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FREE PRODUCTS OF ABELIAN l-GROUPS

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Introduction. All l-groups (lattice-ordered groups) in this paper will be abelian. In [5] the author developed the construction of free products in various varieties of l-groups, and showed in the abelian case that the free product $A \amalg B$ is non-isomorphic to $A \boxplus B$, the cardinal sum, unless $A = 0$ or $B = 0$. For the most part our structure theorems in this context will be for the special case of the free product of two Archimedean o-groups (totally ordered groups). Our main theorems (2.2, 2.3 & 2.11) say that if A and B are Archimedean o-groups then $A \amalg B$ has no singular elements and no basic elements, and each pair $0 < a \in A$ and $0 < b \in B$ have uncountably many values in common. If $A \boxplus B$ satisfies a certain geometric condition then $A \amalg B$ can be represented as an l-group of continuous functions on the closed unit interval, in such a way that all $0 < a \in A$ $(0 < b \in B)$ are given as monotone decreasing (increasing) differentiable functions; this is true in particular of $\boldsymbol{Z} \amalg \boldsymbol{Z}$. Under the same geometric condition, we show that *each* nonzero element of $A \amalg B$ has uncountably many values. We generalize some of our results to arbitrary free products: theorems 2.2 & 2.3 hold for all free products.

The paper is essentially self-contained, except for occasional references to [5]. The terminology and notation is also that of [5], with one exception which we shall point out. The symbols $(\subset) \subseteq$ are used for (proper) containment of sets. If two elements y and z in a partially ordered set (p. o. set) P are incomparable we write $y \parallel z$.

1. Preliminaries. If $\{A_\lambda \mid \lambda \in \Lambda\}$ is a family of l-groups, the free product $A = \amalg\{A_\lambda \mid \lambda \in \Lambda\}$ is an l-group together with l-homomorphisms u_λ: $A_\lambda \to A$ (which turn out to be l-embeddings) called co-projections, having the property that if ϕ_λ: $A_\lambda \to B$ is any family of l-homomorphisms into the l-group B, there is a unique l-homomorphisms ϕ of A into B such that $u_\lambda \phi = \phi_\lambda$ $(\lambda \in \Lambda)$. We shall consider free products of two l-groups A and B, written $A \amalg B$, and we shall suppress the co-projections and think of A and B as l-subgroups of $A \amalg B$.

The following results are proved in [5].

i) In $G = A \amalg B$ a typical element $g \in G$ is of the form

$$g = \vee_{\alpha} \wedge_{\beta} a(\alpha, \beta) + b(\alpha, \beta),$$

with $a(\alpha, \beta) \in A$ and $b(\alpha, \beta) \in B$, and where the indicated joins and meets are over finite sets. This notation is different from [5].

ii) If $0 < a \in A$ and $0 < b \in B$ then $0 < a \wedge b \in G$ and $a \parallel b$.

iii) G contains the cardinal sum $A \boxplus B$ as a subgroup but not as an l-subgroup, unless $A = 0$ or $B = 0$.

iv) $A \boxplus B$ is an l-homomorphic image of G by a map whose kernel K is generated by $\{a \wedge b \mid 0 < a \in A,\ 0 < b \in B\}$. $K \neq 0$ unless $A = 0$ or $B = 0$.

v) G is the l-ideal generated by $A \boxplus B$ in G.

vi) If ϕ is an l-homomorphism of $A \boxplus B$ into the l-group C, it has a unique extension to an l-homomorphism $\bar{\phi}$ of G into C.

A p. o. group A is *semiclosed* if for all $a \in A$, $na \geqq 0$ for some positive integer n implies that $a \geqq 0$. For each semiclosed p. o. group A let $\Phi(A)$ denote the free l-group (in Weinberg's sense) on A. Then

vii) for o-groups A and B, $A \amalg B = \Phi(A \boxplus B)$.

Suppose $F(X)$ denotes the free l-group over the set X; we then have:

viii) if X is the disjoint union of X_1 and X_2 then

$$F(X) = F(X_1) \amalg F(X_2).$$

The proofs to almost all of these are to be found in § 1 of [5].

2. The structure of the free product of two Archimedean o-groups. Throughout this section A and B will be Archimedean o-groups; they are as such by Hölder's theorem, o-subgroups of $\boldsymbol{R}$, the additive group of reals with the natural order. G will denote $A \amalg B$.

In an l-group L an element $0 < s \in L$ is *singular* if $0 \leqq g < s$ implies that $g \wedge s - g = 0$. Notice that a nonzero image of a singular element under an l-homomorphism is singular. If H is a p. o. group we call $x_1, x_2, \ldots, x_n \in H$ *positively independent* if whenever $\sum_{i=1}^{n} k_i x_i \leqq 0$, with each k_i a non-negative integer, then each $k_i = 0$. This is equivalent to saying that all the x_i's are positive in some order extension of the cone of H. The notion of positive independence is due to Bernau ([1]).

Recall that in the free l-group $\Phi(H)$ over the semiclosed p. o. group H a typical element is of the form $x = \vee_{\lambda} \wedge_{\mu} h(\lambda, \mu)$, where each $h(\lambda, \mu) \in H$ and the indicated joins and meets are finite. Bernau has proved the following rather crucial result.

2.1. Lemma. (Bernau, [1]) *Let $\Phi(H)$ be the free l-group over the semiclosed p. o. group H, and $x = \vee_\lambda \wedge_\mu h(\lambda, \mu) \in \Phi(H)$. Then $x^+ > 0$ if and only if there exists a λ_0 such that $\{h(\lambda_0, \mu)\}$ is positively independent.*

A *regular* l-ideal K of an l-group L is one which is maximal with respect to not containing some $0 \neq x \in L$; we say K is a *value* of x. $0 \neq a \in L$ is *special* if it has only one value. We can now state one of the principal results of this section.

2.2. Theorem. *Let $G = A \amalg B$; each $0 < a \in A$ and $0 < b \in B$ have uncountably many values in common.*

Proof. For each positive real number r we consider the following total order on $A \times B$. Put $(a, b) \in Q_r$ if $a + rb > 0$ (as real numbers) or $a + rb = 0$, and then $b \geqq 0$; this certainly defines an admissible total order on $A \times B$. Let $C_r = \{(a, b) \in A \times B \mid a + rb = 0\}$; C_r may be 0 for a given r, but at any rate it is a convex subgroup of $(A \times B, Q_r)$. For each r we may embed A and B as o-subgroups of $(A \times B, Q_r)$ in the obvious way: $a \to (a, 0)$ and $b \to (0, b)$. This extends to an l-homomorphisms ϕ_r of G onto $(A \times B, Q_r)$. We set $K_r = \operatorname{Ker}(\phi_r)$ and $G_r = C_r\phi_r^{-1}$; in general G_r may coincide with K_r but each G_r is a maximal l-ideal of G.

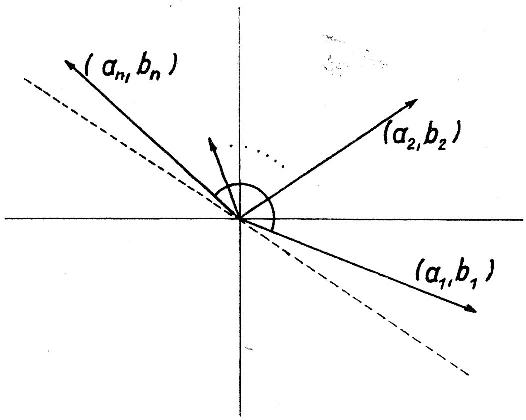
We proceed to show that the G_r are distinct. Since it is clear that each $0 < a \in A$ and $0 < b \in B$ have a value in each G_r our assertion will then be proved. We may suppose $r < s$ and select a rational number m/n $(m, n > 0)$ such that $r < m/n < s$, i.e. $-m + rn < 0$ while $-m + sn > 0$. (There is certainly no loss in generality in assuming 1 and hence $\boldsymbol{Z}$, the integers, to be contained in both A and B.) Then $(-m, n) \in Q_s \setminus C_s$ while $(-m, n) \in -Q_r$. If $z = (-m + n)^+ \in G$ then $z\phi_s \notin C_s$ but $z\phi_r = 0$, so that $z \in G_r \setminus G_s$.

Notice that each G_r is a value of $a \wedge b \in G$ also. Now $(a, 0) < (0, b)$ in $(A \times B, Q_r)$ if and only if $r \geqq a/b$. In particular if $r > a/b$ then $b - a \wedge b \notin G_r$, so $a \wedge b$ and $b - a \wedge b$ have (uncountably many) values in common, and therefore cannot be disjoint. A similar argument works on a, and we've proved

2.3. Theorem. *If $0 < a \in A$ and $0 < b \in B$ then neither are singular in G.*

2.4. Lemma. *Suppose $0 \neq (a_i, b_i) \in A \times B$ $(i = 1, 2, \ldots, n)$ are given. If $\bigwedge_{i=1}^{n} (a_i, b_i) > 0$ relative to some Q_r then $\{(a_i, b_i)\}$ is positively independent. Conversely if this set is positively independent, and no positive real number r exists such that $(a_i, b_i) + r(a_j, b_j) = 0$ with $i \neq j$, then $\bigwedge_{i=1}^{n} (a_i, b_i) > 0$ relative to some Q_r.*

Proof. The first statement is obvious from the definition of positive independence. The converse argument is geometric: suppose $\{(a_i, b_i)\}$ is positively independent, and no $r > 0$ exists as specified above. The total angle in the plane determined by the (a_i, b_i) is less than 180°. Clearly an $r > 0$ can be found such that $\bigwedge_{i=1}^{n} (a_i, b_i) > 0$ relative to Q_r. In fact we can find $r_1, r_2 > 0$ with the property that $\bigwedge_{i=1}^{n} (a_i, b_i) \notin C$. for all $r_1 < r < r_2$.



Call two nonzero elements (a, b) and (c, d) in $A \times B$ *separated* if $(a, b) + r(c, d) = 0$ for some positive real number r. This is equivalent to saying that the origin lies in between the two points *on* the line through them.

2.5. Lemma. *If (a, b) and (c, d) form a positively independent, separated pair, then in G*

$$0 < n[(a + b)^+ \wedge (c + d)^+] < 1 + 1, \quad \text{all} \quad n = 1, 2, \ldots.$$

Proof. $0 < (a + b)^+ \wedge (c + d)^+$ by 2.1. For each $n > 0$ $(na - 1, nb - 1)$ and $(nc - 1, nd - 1)$ are not positively independent. (The angle between them through the first quadrant is larger than 180°.) Thus again by 2.1

$$[n(a + b) - (1 + 1)] \wedge [n(c + d) - (1 + 1)] \leqq 0,$$

or

$$n(a + b) \wedge n(c + d) \leqq 1 + 1.$$

Clearly then $n[(a + b)^+ \wedge (c + d)^+] < 1 + 1$.

The next theorem improves on 2.2 in the absence of separated, positively independent pairs. We still carry the notation of the proof of 2.2 We need a notion due to Bernau ([1]): an l-group L is *uniformly Archimedean* if for each positively independent subset $\{x_1, x_2, \ldots, x_n\}$ of L and $0 \leqq y \in L$ there is a positive integer N such that $\sum_{i=1}^{n} m_i x_i \nleqq y$ for all $m_i \geqq N$.

A subset of regular l-ideals of an l-group L is *plenary* if it has zero intersection and is a dual ideal of the set of all regular l-ideals.

2.6. Theorem. *For $G = A \amalg B$ the following are equivalent.*

1. *G is a subdirect product of reals.*
2. *G is Archimedean.*
3. *$A \boxplus B$ has no separated, positively independent pairs.*
4. *$\{G_r \mid r > 0\}$ is a plenary set.*
5. *$A \boxplus B$ is uniformly Archimedean.*

If any of the above hold each nonzero element of G has uncountably many values.

Proof. The implications 1. → 2. and 4. → 1. are trivial; 2. ↔ 5. is due to Bernau ([1]), and 2. → 3. follows from lemma 2.5. It suffices therefore to prove that 3. → 4.; we show in fact that every $0 < g \in G$ has uncountably many values in $\{G_r \mid r > 0\}$, whence the last statement is also proved.

Select $0 < g = \vee_\lambda \wedge_\mu a(\lambda, \mu) + b(\lambda, \mu)$, with $a(\lambda, \mu) \in A$ and $b(\lambda, \mu) \in B$. Since $G = \Phi(A \boxplus B)$ we have by lemma 2.1 that $\{(a(\lambda_0, \mu), b(\lambda_0, \mu))\}$ is positively independent for a suitable λ_0. Since we have no separated, positively independent pairs we can deduce from lemma 2.4 and its proof that uncountably real numbers $r > 0$ exist such that $\wedge_\mu(a(\lambda_0, \mu) + b(\lambda_0, \mu))^+ \notin G_r$. Clearly then $g \notin G_r$, for all those $r > 0$, and so our proof is complete.

If $A \boxplus B$ has no separated, positively independent pairs, we see from 2.6 that G has a representation as a subdirect product of reals. We shall investigate this embedding more closely; for the moment we study the general situation where $A \boxplus B$ may have separated, positively indepedent pairs. Using the notation established in the proof of 2.2 we consider $E = \cap\{G_r \mid r > 0\}$; E is an l-ideal of G, and in view of what has been said we may characterize it in several ways. First, E is the l-ideal of all elements of G having no values which are maximal l-ideals. Before showing this we need some terminology. Call $0 < g \in G$ *primary* if it is of the form $g = \bigwedge_{i=1}^{n} (a_i + b_i)^+$, and $\{(a_i, b_i) \mid 1 \leqq i \leqq n\}$ contains a separated pair. In view of lemma 2.5 an element of that form cannot have a maximal l-ideal as one of its values. We conclude then that if $g = \bigwedge_{i=1}^{n} (a_i + b_i)^+$ as specified above, then every such expression has the same property; in particular, the term *primary* is well defined. We remark here that if $0 < x \in G$ can be written as a join of primary elements, and $x = \vee_\lambda \wedge_\mu(a(\lambda, \mu) + b(\lambda, \mu))^+$ then each $x_\lambda = \wedge_\mu(a(\lambda, \mu) + b(\lambda, \mu))^+$ is primary if $x_\lambda > 0$.

If $0 < x \in G$ is a join of primary elements it cannot have a value which is a maximal l-ideal, and so $x \in E$. Conversely, if $0 < x \in E$ and $x = \vee_\lambda \wedge_\mu(a(\lambda, \mu) + b(\lambda, \mu))^+$, then by the proof of 2.6 each $x_\lambda = \wedge_\mu(a(\lambda, \mu) + b(\lambda, \mu))^+$ is primary, whenever $x_\lambda > 0$.

We now summarize the above discussion; if S is a subset of a group K, let $\langle S\rangle$ be the subgroup of K generated by S.

2.7. Proposition. *Let $G = A \amalg B$ and $E = \cap\{G_r \mid r > 0\}$. Then*

$$\begin{aligned} E &= \langle 0 < x \in G \mid x \text{ is a join of primary elements}\rangle \\ &= \{x \in G \mid x = 0 \text{ or no value of } x \text{ is a maximal } l\text{-ideal}\} \end{aligned}$$

$E = 0$ if and only if $A \boxplus B$ has no separated, positively independent pairs.

The object we wish to examine is G/E; certainly it is a subdirect product of reals using the representing σ: $G/E \to \Pi\{G/G_r \mid r > 0\}$ induced by the canonical maps σ_r: $G/E \to G/G_r$. We shall look at the real valued functions $(a + E)\,\sigma$ and $(b + E)\,\sigma$ for $0 < a \in A$ and $0 < b \in B$. For each $r > 0$ we have the natural o-isomorphism $G/G_r \simeq (A \times B, Q_r)/C_r$; identifying G/G_r with $(A \times B, Q_r)/C_r$ we see that $(a + E)\,\sigma_r$ is determined by the projection of a on the line $\{(x, rx) \mid x \in \boldsymbol{R}\}$; thus

$$(a + E)\,\sigma_r = a/(r^2 + 1)^{1/2}, \quad \text{likewise} \quad (b + E)\,\sigma_r = br/(r^2 + 1)^{1/2}.$$

It follows that σ represents $a + E$ (resp. $b + E$) as a bounded, monotone decreasing (resp. increasing) differentiable function, and G/E is represented as an l-group of bounded, continuous functions on the positive reals. If we extend the positive reals by 0 and ∞, we may extend the functions to be defined at these two points in a continuous manner. Topologically, this set of "extended" non-negative reals is isomorphic to the closed unit interval $[0, 1]$; we therefore have the following result.

2.8. Theorem. *Let $G = A \amalg B$ and $E = \cap\{G_r \mid r > 0\}$. Then G/E is l-isomorphic to an l-group of continuous real valued functions on the closed unit interval, such that each $0 < a \in A$ $(0 < b \in B)$ is represented as a strictly decreasing (increasing) differentiable function $a(t)$ $(b(t))$ satisfying $a(0) = a$ and $a(1) = 0$, $(b(0) = 0$ and $b(1) = b.)$*

In particular, if $A \boxplus B$ has no separated, positively independent pairs G has such a representation.

In the absence of separated, positively independent pairs it turns out that G has no singular elements. We shall state this result so, as to leave open the question of whether this is true without said assumption.

2.9. Theorem. *In G suppose $0 < x = \bigwedge_{i=1}^{n} (a_i + b_i)^+$ is given. If x is singular it is primary. Therefore, if $E = 0$, G has no singular elements.*

Proof. Suppose x is not primary; we know there are uncountably many $r > 0$ such that $x\phi_r = \bigwedge_{i=1}^{n} (a_i, b_i) \in Q_r \setminus C_r$. We can easily choose a positive real number s and

$(c, d) \in A \times B$ with the properties that $(c, d) \in Q_s \setminus C_s$ and $x\phi_s - (c, d) \in Q_s \setminus C_s$. Hence $y = x \wedge (c + d)^+ > 0$, and both y and $x - y$ have G_s as their value. We conclude that $y \wedge (x - y) > 0$, and so x is not singular.

If $E = 0$ then G has no primary elements, and as every positive element of G is of the form $\bigwedge_{i=1}^{n} (a_i + b_i)^+$ we conclude that G has no singular elements.

By now the reader must be wondering about what *can* be said about primary elements, with regard to their values for example. Suppose $0 < x = \bigwedge_{i=1}^{n} (a_i + b_i)^+$ is primary; we have a real number $r > 0$ such that (after reindexing if necessary) $a_i + rb_i = 0$ for all $i = 1, 2, \ldots, k$, and $\{(a_i, b_i) \mid 1 \leqq i \leqq k\}$ contains a separated pair. *We suppose in addition here that the* (a_i, b_i) $(1 \leqq i \leqq k)$ *are rationally linearly independent.* For each $(k-1)$-tuple $(s_2, s_3, \ldots, s_k)$ of positive real numbers we define a partial order on $A \times B$ by: $(x, y) \in \tilde{Q}(s_2, \ldots, s_k)$ if $x + ry > 0$, or $x + ry = 0$ and then $x = \sum_{i=1}^{k} q_i a_i$ while $q_1 + q_2 s_2 + \ldots + q_k s_k > 0$, or $q_1 + \sum_{i=2}^{k} q_i s_i = 0$ and $\sum_{i=2}^{\kappa} q_i s_i > 0$, or etc. ..., or $q_{k-1} s_{k-1} + q_k s_k = 0$ and $q_k \geqq 0$. (In view of the independence of the (a_i, b_i), $1 \leqq i \leqq k$, this partial order is well defined.) In each $\tilde{Q}(s_2, \ldots, s_k)$ the (a_i, b_i) are all positive, for $i = 1, \ldots, n$, since if $i = k + 1, \ldots, n$ then $a_i + rb_i > 0$. Moreover the (a_i, b_i), $1 \leqq i \leqq k$, are Archimedean equivalent.

Let $Q(s_2, \ldots, s_k)$ be any total order of $A \times B$ extending $\tilde{Q}(s_2, \ldots, s_k)$, and let $C(s_2, \ldots, s_k)$ be the common value in $(A \times B, Q(s_2, \ldots, s_k))$ of the (a_i, b_i), $1 \leqq i \leqq k$. Embed A and B in $(A \times B, Q(s_2, \ldots, s_k))$ in the obvious way, and let $\phi(s_2, \ldots, s_k)$ be the extension of these embeddings to G. Let

$$G(s_2, \ldots, s_k) = C(s_2, \ldots, s_k)\, \phi(s_2, \ldots, s_k)^{-1}.$$

For each $(k-1)$-tuple $G(s_2, \ldots, s_k)$ is a value in G of each $(a_i + b_i)^+$, $1 \leqq i \leqq k$, and hence of x. But the $G(s_2, \ldots, s_k)$ are distinct; the argument is similar to the one used to show the G_r are distinct. Thus x has uncountably many values.

And x is not singular: there are uncountably many $(k-1)$-tuples $(s_2, \ldots, s_k)$ for which

$$0 < (a_2, b_2) - (a_1, b_1) < (a_1, b_1) < (a_2, b_2) < \ldots < (a_k, b_k),$$

relative to $\tilde{Q}(s_2, \ldots, s_k)$. (All that's required is that $1 \leqq s_2 < 2$, and $s_i \geqq s_{i-1}$ for $i > 2$.) So let $y = x \wedge ((a_2 - a_1) + (b_2 - b_1))^+$; if the $k - 1$-tuple is as we have required then $y\phi(s_2, \ldots, s_k) = (a_2, b_2) - (a_1, b_1) > 0$ while $(x - y)\, \phi(s_2, \ldots, s_k) = (a_1, b_1) - [(a_2, b_2) - (a_1, b_1)] > 0$. Thus $(x - y) \wedge y > 0$ in G proving x is not singular.

We recapitulate the preceding:

2.10. Proposition. *Suppose* $0 < x = \bigwedge_{i=1}^{n} (a_i + b_i)^+$ *is primary in G. If the subset* *of (a_i, b_i) satisfying $a_i + rb_i = 0\ (r > 0)$ that contains a separated pair is linearly independent then x has uncountably many values, and is not singular.*

The author strongly suspects that each nonzero element of G does have uncountably many values, and that G has no singular elements at all, without any additional assumptions. The above methods do not seem to generalize to prove this. Apart from proposition 2.10 we can say one more thing about G from a global point of view. Recall that $0 < x$ in an l-group L is *basic* if the set $\{y \in L \mid 0 \leqq y \leqq x\}$ is a chain. The following important observation is now a simple consequence of our previous results.

2.11. Theorem. $G = A \amalg B$ *has no basic elements.*

Proof. It suffices without loss of generality to consider primary elements. Suppose therefore that $0 < x = \bigwedge_{i=1}^{n} (a_i + b_i)^+$ is primary. There is a unique positive real number r with the property that $a_i + rb_i \geqq 0$ for each $i = 1, \ldots, n$, and there is a separated pair among the (a_i, b_i) for which equality holds; we may take this pair to be (a_1, b_1) and (a_2, b_2). Let $(c, d) = (a_1, b_1) - (a_2, b_2)$; then $\{(a_i, b_i) \mid 1 \leqq i \leqq n\} \cup \{(c, d)\}$ and $\{(a_i, b_i) \mid 1 \leqq i \leqq n\} \cup \{-(c, d)\}$ are both positively independent sets. Hence $y = x \wedge (c + d)^+$ and $z = x \wedge (c + d)^-$ are strictly positive and disjoint in G. Clearly then x is not basic.

We close this section with a result on weak order units. (An element $0 < x \in L$, an l-group, is a *weak order unit* if $x \wedge y = 0$ implies $y = 0$.)

2.12. Proposition. *Suppose* $0 < x = \bigwedge_{i=1}^{n} (a_i + b_i)^+$ *in* G; *x is a weak order unit if and only if each $a_i \geqq 0$ and each $b_i \geqq 0$.*

Proof. Suppose each $(a_i, b_i) \geqq 0$ and let $0 < y \in G$, say $y = \vee_\lambda \wedge_\mu (c(\lambda, \mu) + d(\lambda, \mu))^+$. For a suitable λ_0, $\{(c(\lambda_0, \mu), d(\lambda_0, \mu))\}$ is positively independent. When we adjoin the (a_i, b_i) to this positively independent set we still end up with a positively independent set. It follows then that $x \wedge y > 0$, and so x is a weak order unit.

Conversely, suppose without loss of generality that $0 > a_1$. Choose $(c, d) \in A \times B$ as follows: if $r = -a_1/b_1$, pick $c > 0$ and $c + rd < 0$. By lemma 2.4 the pair $\{(a_1, b_1), (c, d)\}$, and hence the set $\{(a_i, b_i) \mid 1 \leqq i \leqq n\} \cup \{(c, d)\}$, is not positively independent, which implies that $x \wedge (c + d)^+ = 0$; x is not a weak order unit.

2.12.1. Corollary. *Suppose* $0 < g \in G$ *and* $g = \vee_\lambda \wedge_\mu (a(\lambda, \mu) + b(\lambda, \mu))^+$ *where for some λ_0 each $(a(\lambda_0, \mu), b(\lambda_0, \mu)) > 0$ in $A \boxplus B$; then g is a weak order unit.*

The converse of 2.12.1 is false; for example, if $A \boxplus B$ has no separated, positively independent pairs it is clear from theorem 2.8 that $g = |(-1 + 1)|$ is represented by a continuous function which is zero at one point in the interior of the interval only; it is therefore a weak order unit. But a representation of g as in corollary 2.12.1 would be represented by a continuous function which is strictly positive everywhere.

3. Generalizations. We attempt here to extend some of the results of § 2 to free products of arbitrary l-groups. To begin we record a fact about divisible hulls. If L is an l-group, its divisible hull $\hat{L}$ is an l-group by defining $\hat{x} \geqq 0$ in $\hat{L}$ if $n\hat{x} \geqq 0$ in L for a suitable positive integer multiple. The containment of L in $\hat{L}$ is as an l-subgroup. Categorically, the divisible hull may be characterized as follows: $\hat{L}$ is a divisible l-group and L is an l-subgroup of $\hat{L}$, while if α is an l-homomorphism of L into the divisible l-group D, there is a unique extension of α to an l-homomorphism $\hat{\alpha}$ of $\hat{L}$ into D. From this observation we can easily prove.

3.1. Lemma. *For l-groups A and B let $G = A \amalg B$. Then $\hat{G} = \hat{A} \amalg \hat{B}$.*

Proof. From the categorical point of view this is obvious. For the functor $L \to \hat{L}$ is co-adjoint to the full embedding functor E that embeds the category of divisible l-groups in the category of all l-groups. The divisible hull functor necessarily preserves all co-limits, and in particular free products. (See [6], p. 67, proposition 12.1.)

The reader is invited to furnish a more mundane proof, using the remark directly preceding this lemma.

The first result we generalize from § 2 is the following.

3.2. Theorem. *Let G be the free product of l-groups A and B, $0 < a \in A$ and $0 < b \in B$. Then a and b have uncountably many values in common, and they are not singular.*

Proof. Suppose the theorem holds whenever A and B are o-groups. If A and B are arbitrary and $0 < a \in A$, $0 < b \in B$, select prime l-ideals M and N of A and B respectively such that $a \notin M$ and $b \notin N$. The canonical l-homomorphisms $A \to A/M$ and $B \to B/N$ induce an l-homomorphism ϕ of G onto $A/M \amalg B/N$; note that $a\phi$, $b\phi > 0$. By our assumptions $a\phi$ and $b\phi$ have uncountably many values in common, and are not singular. The same conclusion must hold for a and b in G.

So we must prove our theorem for the case when A and B are o-groups. In view of lemma 3.1 and the well known one-to-one correspondence between the l-ideals of an l-group and those of its divisible hull, we may assume that A and B are divisible for the proof of the first statement.

Fix then $0 < a \in A$, $0 < b \in B$, and let M (N) be the value in A (B) of a (b) with cover $\overline{M}$ $(\overline{N})$. In view of divisibility $A \simeq A' \overrightarrow{\times} M_1 \overrightarrow{\times} M$ amd $B \simeq B' \overrightarrow{\times} N_1 \overrightarrow{\times} N$, where A' (resp. B') is an o-group isomorphic to $A/\overline{M}$ $(B/\overline{N})$, and M_1 (resp. N_1) is an o-group isomorphic to $\overline{M}/M$ $(\overline{N}/N)$. ($H \overrightarrow{\times} K$ denotes the lexicographic product of H over K.) Let $L = A' \overrightarrow{\times} B' \overrightarrow{\times} (M_1 \amalg N_1)$; we have an o-homomorphism of A into L by $a = (a', m_1, m) \to (a', 0, m_1)$ where $a' \in A'$, $m_1 \in M_1$ and $m \in M$. We have a similar o-homomorphism of B into L. Let $\psi\colon G \to L$ denote the induced l-homomorphism; it is onto since the images of A and B in L generate L. M_1 and N_1 are Archimedean o-groups, and so by theorem 2.2 $a\psi$ and $b\psi$ have uncountably many values in common in $M_1 \amalg N_1$, and hence in L. Thus a and b have the same property in G.

Certainly divisible l-groups have no singular elements, so we cannot take A and B divisible here; but we can embed G in its divisible hull $\hat{A} \amalg \hat{B}$, and then proceed with the construction of the previous paragraph. Using the same notation, e.g. $\psi\colon \hat{A} \amalg \hat{B} \to$ $\to L$ as described there, we recall that it's a consequence of the proof of 2.3 that $a\psi \wedge b\psi = (a \wedge b)\psi$ and $a\psi - a\psi \wedge b\psi$ have a value in common in $M_1 \amalg N_1$, and hence in L, so that $a \wedge b$ and $a - (a \wedge b)$ (both elements of G) have a value in common in $\hat{A} \amalg \hat{B}$. Then in G they must have a common value, proving that a is not singular in G; an identical argument applies to b. Our proof is now completed.

3.3. Theorem. *Suppose $G = A \amalg B$ is a subdirect product of reals. Then G has no basic elements, and so every nonzero element has infinitely many values.*

Moreover, if G is a subdirect product of integers it has no singular elements, and every nonzero element has uncountably many values.

Proof. Select $0 < g \in G$ and a value K of g which is maximal l-ideal in G. If $K \not\supseteq A$ then $K \cap A$ is a maximal l-ideal of A; if $K \supseteq A$ we pick any maximal l-ideal of A. At any rate we have a maximal l-ideal M of A contained in K, and likewise a maximal l-ideal N of B contained in K. Let $\eta\colon G \to A/M \amalg B/N$ be the onto l-homomorphism induced by the two canonical l-homomorphisms. Its kernel $K(M, N)$ is the l-ideal generated by M and N (corollary 5.3.1 in [5]), and we conclude immediately that $K(M, N) \subseteq K$. Thus $g\eta > 0$ and so by 2.11 it is not basic. But then g cannot be basic in G either.

Since G is Archimedean, an element $0 < x \in G$ is basic if and only if it's special. Therefore G has no special elements, and so all nonzero elements have infinitely many values. (This paragraph expresses what is merely standard lore in the theory of l-groups; proofs may be found in [2].)

If G is a subdirect product of integers then in the first paragraph of the proof K, M and N may be taken so that G/K, A/M and B/N are all cyclic. η is then onto $Z \amalg Z$, and since $Z \boxplus Z$ has no separated, positively independent pairs, we conclude from theorems 2.6 and 2.9 that $g\eta$ has uncountably many values and is not singular; the same two facts must be true of g in G.

3.3.1. Corollary. *If $F(X)$ is the free l-group on the set X then each nonzero element of $F(X)$ has uncountably many values, and $F(X)$ has no singular elements.*

Remark. The last assertion was first proved by CONRAD and MCALISTER in [3]. The proof of the corollary is a consequence of Weinberg's theorem to the effect that free l-groups are subdirect product of integers. ([7])

Obviously, if $G = A \amalg B$ is a subdirect product of reals then A and B are also; conversely if A and B are subdirect product of reals, is $A \amalg B$ a subdirect product of reals? Theorem 2.6 says that for Archimedean o-groups the answer is no unless $A \boxplus B$ has no separated, positively independent pairs. What if A and B are subdirect products of *integers*, can we then assert that $G = A \amalg B$ is also a subdirect product of integers? We give some partial affirmative answers.

3.4. Proposition. *Suppose A and B are cardinal sums of Archimedean o-groups. Then $G = A \amalg B$ has no basic elements. If furthermore, A and B are cardinal sums of copies of $\boldsymbol{Z}$ then G is a subdirect product of integers.*

Proof. Let $A = \boxplus_\lambda R_\lambda$ and $B = \boxplus_\mu S_\mu$, where each R_λ and S_μ is an Archimedean o-group. The regular l-ideals of A and B are therefore maximal. So if $0 < g \in G$ and K is a value of g in G we can, as in the proof of 3.3, find maximal l-ideals M and N of A and B respectively, contained in K. If η once again denotes the l-homomorphism of G onto $A/M \amalg B/N$ induced by the two canonical maps, we obtain that $g\eta > 0$. As before $g\eta$ is not basic and hence neither is g in G.

If the R_λ and S_μ are all cyclic, then $A/M \simeq \boldsymbol{Z} \simeq B/N$, and so since $\boldsymbol{Z} \amalg \boldsymbol{Z}$ is a subdirect product of integers (see 4.1, 4.2 and 4.2.1 in [5]), $g\eta$ has a value D such that $(\boldsymbol{Z} \amalg \boldsymbol{Z})/D$ is cyclic. But then g has a value in G with the same property; it follows that G is a subdirect product of integers.

The essential ingredient in the proof of 3.4 is the fact that a regular l-ideal of a cardinal sum of subgroups of $\boldsymbol{R}$ is maximal. Recall that an l-group L is called *hyper-Archimedean* if every l-homomorphic image of L is Archimedean. This is equivalent to the condition that every regular l-ideal of L be maximal ([2], theorem 2.4).

Suppose L is a hyper-Archimedean l-group which can be represented as a subdirect product of integers. Is it true that for each maximal l-ideal M of L, L/M is cyclic? The author has been unable to answer this question. If every regular l-ideal of L has a cyclic factor we call L an *absolute subdirect product of integers.* Small cardinal sums of copies of $\boldsymbol{Z}$ have this property.

3.5. Theorem. *If A and B are hyper-Archimedean l-groups then $G = A \amalg B$ has no basic elements. If moreover, A and B are absolute subdirect products of integers then G is a subdirect product of integers.*

Proof. Same as the proof of 3.4.

4. The functor. $\amalg B$. In this section we prove the following theorem.

4.1. Theorem. *Let C be an l-ideal of the l-group A, and B be any l-group. The l-homomorphism ϕ of $C \amalg B$ into $A \amalg B$ induced by the containment of C in A is one-to-one; however, if $(C \amalg B)\,\phi$ is an l-ideal of $A \amalg B$ then $C = A$.*

Proof. First assume A is an o-group and C is a convex subgroup. Without loss of generality we may assume A and B both are divisible; we can therefore write $A = M \overrightarrow{\times} C$, where M is an o-group isomorphic to A/C. Let $K = M \overrightarrow{\times} (C \amalg B)$; there are obvious l-embeddings of A and B in K. They induce an l-homomorphism θ of $A \amalg B$ into K. If $0 < g = \vee_\lambda \wedge_\mu c(\lambda, \mu) + b(\lambda, \mu) \in C \amalg B$ then

$$g\phi = \vee_\lambda \wedge_\mu c(\lambda, \mu) + b(\lambda, \mu) \in A \amalg B$$

and $g\phi\theta = (0, g) > 0$; it follows that $g\phi > 0$ and hence that ϕ is one-to-one.

If A is not an o-group we proceed as follows: let $0 < g \in C \amalg B = H$ and P be a prime l-ideal of H not containing g. Take a prime l-ideal Q of C contained in P: let $Q = P \cap C$ if $P \not\supseteq C$, otherwise any proper prime of C will do. The canonical map $C \to C/Q$ induces an l-homomorphism α: $H \to C/Q \amalg B$; since $P \supset Q$, $g\alpha > 0$. Let $Q' = \{a \in A \mid |a| \wedge |c| \in Q, \text{all } c \in C\}$; then Q' is a prime of A such that $Q' \cap C = Q$, and we have an o-embedding $C/Q \to A/Q'$; (note that $C/Q \simeq Q' + C/Q'$.) By the first part of our proof the induced l-homomorphism β: $C/Q \amalg B \to A/Q' \amalg B$ is one-to-one so that $g\alpha\beta > 0$. On the other hand if γ: $A \amalg B \to A/Q' \amalg B$ is the l-homomorphism induced by the canonical map $A \to A/Q'$ then $\phi\gamma = \alpha\beta$; the situation is described below by the corresponding commutative diagram.

$$\begin{tikzcd}
C \amalg B \arrow[r, "\phi"] \arrow[d, "\alpha"] & A \amalg B \arrow[d, "\gamma"] \\
C/Q \amalg B \arrow[r, "\beta"] & A/Q' \amalg B
\end{tikzcd}$$

Hence $g\phi\gamma > 0$ and so $g\phi > 0$, which shows ϕ is one-to-one in the this case.

To show that $(C \amalg B)\,\phi$ is not an l-ideal of $A \amalg B$, unless $C = A$, simply examine the following commutative diagram.

$$\begin{tikzcd}
C \amalg B \arrow[r, "\phi"] \arrow[d] & A \amalg B \arrow[d, "\pi"] \\
B \arrow[r] & A/C \amalg B
\end{tikzcd}$$

π is the l-homomorphism induced by the canonical map $A \to A/C$. The remaining two arrows are the "projection" of $C \amalg B$ on B and the containment of B in $A/C \amalg B$. If $(C \amalg B)\,\phi$ were an l-ideal of $A \amalg B$ then B would be an l-ideal of $A/C \amalg B$, which is impossible unless $A/C = 0$, that is $C = A$.

Summarizing then, the functor $.\amalg B$ preserves convex l-embedding as l-embeddings, though except for the trivial case, never as *convex* l-embeddings. It still leaves open the question of whether the functor preserves l-embeddings in general.

In § 4 of [5] we showed that projective (abelian) l-groups are subdirect products of integers; moreover free products of projective l-groups are projective. Applying the theorem just proved we get:

4.1.1. Corollary. *If A and B are l-ideals of projective l-groups then $A \amalg B$ is a subdirect product of integers. There are no singular elements in $A \amalg B$, and any nonzero element has uncountably many values.*

Bibliography

[1] *S. Bernau:* Free abelian lattice groups; Math. Annalen, *180*, pp. 48—59 (1969).
[2] *P. Conrad:* Lattice-ordered groups; Tulane University, 1970.
[3] *P. Conrad & D. McAlister:* The completion of a lattice-ordered group; J. of the Austral. Math. Soc., Vol *IX*, parts 1, 2, pp. 182—208 (1969).
[4] *L. Fuchs:* Partially Ordered Algebraic Systems; Pergamon Press (1963).
[5] *J. Martinez:* Free products in varieties of lattice-ordered groups; Czechoslovak Math. J., *22* (*97*) 1972, 535—553.
[6] *B. Mitchell:* Theory of Categories; Academic Press (1965).
[7] *E. Weinberg:* Free lattice-ordered abelian groups; Math. Annalen, *151*, pp. 187—199 (1963).
[8] *E. Weinberg:* Free lattice-ordered abelian groups II; Math. Annalen, *159*, pp. 217—222 (1965).

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