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k-PATH EULER GRAPHS

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1. INTRODUCTION

Whenever a graph contains an Euler cycle, the graph is referred to as an *Euler graph*. It is well-known that a graph is an Euler graph if and only if it is connected and every vertex has even degree.

An Euler graph is said to be *randomly traceable* from a vertex v if an Euler cycle can be constructed from v by choosing at each step in the path any edge not already chosen. ORE [3] has characterized such graphs. Similarly, CHARTRAND and LICK [1] have characterized randomly traceable diagraphs.

Of course, not every Euler graph is randomly traceable. A graph will be called a *k-path Euler graph* if a random selection of edges will suffice for k steps, or less, but not for more than k steps.

The purpose of this article is to present some of the properties of k-path Euler graphs.

2. PRELIMINARIES

The graphs considered in this paper will all be simple finite graphs. A simple finite graph, $G = [V(G), E(G)]$, is a finite collection $V(G)$, of points or vertices, v_i, together with some subset, $E(G)$, called lines or edges, of all unordered pairs, v_iv_j, of distinct points of $V(G)$. The edge v_iv_j is said to be incident to the points v_i and v_j. A path in the graph G is an alternating sequence of points and edges of the form v_1, v_1v_2, v_2, v_2v_3, ..., $v_{n-1}v_n$, v_n. This path may also be denoted by $v_1v_2v_3 \dots v_n$. It is called an Euler path if all the edges are distinct. If all the edges are distinct and $v_1 = v_n$, it is called a cycle. An Euler cycle of a graph is a cycle which contains all the points and edges of the graph. The length of a path is the number of edges in the path. The complement with respect to a graph G of a path P in G is the subgraph of G

which remains when all the edges of P are removed from G as well as each vertex from which all edges have been removed.

A graph is said to be connected if each pair of points in the graph is connected by a path. A graph is said to be complete if it has every pair of its points adjacent. A component of a graph is a maximal connected subgraph. The star of a vertex is the collection of all edges incident to the vertex. A vertex v of a graph G is called a cutpoint if the removal of v and the star of v leaves a subgraph of G having more components than G. The degree of a vertex is the number of edges which are incident to the vertex.

3. NECESSARY AND SUFFICIENT CONDITIONS

A graph will be called a k-path Euler graph if every Euler path of length k or less is extendible to an Euler cycle and if there is an Euler path of length $k + 1$ in the graph, then at least one Euler path of length $k + 1$ is not contained in an Euler cycle. A vertex v of a graph G is said to have the Euler property for the nonnegative integer l if every Euler path of length not exceeding l emanating from v can be extended to an Euler cycle of G. A vertex v is said to be an m-Euler vertex if $m = \max \{l : v \text{ has the Euler property for } l\}$. Hence, if $V(G) = \{v_i : i = 1, 2, \ldots, n\}$, then G is a k-path Euler graph if

$$k = \min_{1 \leq i \leq n} \{m_i : v_i \text{ is } m_i\text{-Euler}\} .$$

Thus, the property of being a k-path Euler graph is a minimax property. For example, the graph in Figure 1 is 1-path Euler since vertices a, b, c, and d are each 1-Euler, while vertex e is 6-Euler.

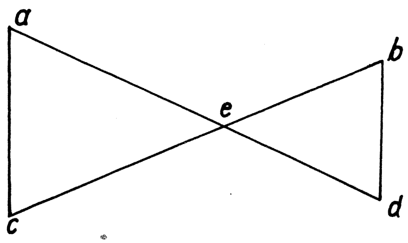

Fig. 1.

Theorem 1. *An Euler graph G is a k-path Euler graph if and only if every Euler path P of length k, or less, satisfies*:

(i) *The complement, G_P, of P with respect to G is a connected graph*;

(ii) *Path P is not a cycle containing the star of any vertex, unless P is an Euler cycle.*

and some path of length $k + 1$, *if such a path exists, does not satisfy conditions* (i) *and* (ii).

Proof. It will be enough to show that conditions (i) and (ii) are necessary and sufficient for every Euler path P of length k, or less, to be contained in an Euler cycle of G. To make this clear, suppose that this has been proved. Then if every Euler path of length not exceeding $k + 1$ in G satisfies conditions (i) and (ii), G is at least a $(k + 1)$-path Euler graph.

Proof of Necessity of (i). Suppose that P is an Euler path of length k, or less, in the Euler graph G such that the complement, G_P, of P with respect to G is disconnected. Since the null graph is considered to be connected, G_P is not the null graph, i.e. P is not an Euler cycle of G.

Suppose that the terminal vertex v_f of the path P is not in G_P. Then the star of v_f is in P, and P can only be extended by repeating some edge. Thus, path P cannot be extended to an Euler cycle of G.

Suppose that the terminal vertex v_f of the path P is in G_P. Since G_P is disconnected, G_P consists of two or more components. Then v_f can only be in one component, and in order to extend P to another component of G_P, at least one edge of P would have to be repeated. Thus, path P cannot be extended to an Euler cycle of G.

Proof of Necessity of (ii). Suppose that P is a cycle of length k, or less, containing the star of a vertex v in an Euler graph G, and P is not an Euler cycle of G. Then the cycle P can be rearranged into a cycle P' having vertex v as both its initial and terminal vertex. The cycle P' cannot be extended to an Euler cycle of G, since every edge adjacent to v is contained in P'.

Proof of Sufficiency. Suppose that conditions (i) and (ii) hold for each Euler path of length not exceeding k in the Euler graph G. Let P be an Euler path of length k, or less, in G. Then the complement, G_P, of P with respect to G is connected. If G_P is the null graph, P is an Euler cycle of G.

Otherwise, since condition (ii) holds, the initial vertex v_i and the terminal vertex v_f of P must both lie in G_P. Since G_P is connected, there is an Euler path Q, possibly the null path, in G_P which has initial vertex v_f and terminal vertex v_i. Then, the complement of the Euler path $P \cup Q$ with respect to G consists of components, possibly one or none, with each vertex having even degree, since each vertex in both G and $P \cup Q$ has even degree. Hence, each component is an Euler graph. Also, each component contains at least one vertex of Q. Thus, the Euler path P can be extended to an Euler cycle of G.

Note that it follows immediately from Theorem 1 that a k-path Euler graph can contain no cycle of length k, or less, which passes through a vertex of degree two, unless that cycle is an Euler cycle.

4. SOME OTHER PROPERTIES

Given any graph, $G = [V(G), E(G)]$, let A denote some subset of $V(G)$, and let $\bar{A}$ denote the complement of A in $V(G)$. Then the sets A and $\bar{A}$ divide $E(G)$ into three distinct categories:

(1) interior edges of A;

(2) edges of attachment to A;

(3) edges exterior to A.

An edge of the form $v_i v_j$ is:

(1) an interior edge of A if both v_i and v_j are in A;

(2) an edge of attachment if one is in A and the other is in $\bar{A}$;

(3) an exterior edge to A if both v_i and v_j are in $\bar{A}$.

The edge attachment number of the set A, denote by $\varrho(A, \bar{A})$, is defined to be the number of edges of attachment to A. Theorem 5.1.1 of [2] states that if the degree of every vertex of a graph G is even, then every subset of $V(G)$ has an even attachment number. Thus, if G is an Euler graph, then every subset of $V(G)$ has an even attachment number.

Theorem 2. *If a graph G is a k-path Euler graph* ($k \geqq 2$), *then G has no cutpoints of degree less than eight.*

Proof. Since the degree of every vertex of an Euler graph is even, a cutpoint of degree 3, 5 or 7 is not possible. No Euler graph can have a cutpoint of degree two, since the attachment number of every subset of an Euler graph must be even. If an Euler graph contains a cutpoint p of degree four, then the removal of p and the star of p leaves two components. These components were connected to p by two edges each, as in Figure 2, since the edge attachment numbers must be even.

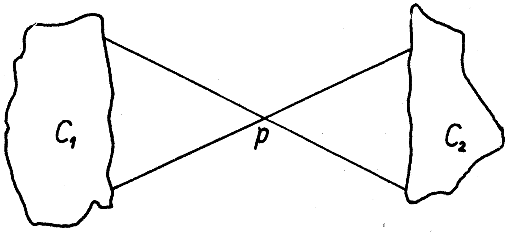

Fig. 2.

There is clearly a path of length two whose complement is disconnected. Hence, condition (i) of Theorem 1 is violated, and the graph is not a k-path Euler graph for any k greater than one.

If an Euler graph contains a cutpoint p of degree six, then the removal of p and the star of p leaves either two or three components, since the edge attachment numbers must be even. There can be two components with attachment numbers of two and four, as in Figure 3.

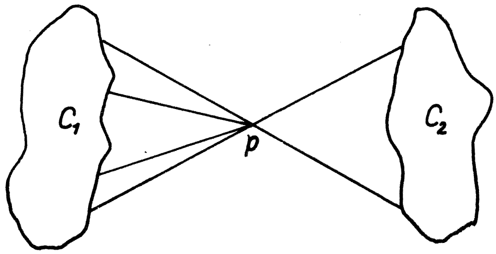

Fig. 3.

Or there can be three components with each having an attachment number of two, as in Figure 4.

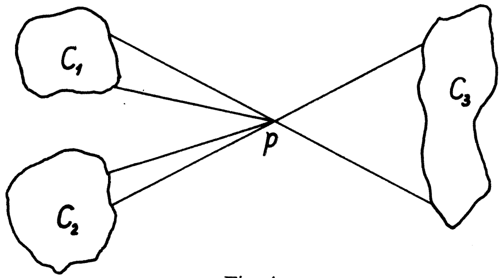

Fig. 4.

In either case there is clearly a path of length two whose complement is disconnected. Again, condition (i) of Theorem 1 is violated, and the graph is not a k-path Euler graph for any k greater than one.

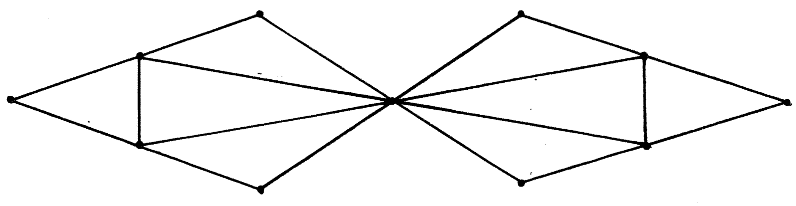

Fig. 5.

Note that Figure 5 is a 2-path Euler graph containing a cutpoint of degree eight. Thus the bound in Theorem 2 is, in a sense, as sharp as possible.

Theorem 3. *The complete graph on* $2n + 1$ *points* $(n > 1)$ *is a* $(3n - 1)$*-path Euler graph.*

Proof. Note that the degree of each vertex of the complete graph G on $2n + 1$ points in $2n$, so G is an Euler graph. Let $\{v_i: i = 1, 2, \ldots, 2n + 1\}$ be the vertices of G. Then the path $v_1v_2v_3v_1v_4v_5v_1 \ldots v_1v_{2n}v_{2n+1}v_1$ is readily seen to have $3n$ edges. It contains all $2n + 1$ edges emanating from v_1. Thus, condition (ii) of Theorem 1 is violated, and G is not a k-path Euler graph for any $k \geqq 3n$.

Conversely, no path of length $3n - 1$ or less can contain the star of any vertex. Suppose P is a path of length not exceeding $3n - 1$ and the complement G_P of P in G consists of two or more components, each having at least two vertices. Then, each of the $2n + 1$ vertices in G are in G_P. Let A be a component of G_P with a minimum number of edges and let $\bar{A}$ be the complement of A in G. Then if A has α vertices and $\bar{A}$ has $\bar{\alpha}$ vertices, we have

$$\alpha \geqq 2 ; \quad \bar{\alpha} \geqq 2 ;$$

$$\alpha + \bar{\alpha} = 2n + 1$$

and

$$\varrho(A, \bar{A}) = \alpha\bar{\alpha} .$$

Under these conditions $\varrho(A, \bar{A})$ is a minimum when $\alpha = 2$, and then $\varrho(A, \bar{A}) = 2(2n - 1) > 3n - 1$, since $n > 1$. So P cannot contain all the edges of attachment of A and hence G_P is connected. By Theorem 1, G is seen to be a $(3n - 1)$-path Euler graph.

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