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A NONCOMPACT h-COBORDISM THEOREM

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1. INTRODUCTION

The object of this paper is to generalize a theorem due to E. Luft [2], and then to apply it to strengthen a topological h-cobordism theorem of E. H. Connell [1]. Our techniques and arguments apply equally well to the piecewise linear and smooth categories of manifolds, but we shall state our theorems in terms of topological manifolds, since it is in the topological category that all of our results are believed to be new. Our generalization of Luft's theorem is:

Theorem 1. *Let M be a topological n-manifold, and let $U_1, \ldots, U_m$ be open subsets of M such that $U_i = \bigcup_{j=1}^{\infty} V_{i,j}$, where $V_{i,j}$ is open in M, $\mathrm{Cl}\, V_{i,j} \subset V_{i,j+1}$, $(M - \mathrm{Cl}\, V_{i,j}, V_{i,j+1} - \mathrm{Cl}\, V_{i,j})$ is k_i-connected, $k_i \leqq n - 3$ if $k_i > 0$, $j \geqq 1$, $1 \leqq i \leqq m$, and $\partial M \subset \bigcup_{i=1}^{m} V_{i,1}$. Then, if $k_i + \ldots + k_m + m \geqq n + 1$, there are homeomorphisms h_i of M onto itself such that*

$$h_i|_{\mathrm{Cl}\, V_{i,1} \cup \partial M} = \mathrm{id}_{\mathrm{Cl}\, V_{i,1} \cup \partial M}\,, \quad 1 \leqq i \leqq m\,, \quad \text{and} \quad M = \bigcup_{i=1}^{m} h_i(U_i)\,.$$

When applied to noncompact h-cobordisms, this gives:

Corollary 1. *Let M be a connected topological n-manifold, $n \geqq 5$, with two boundary components N_1 and N_2 such that the inclusion of N_i into M is a homotopy equivalence, $i = 1, 2$. Then there are homeomorphisms f_i of $N_i \times [0, \infty)$ into M such that $f_i(x, 0) = x$ for all $x \in N_i$, $i = 1, 2$, and $M = f_1(N_1 \times [0, \infty)) \cup f_2(N_2 \times \times [0, \infty))$.*

This was proved by E. H. Connell with the condition that N_1 and N_2 be compact. A repeated application of Corollary 1 gives:

Theorem 2. *Under the hypotheses of Corollary* 1, *there is a homeomorphism* f *of* $N_1 \times [0, \infty)$ *onto* $M - N_2$ *such that* $f(x, 0) = x$ *for all* $x \in N_1$.

This was proved by E. H. Connell for M compact. Of course these theorems are also true in the piecewise linear and smooth categories, but in the compact case there are much stronger results available: the well known h-cobordism theorem of Smale states that M is a product of N_1 with $[0, 1]$, so N_1 is homeomorphic to N_2, if M has no Whitehead torsion [4].

2. PRELIMINARIES

By a topological manifold M we mean a separable Hausdorff space such that each point of M has an open neighborhood homeomorphic to an open subset of the half-open subspace $H^n = \{(x_1, \ldots, x_n) \in \boldsymbol{R}^n : x_n \geqq 0\}$ of $\boldsymbol{R}^n$.

The image $f(\boldsymbol{R}^n)$ of $\boldsymbol{R}^n$, where $f : \boldsymbol{R}^n \to M$ is a homeomorphism into the topological n-manifold M, is called an open topological n-cell in M. Let

$$C_\alpha^n = \{(x_1, \ldots, x_n) \in \boldsymbol{R}^n : |x_i| \leqq \alpha, \ 1 \leqq i \leqq n\},$$

where $\alpha \geqq 0$.

A topological space (X, A) is said to be k-connected, $k \geqq 0$, if $\pi_i(X, A) = 0$, $0 \leqq i \leqq k$.

We shall need the following version of M. H. A. Newman's Engulfing Theorem:

Let M *be a topological n-manifold, and let* $q : \boldsymbol{R}^n \to M$ *be a homeomorphism. Let* P *be a k-dimensional subpolyhedron of* $\boldsymbol{R}^n$, *not necessarily compact, such that* $g(P)$ *is closed, and let* $U \subset M$ *be an open set such that* $g(P) - U$ *is compact. Let* $E \supset \partial M$ *be a closed set such that* $E \subset U$, *and* $(M - E, U - E)$ *is k-connected. If* $k \leqq n - 3$, *there is a compact set* $C \subset M - E$, *and a homeomorphism* h *of* M *onto itself, such that* $h(U) \supset g(P)$, *and* $h(x) = x$ *if* $x \notin C$.

Note that $h_{|E} = \mathrm{id}_E$, and in particular, that h is the identity on a neighborhood of ∂M.

This theorem is an immediate consequence of Theorem 4 of [5] when stated in terms of relative homotopy. The introduction of relative homotopy causes no new complications. Another version, in the differentiable case, appears in the author's thesis.

We shall also need the following "stretching theorem": Let K be a simplicial complex in $\boldsymbol{R}^n$, L a full finite subcomplex of K, and $L^c = \{\Delta \in K : \Delta \cap L = \emptyset\}$ the subcomplex of K complementary to L. Let U and V be open sets in $\boldsymbol{R}^n$ such that $|L| \subset U$ and $|L^c| \subset V$. Let $F \subset \boldsymbol{R}^n$ be a closed set such that $F \cap |K| \subset |L| \cup |L^c|$. Then there is a compact set $C \subset \boldsymbol{R}^n - F$ and a homeomorphism S of $\boldsymbol{R}^n$ onto itself such that:

1) $S(x) = x$ if $x \notin C$, and $|K| \subset S(U) \cup V$.
2) $S(\Delta) = \Delta$ for all $\Delta \in K$.

An outline of the proof may be found in [6]. A proof for the differentiable case appears in [3].

3. PROOF OF THEOREM 1

We shall need the following lemma:

Lemma 1. *Let M be a topological n-manifold, let $U_1, \ldots, U_m, V_1, \ldots, V_m$ be open subsets of M such that $\operatorname{Cl} V_i \subset U_i$ and $(M - \operatorname{Cl} V_i, U_i - \operatorname{Cl} V_i)$ is k_i-connected, where $k_i \leqq n - 3$ if $k_i > 0$, $1 \leqq i \leqq m$. Let $E_1, \ldots, E_m$ be closed subsets of M such that $E_i \subset V_i$ and $\partial M \subset \bigcup_{i=1}^{m} E_i$. Let $g : \mathbf{R}^n \to M$ be a homeomorphism and let $0 < \alpha < 1$. If $k_1 + \ldots + k_m + m \geqq n + 1$, there are compact sets $C_1, \ldots, C_m$ in M such that $C_i \cap (E_i \cup \partial M) = \emptyset$, $1 \leqq i \leqq m$ and homeomorphisms h_i of M onto itself such that $h_i(x) = x$ if $x \notin C_i$, $1 \leqq i \leqq m$, and $g(C^n_\alpha) \subset \bigcup_{i=1}^{m} h_i(U_i)$.*

Proof. Let G be the simplicial complex determined by a simplicial subdivision of $\mathbf{R}^n$ such that C^n_a is the set of points of a subcomplex K of G, $|N(K, G)| \subset \operatorname{Int} C^n_1$, and for any simplex $\Delta \in G$ such that $g(\Delta) \cap E_i \neq \emptyset$, we have $g(\Delta) \subset V_i$, i.e.: $g(|N(g^{-1}(E_i), G)|) \subset V_i$, $1 \leqq i \leqq m$.

Let $L_0 = K$. We construct inductively two sequences $L_0, \ldots, L_{m-1}$, and $K_1, \ldots, K_{m-1}$ of simplicial complexes as follows: suppose L_{i-1} is defined. Let $K_i = \beta(L^{(k_i)}_{i-1})$, and let L_i be the complementary complex of K_i in $\beta(L_{i-1})$, $1 \leqq i \leqq m - 1$. Then $\dim L_i = n - i - (k_1 + \ldots + k_i)$. Thus $\dim L_{m-1} = n - m + 1 - (k_1 + \ldots + k_{m-1}) \leqq k_m$. Let $K_m = L_{m-1}$.

We now apply the Engulfing Theorem with respect to each K_i. Let $P_i = g^{-1}(g(|K_i| - \operatorname{Cl} V_i))$. Then P_i is a k_i-dimensional polyhedron in $\operatorname{Int} C^n_1 - g^{-1}(\operatorname{Cl} V_i)$, $g(P_i)$ is closed in $M - \operatorname{Cl} V_i$, and $g(P_i) - U_i$ is compact, so there are homeomorphisms h'_i of M onto itself and compact sets C'_i, $1 \leqq i \leqq m$, such that $h'_i(x) = x$ if $x \in C'_i$, and $g(P_i) \subset h'_i(U_i)$.

Let $W_i = g^{-1}(h'_i(U_i))$, $1 \leqq i \leqq m$. Then $|K_i| \subset W_i$. The barycentric subdivisions used in the definitions of the K_i and L_i imply that K_i and L_i are full subcomplexes of $\beta(L_{i-1})$, $1 \leqq i \leqq m - 1$. Applying the "stretching theorem", we construct inductively a sequence of homeomorphisms $S_{m-i} : C^n_1 \to C^n_1$ such that S_{m-i} is the identity on $C^n_1 - |N(K, G)| \cup |N(g^{-1}(E_i), G)|$, $1 \leqq i \leqq m - 1$, and

$$
\begin{aligned}
&|L_{m-2}| \subset S_{m-1}(W_{m-1}) \cup W_m\,,\\
&|L_{m-3}| \subset S_{m-2}(W_{m-2}) \cup S_{m-1}(W_{m-1}) \cup W_m\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&|K| = |L_0| \subset S_1(W_1) \cup \ldots \cup S_{m-1}(W_{m-1}) \cup W_m\,.
\end{aligned}
$$

For example, we construct S_{m-1}. In the notation of the "stretching theorem", let $U = W_{m-1}$, $V = W_m$,

$$L = K_{m-1} \cup \{\beta(\Delta) : \Delta \in L_{m-1} \text{ and } g(\Delta) \cap E_{m-1} \neq \emptyset\},$$

$$L^c = \{\Delta \in \beta(L_{m-2}) : \Delta \cap L = \emptyset\}, \quad F = (R^n - |N(K, G)|) \cup g^{-1}(E_{m-1}).$$

Note that $|L| \subset W_m$, L is full in $\beta(L_{m-2})$, and $\operatorname{Fr} |\beta(L_{m-2})| \subset |L| \cup |L^c|$. Let S_{m-1} be the homeomorphism S obtained in the "stretching theorem".

We lift the homeomorphisms S_i onto M: let $\hat{S}_i : M \to M$ be defined by $\hat{S}_i(p) = g \circ S_i \circ g^{-1}(p)$, if $p \in g(C^n_1)$, and $\hat{S}_i(p) = p$, otherwise, $1 \leqq i \leqq m$. It follows that

$$g(C^n_\alpha) = g(|K|) \subset \hat{S}_1 \circ h'_1(U_1) \cup \ldots \cup \hat{S}_{m-1} \circ h'_{m-1}(U_{m-1}) \cup h'_m(U_m).$$

Let $h_i = \hat{S}_i \circ h'_i$, $1 \leqq i \leqq m - 1$, and let $h_m = h'_m$. Let $C_i = C'_i \cup \operatorname{Cl} g(|N(K, G) - N(g^{-1}(E_i), G)|)$.

Proof of Theorem 1. Let $g_j : R^n \to M$, $j = 1, 2, \ldots$, be a sequence of homeomorphisms such that $\operatorname{Int} M = \bigcup_{j=1}^{\infty} g_j(C^n_{1/2})$. Suppose we have constructed m sequences $\{f_{i,0}, \ldots, f_{i,k}\}$, $1 \leqq i \leqq m$, of homeomorphisms of M onto itself such that

$$\bigcup_{j=1}^{k} g_j(C^n_{1/2}) \subset \bigcup_{i=1}^{m} f_{i,k}(V_{i,2k}), \quad \text{and} \quad f_{i,j|V_{i,2j-2}} = f_{i,j-1|V_{i,2j-2}},$$

$1 \leqq j \leqq k$, where $f_{i,0} = \mathrm{id}_M$, $1 \leqq i \leqq m$.

We apply Lemma 1 with $E_i = \operatorname{Cl} V_{i,2k}$, $V_i = V_{i,2k+1}$, $U_i = V_{i,2k+2}$, and $g = g_{k+1}$ to get homeomorphisms $f_{i,k+1}$, $1 \leqq i \leqq m$, of M onto itself such that

$$\bigcup_{j=1}^{k+1} g_j(C^n_{1/2}) \subset \bigcup_{i=1}^{m} f_{i,k+1}(V_{i,2k+2}) \quad \text{and} \quad f_{i,k+1|V_{i,2k}} = f_{i,k|V_{i,2k}}.$$

Let $h_i(x) = \lim_{k \to \infty} f_{i,k}(x)$ for all $x \in M$.

4. PROOF OF THEOREM 2

Let $g_j : R^n \to M$, $j = 1, 2, \ldots$, be a sequence of homeomorphisms such that $\operatorname{Int} M = \bigcup_{j=1}^{\infty} g_j(C^n_{1/2})$. Let f_0 be the h_1 of Corollary 1. We construct inductively a sequence $f_0, f_1, f_2, \ldots$ of homeomorphisms of $N_1 \times [0, \infty)$ into M such that for each $j \geqq 1$, $\bigcup_{i=1}^{j} g_i(C^n_{1/2}) \subset f_j(N_1 \times [0, j + 1])$, and $f_{j|N_1 \times [0,j]} = f_{j-1|N_1 \times [0,j]}$. Let $h : N_2 \times [0, \infty) \to M$ be a collaring such that $h(N_2 \times [0, \infty)) \cap (g_{j+1}(C^n_{1/2}) \cup f_j(N_1 \times [0, j + 2])) = \emptyset$. Let $M_j = M - f_j(N_1 \times [0, j + 1))$. By Theorem 1, there are homeomorphisms V_1 and V_2 of M_j onto itself which are the identity on a neighborhood of the boundary of M_j such that $M_j \subset V_1(f_j(N_1 \times [j + 1, j + 2)) \cup \cup V_2(h(N_2 \times [0, \infty)))$. Let $f_{j+1|N_1 \times [0,j+1]} = f_{j|N_1 \times [0,j+1]}$, $f_{j+1|N_1 \times [j+1,\infty)} = V_2^{-1} \circ \circ V_1 \circ f_{j|N_1 \times [j+1,\infty)}$. Then $M = f_{j+1}(N_1 \times [0, j + 2)) \cup h(N_2 \times [0, \infty))$. Since $g_{j+1}(C^n_{1/2}) \cap h(N_1 \times [0, \infty)) = \emptyset$, we have $\bigcup_{i=1}^{j+1} g_i(C^n_{1/2}) \subset f_{j+1}(N_1 \times [0, j + 2))$. Let $f = \lim_{j \to \infty} f_j$. Then $M - N_2 = f(N_1 \times [0, \infty))$.

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