lambda$ be the distinct real parts of eigenvalues $\lambda_i(A)$ of A and let m_i be the maximum order of those blocks in A which correspond to eigenvalues with real part μ_i. Denote $m_s = m$; $p = m_j$ if $\mu_j = 0$, $p = 1$ if no μ_j equals zero.

Without loss of generality we will suppose that $A = \operatorname{diag}(A_1, A_2)$, where A_1 and A_2 are square matrices such that

$$\operatorname{Re} \lambda_j(A_1) \leqq -\alpha = \max_j \operatorname{Re} \lambda_j(A_1) < 0\,; \quad m^* = m_i \quad \text{if} \quad \mu_i = -\alpha\,;$$
$$\operatorname{Re} \lambda_j(A_2) \geqq 0 \quad \text{for all } j\,.$$

Then $Y(t) = \operatorname{diag}(e^{tA_1}, e^{tA_2})$ is the fundamental matrix of (2). Let be

$$Y_1(t) = \operatorname{diag}(e^{tA_1}, 0),\ Y_2(t) = \operatorname{diag}(0, e^{tA_2})\,.$$

Then

$$(4) \qquad Y(t) = Y_1(t) + Y_2(t)\,, \quad Y(t)\, Y^{-1}(s) = Y_1(t)\, Y_1^{-1}(s) + Y_2(t)\, Y_2^{-1}(s)\,,$$
$$Y_i(t)\, Y_i^{-1}(s) = Y_i(t - s)\,, \quad i = 1, 2\,,$$

and

$$(5) \qquad |Y_1(t)| \leqq a e^{-\alpha t} \chi_{m^*}(t)\,, \quad t \geqq 0\,,$$
$$|Y_2^{-1}(t)| = |Y_2(-t)| \leqq b\, \chi_p(t)\,, \quad t \geqq 0\,,$$

where

$$(6) \qquad \chi_k(t) = \begin{cases} t^{k-1}\,, & t \geqq 1 \\ 1\,, & 0 \leqq t \leqq 1\,, \end{cases}$$

and a, b are suitable constants.

The following lemma will be useful in what follows.

Lemma 1. (see Brauer [8]). *Let σ be a positive constant and let $g(x) \geqq 0$ be continuous on $0 \leqq t < \infty$ and such that either*

$$\int_0^\infty g(t)\, dt < \infty \quad \textit{or} \quad \lim_{t\to\infty} g(t) = 0\,.$$

Then

$$\lim_{t\to\infty} e^{-\sigma t} \int_0^t e^{\sigma s}\, g(s)\, ds = 0\,.$$

Now we are able to prove the following theorem.

Theorem 2. *If*

$$\int_0^\infty t^{p-1} |f(t)|\, dt < \infty\,, \tag{7}$$

then the equation

$$x' = Ax + f(t) \tag{8}$$

has at least one solution converging to zero as $t \to \infty$.

Proof. Let $x(t)$, $x(0) = x_0$, be the solution of (1). x_0 will be chosen later. Using (4) and the formula of variation of constants we get for $x(t)$ the expression

$$x(t) = Y(t)\, x_0 + Y_1(t) \int_0^t Y_1^{-1}(s) f(s)\, ds + Y_2(t) \int_0^t Y_2^{-1}(x) f(s)\, ds\,. \tag{9}$$

Respecting (7) and (5) we have that $|\int_0^\infty Y_2^{-1}(s) f(s)\, ds| < \infty$. Therefore we can write (9) in the form

$$x(t) = Y_1(t)\, x_0 + Y_2(t) \left[x_0 + \int_0^\infty Y_2^{-1}(s) f(s)\, ds\right] + Y_1(t) \int_0^t Y_1^{-1}(s)\, f(s)\, ds -$$
$$- Y_2(t) \int_t^\infty Y_2^{-1}(s) f(s)\, ds\,.$$

We choose x_0 such that $x_0 + \int_0^\infty Y_2^{-1}(s) f(s)\, ds = 0$. Then our solution $x(t)$ is given by

$$x(t) = Y_1(t)\, x_0 + Y_1(t) \int_0^t Y_1^{-1}(s) f(s)\, ds - Y_2(t) \int_t^\infty Y_2^{-1}(s) f(s)\, ds\,. \tag{10}$$

We will prove that this solution has the property: $\lim x(t) = 0$ as $t \to \infty$. It follows from (5) that $\lim Y_1(t)\, x_0 = 0$ as $t \to \infty$ and

(11) $$\left|Y_1(t) \int_0^t Y_1^{-1}(s) f(s)\, ds\right| = \left|\int_0^t Y_1(t - s) f(s)\, ds\right| \leqq$$
$$\leqq a \int_0^t e^{-\alpha(t-s)} \chi_{m*}(t - s)\, |f(s)|\, ds =$$
$$= a\left[\int_0^{t/2} e^{-\alpha(t-s)} \chi_{m*}(t - s)\, |f(s)|\, ds + \int_{t/2}^t e^{-\alpha(t-s)} \chi_{m*}(t - s)\, |f(s)|\, ds\right] =$$
$$= a[J_1 + J_2]\,.$$

But for $t \geqq 2$, $0 \leqq s \leqq \frac{1}{2}t$ we have $t \geqq t - s \geqq \frac{1}{2}t \geqq 1$. Therefore using the definition of the function χ_k we get

$$J_1 = \int_0^{t/2} e^{-\alpha(t-s)} (t - s)^{m^*-1}\, |f(s)|\, ds =$$
$$= \int_0^{t/2} e^{-\alpha(t-s)/2} e^{-\alpha(t-s)/2} (t - s)^{m^*-1}\, |f(s)|\, ds \leqq B_1 \int_0^{t/2} e^{-\alpha(t-s)/2} |f(s)|\, ds$$

using the fact that $e^{-\alpha u/2} u^{m^*-1} \leqq B_1$ for $u \geqq 0$.

Now the application of lemma 1 gives that $\lim J_1 = 0$ as $t \to \infty$. Using the same fact that $e^{-\alpha(t-s)} \chi_{m*}(t - s) \leqq B_2$ for $t - s \geqq 0$ we have

$$J_2 \leqq B_2 \int_{t/2}^t |f(s)|\, ds \leqq B_2 \int_{t/2}^\infty |f(s)|\, ds \to 0 \quad \text{as} \quad t \to \infty\,.$$

In such a way we have proved that also the second term on the right in (10) tends to zero as $t \to \infty$.

We are going to estimate the last term in (10).

Suppose that $p > 1$. Then using (5) we get

$$\left|Y_2(t) \int_t^\infty Y_2^{-1}(s) f(s)\, ds\right| = \left|\int_t^\infty Y_2(-(s - t)) f(s)\, ds\right| \leqq b \int_t^\infty \chi_p(s - t)\, |f(s)|\, ds$$

respecting the fact that $s - t \geqq 0$. But

$$\int_t^\infty \chi_p(s - t)\, |f(s)|\, ds = \int_t^{t+1} |f(s)|\, ds + \int_{t+1}^\infty (s - t)^{p-1}\, |f(s)|\, ds \leqq$$
$$\leqq \int_t^\infty |f(s)|\, ds + \int_{t+1}^\infty s^{p-1} |f(s)|\, ds \to 0$$

as $t \to \infty$ respecting (7). Thus for $p > 1$ the last term in (10) converges also to zero as $t \to \infty$. For $p = 1$ the proof of the same fact is obvious. This completes the proof.

We note that the condition (7) is the best in such sense that there are systems of the type (8) with a solution converging to zero as $t \to \infty$ and this fact implies that (7) holds.

Example. The system

$$x_1' = x_2\,,$$
$$x_2' = \quad f_2(t)$$

has the general solution:

$$x_2 = c_2 + \int_0^t f_2(s)\,\mathrm{d}s\,,$$

$$x_1 = c_1 + c_2 t + \int_0^t (t - s) f_2(s)\,\mathrm{d}s\,.$$

If $x_1(t) \to 0$, $x_2(t) \to 0$ as $t \to \infty$, then it must be $c_2 = -\int_0^\infty f_2(s)\,\mathrm{d}s$ and $c_1 - t \int_t^\infty f_2(s)\,\mathrm{d}s - \int_0^t s f_2(s)\,\mathrm{d}s \to 0$ as $t \to \infty$. If we assume that $f_2(t) \geqq 0$, then the last condition implies that $\int_0^\infty s f_2(s)\,\mathrm{d}s < \infty$.

Now applying theorem 1 and 2 we get immediately the proofs of the following two theorems.

Theorem 3. *Let $F(t, u)$ be a function continuous in t, u on $[0, \infty) \times [0, \infty)$ and nondecreasing in u, $u \in [0, \infty)$. Let be*

$$|f(t, x)| \leqq F(t, |x|) \tag{12}$$

for each $t \in [0, \infty)$ and $|x| < \infty$. Let be

$$\int_0^\infty t^{p-1} F(t, c)\,\mathrm{d}t < \infty \quad \textit{for each} \quad c \geqq 0\,. \tag{13}$$

Let (1) has the solutions existing and bounded on infinite intervals $[t_0, \infty)$. Then to each such bounded solution $x(t)$ of (1) there exists a solution of (2) such that (3) holds.

Proof. Let $x(t)$ be a bounded solution of (1). Then $|x(t)| \leqq K$, $t \geqq 0$. Let $y(t)$ be an arbitrary solution of (2). Then $z(t) = x(t) - y(t)$ is a solution of

$$z' = Az + f(t, x(t)) \tag{14}$$

and if $z(t)$ is an arbitrary solution of (14), then $y(t) = x(t) - z(t)$ is a solution of (2). Using (12) and (13) and the fact that $|x(t)| \leqq K$, $t \geqq 0$ we get

$$\int_0^\infty t^{p-1} |f(t, x(t))|\,\mathrm{d}t \leqq \int_0^\infty t^{p-1} F(t, K)\,\mathrm{d}t < \infty\,.$$

Therefore by theorem 2 the system (14) has a solution $z_0(t)$ such that $z_0(t) \to 0$ as $t \to \infty$. Then $y_0(t) = x(t) - z_0(t)$ is a solution of (2) and $|x(t) - y_0(t)| = |z_0(t)| \to 0$ as $t \to \infty$.

We note that theorem 3 is a generalization of the theorem 3 of Jakubovič [6]. In the same way we can prove

Theorem 4. *Let $x(t)$ be a solution of (1) such that*

$$|x(t)| \leqq K e^{\mu t} t^h\,, \quad t \geqq 0 \tag{15}$$

where $K \geqq 0$, $\mu \geqq 0$, $h \geqq 0$ be constants.

Let (12) and

$$\int_0^\infty t^{p-1} F(t, K e^{\mu t} t^h)\, \mathrm{d}t < \infty \tag{16}$$

be satisfied. Then there exists a solution $y(t)$ of (2) such that $|x(t) - y(t)| \to 0$ as $t \to \infty$.

Now suppose that (1) and (2) are asymptotically equivalent. Then necessarily the asymptotic behavior of the solution $x(t)$ of (1) must be the same as of the corresponding solution $y(t)$ of (2). More precisely, let $Y(t)$, $Y(0) = E$, be the fundamental matrix of (2). Then

$$|Y(t)| \leqq c_0 e^{\lambda t} \chi_m(t)\,, \quad t \geqq 0\,, \tag{17}$$

and

$$|y(t)| = |Y(t)\, y(0)| \leqq c_0 |y(0)|\, e^{\lambda t} \chi_m(t)\,, \quad t \geqq 0\,. \tag{18}$$

Therefore, if (1) and (2) are asymptotically equivalent, the solutions $x(t)$ of (1) satisfy the estimate

$$|x(t)| \leqq d e^{\lambda t} \chi_m(t) + o(1) \quad \text{for} \quad t \geqq t_0 \geqq 0\,,$$

where d and t_0 are suitable constants. Our aim is to establish the conditions which guarantee that for the solutions $x(t)$ of (1) the bounds

$$|x(t)| \leqq D e^{\lambda(t-t_0)} \chi_m(t - t_0)\,, \quad t \geqq t_0 \geqq 0 \tag{19}$$

are valid. We will prove the following theorem.

Theorem 5. *Let (12) be satisfied and let be*

$$\int_0^\infty e^{-\lambda t} F(t, c e^{\lambda t} \chi_m(t))\, \mathrm{d}t < \infty \tag{20}$$

for each $c \geqq 0$ *and*

$$\sup_{[1,\infty)} \frac{1}{c} \int_{t_0}^{\infty} e^{-\lambda t} F(t, ce^{\lambda t} \chi_m(t)) \, \mathrm{d}t = S < \frac{1}{c_0}. \tag{21}$$

Then each solution $x(t)$, $x(t_0) = x_0$, $t_0 \geqq 0$, *of* (1) *exists on* $[t_0, \infty)$ *and the estimate* (19) *holds.*

Proof. Let $x(t)$, $x(t_0) = x_0$, be a solution of (1). Then this solution satisfies the integral equation

$$x(t) = Y(t - t_0) x_0 + \int_{t_0}^{t} Y(t - s) f(s, x(s)) \, \mathrm{d}s \tag{22}$$

and conversely, each solution $x(t)$ of this integral equation is a solution of (1) satisfying the condition $x(t_0) = x_0$.

First we assume that $x(t)$ is a solution of (1) on $[t_0, \infty)$. Then using (17) and (12) we get from (22) that

$$|x(t)| \leqq c_0 e^{\lambda(t-t_0)} \chi_m(t - t_0) |x_0| + c_0 \int_{t_0}^{t} e^{\lambda(t-s)} \chi_m(t - s) F(s, |x(s)|) \, \mathrm{d}s .$$

The function $\chi_m(t)$ is nondecreasing for $t \geqq 0$. Therefore for $t \geqq s \geqq t_0$ we have $t - s \leqq t - t_0$ and $\chi_m(t - s) \leqq \chi_m(t - t_0)$. Then we get

$$|x(t)| \leqq c_0 e^{\lambda(t-t_0)} \chi_m(t - t_0) \left[|x_0| + \int_{t_0}^{t} e^{-\lambda(s-t_0)} F(s, |x(s)|) \, \mathrm{d}s \right]$$

and

$$|x(t)| \leqq c_0 e^{\lambda t} \chi_m(t - t_0) \left[|x_0| \, e^{-\lambda t_0} + \int_{t_0}^{t} e^{-\lambda s} F(s, |x(s)|) \, \mathrm{d}s \right].$$

We denote

$$A_T = |x_0| \, e^{-\lambda t_0} + \int_{t_0}^{T} e^{-\lambda s} F(s, |x(s)|) \, \mathrm{d}s , \quad T \geqq t_0 . \tag{23}$$

Then

$$|x(t)| \leqq c_0 e^{\lambda t} \chi_m(t - t_0) A_T , \quad t_0 \leqq t \leqq T. \tag{24}$$

We will consider two cases:

a) Let for each $T \geqq t_0$ be $A_T < 1/c_0$. Then we have from (24) that

$$|x(t)| \leqq e^{\lambda t_0} e^{\lambda(t-t_0)} \chi_m(t - t_0) \quad \text{for} \quad t \geqq t_0 .$$

Thus (19) is valid.

b) Let there exist such $T_0 \geqq t_0$ that $A_{T_0} c_0 \geqq 1$. Since A_T is a nondecreasing function of T we have that $A_T c_0 \geqq 1$ for each $T \geqq T_0$.

Now from the condition (21) it follows that for $T \geqq T_0$

$$\int_{t_0}^{\infty} e^{-\lambda s} F(s, c_0 A_T e^{\lambda s} \chi_m(s)) \, ds \leqq S c_0 A_T . \tag{25}$$

Substituting (24) in (23) and respecting the monotonicity of $F(t, u)$ in u we get

$$\begin{aligned} A_T &\leqq |x_0| \, e^{-\lambda t_0} + \int_{t_0}^{T} e^{-\lambda s} F(s, c_0 A_T e^{\lambda s} \chi_m(s)) \, ds \leqq \\ &\leqq |x_0| \, e^{-\lambda t_0} + \int_{t_0}^{\infty} e^{-\lambda s} F(s, c_0 A_T e^{\lambda s} \chi_m(s)) \, ds \end{aligned} \tag{26}$$

and respecting also (25)

$$A_T \leqq |x_0| \, e^{-\lambda t_0} + S c_0 A_T .$$

Hence we have

$$A_T \leqq \frac{|x_0|}{1 - S c_0} \, e^{-\lambda t_0} . \tag{27}$$

Substituting in (24) we obtain that

$$|x(t)| \leqq c_0 \frac{|x_0|}{1 - S c_0} \, e^{\lambda(t - t_0)} \chi_m(t - t_0) , \quad t_0 \leqq t \leqq T . \tag{28}$$

But the second term in this inequality does not depend on T. Therefore we conclude that this inequality is valid for all $t \geqq t_0$. Thus the estimate (19) is valid also in this case b).

Now suppose that $x(t)$, $x(t_0) = x_0$, is a solution of (1) existing only on $[t_0, t_1)$, $t_1 < \infty$. Then it must be $|x(t)|$ unbounded on $[t_0, t_1)$. If we make the same considerations as above for $T \in [t_0, t_1)$ we get that for $|x(t)|$ the estimate (19) is valid for $t_0 \leqq t < t_1$. But this contradicts the above statement. This proves that the conditions of theorem 5 and the continuity of $f(t, x)$ on $t \geqq 0$, $|x| < \infty$ guarantee the existence of the solution $x(t)$, $x(t_0) = x_0$, of (1) on $[t_0, \infty)$ and the validity of (19). This completes the proof.

Remark. *The condition* (21) *can be substituted by the condition*

$$\lim_{t_0 \to \infty} \frac{1}{c} \int_{t_0}^{\infty} e^{-\lambda t} F(t, c e^{\lambda t} \chi_m(t)) \, dt = 0 \tag{21'}$$

uniformly with respect to $c \in [1, \infty)$. In fact, if $(21')$ is satisfied then it is possible to choose t_0 in such a way that (21) will be satisfied.

Theorem 5 gives us the main tool to prove the following theorem.

Theorem 6. *Let be satisfied* (12) *and let be*

$$\int_0^\infty t^{p-1} F(t, ce^{\lambda t} \chi_m(t)) \, dt < \infty \tag{29}$$

for each $c \geqq 0$ *if* $\lambda \geqq 0$ *and let be*

$$\int_0^\infty e^{-\lambda t} F(t, ce^{\lambda t} \chi_m(t)) \, dt < \infty \tag{29'}$$

for each $c \geqq 0$ *if* $\lambda < 0$. *Furthermore let be* $(21')$ *satisfied. Then the systems* (1) *and* (2) *are asymptotically equivalent.*

Proof. First we observe that the condition (29) and $(29')$ respectively imply the condition (20). Thus the theorem 5 is applicable. Consider the case that $\lambda < 0$. It means that each solution $y(t)$ of (2) converges to zero as $t \to \infty$ and by the inequality (19) the same is true also for the solutions $x(t)$ of (1). The asymptotic equivalence of (1) and (2) in this case is obvious.

Let us now suppose that $\lambda \geqq 0$. Let $x(t)$ be a solution of (1), and consider the equation (30)

$$z' = Az + f(t, x(t)) \, . \tag{30}$$

It follows from the theorem 5 and from (29) that

$$\int_0^\infty t^{p-1} |f(t, x(t))| \, dt < \infty \, .$$

In fact,

$$\int_0^\infty t^{p-1} |f(t, x(t))| \, dt = \int_0^{t_0} t^{p-1} |f(t, x(t))| \, dt + \int_{t_0}^\infty t^{p-1} |f(t, x(t))| \, dt$$

and

$$\int_{t_0}^\infty t^{p-1} |f(t, x(t))| \, dt \leqq \int_{t_0}^\infty t^{p-1} F(t, ce^{\lambda t} \chi_m(t)) \, dt < \infty \, ,$$

where t_0 is such that (21) is satisfied. Now by theorem 2 the equation (30) has a solution $z_0(t)$ such that $z_0(t) \to 0$ as $t \to \infty$. Then $y(t) = x(t) - z_0(t)$ is a solution of (2) and $x(t) - y(t) = z_0(t) \to 0$ as $t \to \infty$.

We are going to prove the second part of the theorem. Let $y(t)$, $y(0) = y_0$, be a solution of (2). Then consider the integral equation

$$(31)\quad x(t) = y(t) + Y_1(t)\int_{t_0}^{t} Y_1^{-1}(s)\,f(s, x(s))\,\mathrm{d}s - Y_2(t)\int_{t}^{\infty} Y_2^{-1}(s)\,f(s, x(s))\,\mathrm{d}s\,,$$

$t \geqq t_0$. The number t_0 will be chosen later. It easy to see that each solution of this integral equation is a solution of (1) for $t \geqq t_0$. We need to prove that this integral equation has a solution for $t \geqq t_0$.

Let $B = \{\varphi(t) \mid |\varphi(t)| \leqq De^{\lambda t}\chi_m(t),\ t \geqq t_0, \text{ for all } D \geqq 0\}$ be the set of all vector-valued functions $\varphi(t)$ continuous on $[t_0, \infty)$, $t_0 \geqq 0$ and bounded in norm by $De^{\lambda t}\chi_m(t)$. Let

$$\|\varphi(t)\| = \sup_{[t_0,\infty)} \frac{|\varphi(t)|}{e^{\lambda t}\chi_m(t)}\,.$$

Then by this norm is B a Banach space.

Using (5), (12) and the monotonicity of F and χ_m we get

$$\left|\int_{t_0}^{\infty} Y_2^{-1}(s)\,f(s, \varphi(s))\,\mathrm{d}s\right| \leqq \int_{t_0}^{\infty} |Y_2(-s)|\,F(s, |\varphi(s)|)\,\mathrm{d}s \leqq$$

$$\leqq b\int_{t_0}^{\infty} \chi_r(s)\,F(s, De^{\lambda s}\chi_m(s))\,\mathrm{d}s < \infty\,.$$

Thus the operator

$$T\varphi = y(t) + Y_1(t)\int_{t_0}^{t} Y_1^{-1}(s)\,f(s, \varphi(s))\,\mathrm{d}s - Y_2(t)\int_{t}^{\infty} Y_2^{-1}(s)\,f(s, \varphi(s))\,\mathrm{d}s$$

is defined on B. Let be

$$G(t) = Y_1(t)\int_{t_0}^{t} Y_1^{-1}(s)\,f(s, \varphi(s))\,\mathrm{d}s\,.$$

Then, using (5), (12) and the monotonicity of F we have

$$|G(t)| \leqq a\int_{t_0}^{t} e^{-\alpha(t-s)}\,\chi_{m*}(t-s)\,F(s, De^{\lambda s}\chi_m(s))\,\mathrm{d}s\,.$$

The same considerations as in the proof of the theorem 2 give

$$|G(t)| \leqq a\int_{t_0}^{(t+t_0)/2} e^{-\alpha(t-s)}\,\chi_{m*}(t-s)\,F(s, De^{\lambda s}\chi_m(s))\,\mathrm{d}s +$$

$$+ a\int_{(t+t_0)/2}^{t} e^{-\alpha(t-s)}\,\chi_{m*}(t-s)\,F(s, De^{\lambda s}\chi_m(s))\,\mathrm{d}s = a(Z_1 + Z_2)\,.$$

For Z_1 we have

$$Z_1 = \int_{t_0}^{(t+t_0)/2} e^{-\alpha(t-s)}(t-s)^{m^*-1} F(s, De^{\lambda s}\chi_m(s))\,\mathrm{d}s =$$

$$\leqq k_1 \int_{t_0}^{(t+t_0)/2} e^{-\alpha(t-s)/2} F(s, De^{\lambda s}\chi_m(s)) \to 0$$

as $t \to \infty$ applying lemma 1. Here $k_1 = \max_{u \geqq 0} e^{-\alpha u/2} u^{m^*-1} < \infty$. Using the fact that $k_2 = \max_{u \geqq 0} e^{-\alpha u}\chi_{m^*}(u) < \infty$ we have that

$$Z_2 \leqq k_2 \int_{(t+t_0)/2}^{\infty} F(s, De^{\lambda s}\chi_m(s))\,\mathrm{d}s \to 0$$

as $t \to \infty$. So we have proved that

$$\lim_{t \to \infty} G(t) = 0 \tag{32}$$

and therefore $G(t)$ is bounded on $[t_0, \infty)$ by a constant depending on D.

Let now $B_\varrho = \{\varphi \in B \mid \|\varphi\| \leqq \varrho\}$, $\varrho \geqq 2c_0|y_0|$. From (5) and monotonicity of $F(t, u)$ in u it follows that

$$|T\varphi| \leqq c_0 e^{\lambda t}\chi_m(t)\,|y_0| + \int_{t_0}^{t} |Y_1(t-s)|\,|f(s, \varphi(s))|\,\mathrm{d}s +$$

$$+ \int_{t}^{\infty} |Y_2(t-s)|\,|f(s, \varphi(s))|\,\mathrm{d}s \leqq$$

$$\leqq c_0 e^{\lambda t}\chi_m(t)\,|y_0| + a\int_{t_0}^{t} e^{-\alpha(t-s)}\chi_{m^*}(t-s)\,F(s, \varrho e^{\lambda s}\chi_m(s))\,\mathrm{d}s +$$

$$+ b\int_{t}^{\infty} \chi_p(s-t)\,F(s, \varrho e^{\lambda s}\chi_m(s))\,\mathrm{d}s \leqq$$

$$\leqq e^{\lambda t}\chi_m(t)\left[c_0|y_0| + ak_2\int_{t_0}^{\infty} F(s, \varrho e^{\lambda s}\chi_m(s))\,\mathrm{d}s + b\int_{t_0}^{\infty} \chi_p(s)\,F(s, \varrho e^{\lambda s}\chi_m(s))\,\mathrm{d}s\right].$$

We choose t_0 such that

$$ak_2\int_{t_0}^{\infty} F(s, \varrho e^{\lambda s}\chi_m(s))\,\mathrm{d}s + b\int_{t_0}^{\infty} \chi_p(s)\,F(s, \varrho e^{\lambda s}\chi_m(s))\,\mathrm{d}s \leqq \tfrac{1}{2}\varrho\,,$$

which can be done because of (29). Then we get

$$|T\varphi| \leqq \varrho e^{\lambda t}\chi_m(t)\,,$$

which means that $TB_\varrho \subset B_\varrho$.

Next we show that T is continuous on B_ϱ. Let be $\varphi_n, \varphi \in B_\varrho$, $\|\varphi_n - \varphi\| \to 0$ as $n \to \infty$. Then $\varphi_n(t) \to \varphi(t)$ uniformly on any finite interval $[t_0, t_1]$. For $T\varphi_n - T\varphi$ we have

$$|T\varphi_n - T\varphi| = \left| Y_1(t) \int_{t_0}^{t} Y_1^{-1}(s) \left[f(s, \varphi_n(s)) - f(s, \varphi(s)) \right] ds - \right.$$

$$\left. - Y_2(t) \int_t^\infty Y_2^{-1}(s) \left[f(s, \varphi_n(s)) - f(s, \varphi(s)) \right] ds \right| \leqq$$

$$\leqq a \int_{t_0}^{t} e^{-\alpha(t-s)} \chi_{m^*}(t - s) \left| f(s, \varphi_n(s)) - f(s, \varphi(s)) \right| ds +$$

$$+ b \int_t^\infty \chi_p(s - t) \left| f(s, \varphi_n(s)) - f(s, \varphi(s)) \right| ds \leqq$$

$$\leqq ak_2 \int_{t_0}^{t_1} \left| f(s, \varphi_n(s)) - f(s, \varphi(s)) \right| ds + 2ak_2 \int_{t_1}^\infty F(s, \varrho e^{\lambda s} \chi_m(s)) \, ds +$$

$$+ b \int_{t_0}^{t_1} \chi_p(s) \left| f(s, \varphi_n(s)) - f(s, \varphi(s)) \right| ds + 2b \int_{t_1}^\infty \chi_p(s) \, F(s, \varrho e^{\lambda s} \chi_m(s)) \, ds \, .$$

Let $\varepsilon > 0$ be chosen arbitrarily. We choose t_1 such that

$$2ak_2 \int_{t_1}^\infty F(s, \varrho e^{\lambda s} \chi_m(s)) \, ds + 2b \int_{t_1}^\infty \chi_p(s) \, F(s, \varrho e^{\lambda s} \chi_m(s)) \, ds < \tfrac{1}{2}\varepsilon \, .$$

From the continuity of $f(t, x)$ and from the uniform convergence of $\varphi_n(s)$ to $\varphi(s)$ on $[t_0, t_1]$ we get that for $\varepsilon > 0$ there exists an integer $n_0(\varepsilon)$ such that for each $n \geqq n_0(\varepsilon)$

$$\left| f(s, \varphi_n(s)) - f(s, \varphi(s)) \right| < \frac{\varepsilon}{2\left[(t_1 - t_0) \, ak_2 + b \int_{t_0}^{t_1} \chi_p(s) \, ds \right]} \, .$$

But then we have that for $n \geqq n_0(\varepsilon)$

$$\left| T\varphi_n(t) - T\varphi(t) \right| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon \, , \quad t \geqq t_0$$

and also

$$\left\| T\varphi_n(t) - T\varphi(t) \right\| \leqq \varepsilon$$

because

$$\left| T\varphi_n(t) - T\varphi(t) \right| e^{-\lambda t} \chi_m^{-1}(t) \leqq \left| T\varphi_n(t) - T\varphi(t) \right| , \quad t \geqq t_0 \, .$$

But this means the continuity of T on B_ϱ.

Now, from the fact that $TB_\varrho \subset B_\varrho$ it follows that the functions of TB_ϱ are uniformly bounded (in norm). It is also easy to show that $T\varphi = v$, $\varphi \in B_\varrho$, is a solution of the equation

$$v' = Av + f(t, \varphi(t)).$$

Hence we get that

$$|v'| = |A|\,|v| + F(t, \varrho e^{\lambda t} \chi_m(t)) \leqq |A|\,\varrho e^{\lambda t} \chi_m(t) + F(t, \varrho e^{\lambda t} \chi_m(t)).$$

Thus v has a uniformly bounded derivative on any finite interval $[t_0, t_1]$ and therefore the family of functions TB_ϱ is equicontinuous. So all hypotheses of the Schauder fixed point theorem are satisfied. Consequently, the operator T has a fixed point $\bar{\varphi}(t)$ in B_ϱ, i.e. $T\bar{\varphi} = \bar{\varphi}$, which means that $\bar{\varphi}$ is a solution of (31), and also a solution of (1). Further we have to prove that $|\bar{\varphi} - y(t)| \to 0$ as $t \to \infty$.

From (31) we get

$$|\bar{\varphi}(t) - y(t)| = \left| Y_1(t) \int_{t_0}^{t_1} Y_1^{-1}(s) f(s, \bar{\varphi}(s))\, ds - Y_2(t) \int_t^\infty Y_1^{-1}(s) f(s, \bar{\varphi}(s))\, ds \right| \leqq$$

$$\leqq |G(t)| + b \int_t^\infty \chi_p(s - t)\, F(s, \varrho e^{\lambda s} \chi_m(s))\, ds \leqq$$

$$\leqq |G(t)| + b \int_t^{t+1} F(s, \varrho e^{\lambda s} \chi_m(s))\, ds + b \int_{t+1}^\infty s^{p-1} F(s, \varrho e^{\lambda s} \chi_m(s))\, ds \leqq$$

$$\leqq |G(t)| + b \int_t^\infty F(s, \varrho e^{\lambda s} \chi_m(s))\, ds + b \int_t^\infty s^{p-1} F(s, \varrho e^{\lambda s} \chi_m(s))\, ds \to 0$$

as $t \to \infty$. This completes the proof.

After this we are able to prove the following theorem about the asymptotic equivalence of the sets of bounded solutions of (1) and (2).

Theorem 7. *Let the hypotheses of the theorem* 3 *be valid. Then there exists the asymptotic equivalence between the set of all bounded solutions of* (1) *and the set of all bounded solutions of* (2).

Proof. The proof in one direction is given by theorem 3. Thus to each bounded solution $x(t)$ of (1) there exists a solution $y(t)$ of (2) such that (3) holds.

Let now $y(t)$ be a bounded solution of (2). We will prove that the integral equation (31) has a bounded solution under the conditions of the theorem 3. Let H be the set of all vector-valued functions $\varphi(t)$ continuous and bounded on $[t_0, \infty)$. Let $\|\varphi(t)\| = \sup_{[t_0,\infty)} |\varphi(t)|$. Then H is a Banach space. Using (5), (12) and the monotonicity of

$F(t, u)$ in u we get for $\varphi(t) \in H$ that

$$\left| \int_t^\infty Y_2^{-1}(s) f(s, \varphi(s))\, ds \right| \leqq b \int_t^\infty \chi_p(s)\, F(s, c)\, ds < \infty .$$

Therefore we can define on H the operator T_1:

$$T_1 \varphi = y(t) + Y_1(t) \int_{t_0}^t Y_1^{-1}(s) f(s, \varphi(s))\, ds - Y_2(t) \int_t^\infty Y_2^{-1}(s) f(s, \varphi(s))\, ds .$$

Now in the same way as in the proof of the theorem 6 it is possible to prove via Schauder fixed point theorem the existence of a fixed point of T_1 in H. It means that the integral equation (31) has a bounded solution $x_1(t)$. It is obvious that this solution is also a solution of (1). Now, from (31) we have

$$|x_1(t) - y(t)| = \left| Y_1(t) \int_{t_0}^t Y_1^{-1}(s) f(s, x_1(s))\, ds - Y_2(t) \int_t^\infty Y_2^{-1}(s) f(s, x_1(s))\, ds \right| \leqq$$

$$\leqq |G(t)| + b \int_t^\infty \chi_p(s - t)\, F(s, c)\, ds \leqq$$

$$\leqq |G(t)| + b \int_t^{t+1} F(s, c)\, ds + b \int_{t+1}^\infty s^{p-1} F(s, c)\, ds \to 0$$

as $t \to \infty$ using the similar arguments as at the end of the proof of the theorem 6.

We note that the theorems 6 and 7 generalize the theorems 2 and 3 of Jakubovič [6] and the theorem 1 of Brauer and Wong [10] and others.

We note also that the theorems 2–7 remain valid in the case that $A = A(t)$ is a matrix transformable in a constant matrix B, i.e. if there exists a matrix of Ljapunov $L(t) \in C_1([0, \infty))$ such that

$$M = \sup_{[0,\infty)} |L(t)| < \infty , \quad M' = \sup_{[0,\infty)} |L'(t)| < \infty , \quad |\det L(t)| \geqq r > 0$$

and the substitution $u = L(t)\, y$ transforms (2) in $u' = Bu$, where $B = (L' + LA)\, L^{-1}$ is a constant matrix. Then $v = L(t)\, x$ transforms (1') in $v' = Bv + Lf(t)$ and (1) in $v' = Bv + Lf(t, L^{-1}v)$. The constants p, m, λ in the theorems 2–7 must be now those which belong to matrix B.

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Author's address: 816 31 Bratislava, Mlynská dolina, ČSSR (Prírodovedecká fakulta UK).