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ON SUMS IN GENERALIZED ALGEBRAIC CATEGORIES

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The generalized algebraic category is defined as follows: Let F and G be covariant set functors, i.e. functors from the category ***Set*** of all sets and mappings into itself. The generalized algebraic category $A(F, G)$ has as its objects (called algebras) all the pairs (X, ω_X) where X is a set and ω_X is a mapping $XF \to XG$ (which is called an operation). Morphisms from (X, ω_X) to (Y, ω_Y) are all mappings $f : X \to Y$ such that $\omega_X fG = fF\omega_Y$. We remark that the composition of $f : A \to B$ and $g : B \to C$ is written as fg and the image of a set X mapped by a functor F will be written XF.

The notion of the generalized algebraic category arose as a generalization of universal algebras. But in contrary to the categories of universal algebras these categories need not necessarily be always complete and cocomplete – the existence of limits and colimits depending on both the functors F and G. The products, equalizers and coequalizers have been investigated by V. TRNKOVÁ and P. GORALČÍK, P. PTÁK, and J. ADÁMEK and V. KOUBEK in their papers. (See [1], [2], [3], [5].)

A necessary and sufficient condition for the functors F and G is given in this paper so that the category $A(F, G)$ may have sums and another one for this category to have finite sums.

In Section I these conditions are stated together with several definitions and conventions necessary for the understanding of their formulation. In the following sections we give the proofs of these conditions. In Section II we introduce the other notions and conventions used, and we recall some known facts; in Section III we give some propositions which will be useful in Sections IV and V, where we give the proof proper of the above mentioned conditions.

Let us remark that generalized algebraic categories are usually defined more generally: instead of a single unary operation a whole set of operations of arbitrary arity is considered. Namely the categories $A(F, G, \{\delta_\lambda, \lambda \in \Lambda\})$ are considered, where Λ is a set, δ_λ are ordinal numbers, and F, G covariant set functors. For an arbitrary set M let us denote by Q_M the covariant functor $Hom(M, -) : \boldsymbol{Set} \to \boldsymbol{Set}$. Then we can define the category $A(F, G, \{\delta_\lambda, \lambda \in \Lambda\})$ as follows: the objects are all the pairs $(X, \{\omega^X_\lambda, \lambda \in \omega\})$, where X is a set and ω^X_λ are mappings $\omega^X_\lambda : XFQ_{\delta_\lambda} \to XG$. Morphisms from $(X, \{\omega^X_\lambda, \lambda \in \Lambda\})$ to $(Y, \{\omega^Y_\lambda, \lambda \in \Lambda\})$ are all mappings $f : X \to Y$ such that for every $\lambda \in \Lambda$, $\omega^X_\lambda fG = fFQ_{\delta_\lambda}\omega^Y_\lambda$.

It is easily seen, however, that the category $A(F, G, \{\delta_\lambda, \lambda \in \Lambda\})$ is isomorphic to the category $A(F', G)$, where $F' = \bigvee_{\lambda \in \Lambda} FQ_{\delta_\lambda}$. Limits and colimits exist in one of the two isomorphic categories if and only if they exist in the other. Therefore it suffices to treat the categories $A(F, G)$ defined above.

I wish to express my gratitude to V. Trnková for her helpful advice and to V. Koubek, who independently proved III.7.

I.

Let us denote by $\square$ the natural forgetful functor $\square : A(F, G) \to \textbf{\textit{Set}}$ such that $(X, \omega_X) \square = X$. All functors in this paper with exception of $\square$ are set functors.

Let us denote by C_0 the constant functor to the empty set O, for an arbitrary set M with $1 \notin M$ by C_{M1} the functor such that $OC_{M1} = M$ and for every $X \neq 0$, $XC_{M1} = 1$. Because the category $A(F, C_0)$ is either empty or its all objects are isomorphic (which depends on OF) we shall in this paper restrict our attention to the categories $A(F, G)$ where $G \neq C_0$.

We shall write "F preserves $\bigcup$" if the functor F preserves unions of arbitrary systems of sets, "F preserves $\bigvee$" if F preserves sums in $\textbf{\textit{Set}}$, "F preserves $\bigcup^*$" if F preserves counions (definition see in II), and "F preserves $\prod$" if F preserves products. Let us recall here that the functors Q_M where M is arbitrary, C_0, C_{01} and the functors naturally equivalent with them preserve $\prod$ and that they are the only ones with this property (see [6]). We recall the notion of small and big functors: a functor is small iff it is a colimit of a diagram the objects of which are covariant homfunctors. A functor is big iff it is not small.

Let us denote for a set X by $|X|$ the cardinality of X. We shall call a functor excessive iff there is a cardinal α such that for every set X, $|X| \geqq \alpha$ implies $|XF| > |X|$.

No small functor is excessive (see III), but there exist also big functors which are not excessive, e.g. some functors introduced in [4].

Let us remark that by sums we mean the sums of non-empty systems of objects in the usual categorical sense.

I.1. Theorem. *A necessary and sufficient condition for the category $A(F, G)$, where $G \neq C_0, C_{M1}$, to have (finite) sums is described in Table 1, where + means "it has (finite) sums", – means "it has not (finite) sums", and $\square$ "it has (finite) sums preserved by $\square$".*

Table 1.

$G \neq C_0, C_{M1}$		G preserves Π	G does not preserve Π	
			G preserves $\bigcup^*$	G does not preserve $\bigcup^*$
F preserves (finite) $\bigvee$		□	□	□
F does not preserve (finite) $\bigvee$	F preserves (finite) $\bigcup$	$+$	$+$	$-$
	F does not preserve (finite) $\bigcup$	$+$ iff F is not excessive	$-$	$-$

Remark. The case of categories $A(F, C_{M1})$ is easy – see IV.

II.

In this paper we shall work in the Gödel-Bernays set theory sometimes assuming the generalized continuum hypothesis, which will be indicated by GCH. As usual the class of all cardinal numbers will be denoted by Cn and the class of all ordinal numbers by On. A mapping $j : X \to Y$ is an inclusion iff for every $x \in X$ $xj = x$.

II.1. Definition. *A functor F preserves non-trivial inclusions* iff for every $j : X \to Y$ inclusion mapping such that $X \neq 0$, jF is an inclusion.

II.2. Lemma. *Every functor is naturally equivalent to a functor preserving non-trivial inclusions.*

Proof. see [1].

Because $F \simeq F'$ implies that the categories $A(F, G)$ and $A(F', G)$ with G arbitrary are isomorphic, we shall in this paper restrict our attention to the categories $A(F, G)$ where F preserves inclusions and $G \neq C_0$, and we shall not mention these assumptions in our propositions.

The sum of X and Y will be denoted by $\langle X \vee Y, i_X, i_Y \rangle$.

Let X be a set, $x \in X$. Then for an arbitrary Y, $k_x : Y \to X$ will denote the constant mapping to x.

II.3. Definition. Let $\{f_\alpha, \alpha \in A\}$ be a set of surjections with the same domain X, $X \neq 0$. *A surjection f with the domain X* is called a counion of $\{f_\alpha, \alpha \in A\}$ iff $(\forall x, y \in X)((xf = yf) \equiv (\forall \alpha \in A)(xf_\alpha = yf_\alpha))$.

We recall some properties of set functors:

II.4. Lemma. *A functor is small iff it is a factorfunctor of a disjoint union of a set of covariant homfunctors.*

II.5. Lemma. *Every functor preserves monomorphisms with non-empty domain and epimorphisms.*

Proof is easy.

II.6. Definition. *A functor F is connected* iff $|1F| = 1$.

Note. If $F \neq C_0$ and $a \in 1F$ put $XF_a = \{x \in XF, xf_X F = a\}$, where $f_X : X \to 1$. It is easily seen that F_a is a functor and $F = \bigvee_{a \in 1F} F_a$. (See [7].)

II.7. Definition. Let $\{f_\alpha, \alpha \in A\}$ be a system of mappings with the same domain X. It is called a *separating system* iff $A \neq 0$ and the mappings $\{f_\alpha, \alpha \in A\}$ are collection-wise monomorphic, i.e. for every $x, y \in X$, $x \neq y$ there exists $\alpha \in A$ such that $xf_\alpha \neq$ $\neq yf_\alpha$.

A functor F preserves separating systems iff $\{f_\alpha, \alpha \in A\}$ is a separating system implies that $\{f_\alpha F, \alpha \in A\}$ is a separating system.

II.8. Definition. A functor F is separating iff $A \cap B = 0$ implies $AF\, j_A F \cap$ $\cap\, BF\, j_B F = 0$, where $j_A : A \to A \cup B$ and $j_B : B \to A \cup B$ are inclusions.

Note. F preserves sums iff F is separating and preserves $\bigcup$.

Let us express 2 as a sum $\langle 1 \vee 1, \{i, i'\}\rangle$.

II.9. Definition. A point $u \in 1F$ is called a *distinguished point of a functor F* iff $u\, iF = u\, i'F$.

Lemma. *Let u be a distinguished point of F. Then for every $f, g : 1 \to X$, $u\, fF =$ $= u\, gF$.*

Proof is easy.

Convention. Let u be a distinguished point of F. Denote $u_X = u\, fF$ for an arbitrary $f : 1 \to X$.

Proposition. *Let $A \cap B = 0$ and $A \cup B \neq 0$, then $x \in AF\, j_A F \cap BF\, j_B F$ iff there exists a distinguished point $u \in 1F$ such that $x = u_{A \cup B}$.*

Proof. see (7).

Corollary. *A functor is separating iff it has no distinguished point.*

Let us denote by C_M a constant functor to a set M.

II.10. Proposition. *A functor F preserves $\bigvee$ iff there exists a set M such that $F \simeq I \times C_M$.*

A functor F preserves $\bigcup$ iff there exist sets M and L such that $F \simeq (I \times C_M) \vee C_L$.

Proof is evident.

II.11. Definition. A cardinal $\alpha \geqq 1$ is said to be *an unattainable cardinal of a functor F iff $\alpha F \neq \bigcup_{f:X\to\alpha,|X|<\alpha} XF\,fF$.*

We recall from [2]:

II.12. Notation. Let F be an arbitrary functor, $x \in XF$. Denote $\mathscr{I}_{x,X} = \{Y \subseteq X, x \in YF\, j_Y F\}$, where $j_Y : Y \to X$ is an inclusion.

Lemma. For every $x \in XF$ $\mathscr{I}_{x,X}$ is either a filter or $\mathscr{I}_{x,X} = \exp X$.

II.13. Notation. Denote $\|\mathscr{I}_{x,X}\| = \min \{|Y|, Y \in \mathscr{I}_{x,X}\}$. The number $\|\mathscr{I}_{x,X}\|$ will be called the *essential cardinality* of $\mathscr{I}_{x,X}$.

Lemma. *Let F be a functor, $x \in XF$. Then $\|\mathscr{I}_{x,X}\|$ is an unattainable cardinal of F.*

Proof. Let $\alpha = \|\mathscr{I}_{x,X}\|$, then there exists $Y \subseteq$ R with $|Y| = \alpha$ and $x \in YF\, j_Y F$ and thus $x = y\, j_Y F$ for some $y \in YF$. We shall show that $y \in YF - \bigcap_{f:Z\to Y,|Z|<|Y|} ZF\,fF$. Presume that there exists $f : Z \to Y$ with $|Z| < |Y|$ and $y \in ZF\,fF$. Then $y \in (Zf)\,F\,jF$ where $j : Zf \to Y$ is an inclusion. Thus $x \in (Zf)\,F(jj_Y)\,F$ which is a contradiction, for $|ZF| < |Y|$.

Notation. For a functor F let us denote by A_F the class of all unattainable cardinals of F.

II.14. Proposition. *A functor is small iff A_F is a set.*

Proof. see [4].

II.15. Proposition. (GCH). *Let $\alpha \geqq \aleph_0$ be an unattainable cardinal of F. Then $|\alpha F| \geqq 2^\alpha$.*

Proof. see [4].

II.16. Definition. Put $XF = \{\mathscr{I}, \mathscr{I}$ is a filter on $X\} \cup \{\exp X\}$. If $f : X \to Y$ then $U \in (\mathscr{I})\,fF$ iff $(\exists V \in \mathscr{I})\,(Vf \subseteq U \subseteq Y)$. Clearly F is a functor.

Lemma. Let F be a functor, $x \in XF$ and $f : X \to Y$ and let there exists $Z \in \mathscr{I}_{x,X}$ such that f/Z is a monomorphism. Then $\mathscr{I}_{xfF,Y} = (\mathscr{I}_{x,X}) fF$.

Proof. see [4].

II.17. Definition. Let F be a functor. If A, X are sets, $A \subseteq XF$, denote by $F_{\langle A,X \rangle}$ the following subfunctor G of F: for every set Y, $YG = \{y \in YF, (\exists a \in A)(\exists f : X \to Y)(y = a\, fF)\}$; if $g : Y \to Y'$ is a mapping, then $gG : YG \to Y'G$ is the domain-range restriction of gF.

III.

Let f and g be mappings with the same domain. We shall write $f \leqq g$ iff there exists a mapping h with $fh = g$. Notice that $\leqq$ defines a quasiordering.

III.1. Proposition. *The following properties of a functor F are equivalent*:

(a) *F is connected and preserves separating systems*;

(b) *F is connected and preserves counions*;

(c) *let X be a set and f be a mapping with domain XF, then there exists a mapping g with domain X such that $f \leqq gF$ and whenever $f \leqq g'F$ for some g' with domain X, then $g \leqq g'$*;

(d) *let X and Y be sets, $x \in XF$, $y \in YF$, then there exists a mapping f with domain $X \vee Y$ such that $x(i_X f)\, F = y(i_Y f)\, F$ and whenever $x(i_X g)\, F = y(i_Y g)\, F$ for some g with domain $X \vee Y$, then $f \leqq g$*;

(e) *let X be a set with $X \neq 0$, $x, y \in XF$, then there exists a set M such that $F_{\langle \{x,y\},X \rangle}$ is a subfunctor of Q_M.*

Proof. (a) $\equiv$ (b) $\equiv$ (c) $\equiv$ (d) see [2]. (a) $\equiv$ (e) follows easily from Lemma 5.1. and Definition 4.1. in [6].

III.2. Lemma. *Let F be for an arbitrary M a factorfunctor of Q_M, let F preserve counions and let F be connected. Then $F \simeq Q_N$ for some N.*

Proof. Let $\varepsilon : Q_M \to F$ be an epitransformation. Clearly $F = F_{\langle \{1_M \varepsilon^M\},M \rangle}$ and therefore by III.1. there exists a monotransformation $\mu : F \to Q_L$ for some L. Denote $(1_M)\, \varepsilon^M \mu^M = f : L \to M$ and $N = Lf$, and $j : N \to M$ the inclusion. Thus there exists unique f' with $f = f'j$. Let $r : M \to N$ be the retraction, i.e. $jr = 1_N$, put $r\varepsilon^N = n$. Then $n\mu^N = f'$. By the lemma of Yoneda there exists a transformation $\tau : Q_N \to F$ such that $(1_N)\, \tau^N = n$. As μ is a monotransformation, $(1_M)\, \varepsilon^M \mu^M = f = f'j = n\mu^N jQ_L = n\, jF\mu^M$ implies that $(1_M)\, \varepsilon^M = n\, jF$. Hence we obtain easily that since ε is an epitransformation so is τ. Further let $g\tau^X = h\tau^X$ for some $g, h : N \to X$. Then $f'h = f'hQ_L = n\mu^N hQ_L = n\, hF\mu^X = h\tau^X \mu^X = g\tau^X \mu^X = n\mu^N gQ_L = f'g$. Since f is an epimorphism, it follows that $g = h$. Thus τ is also a monotransformation and $F \simeq Q_N$.

III.3. Construction. For a functor F preserving inclusions and a set X with a collectionwise epimorphic system of mappings $\{v_i : X_i \to X,\ i \in I\}$ and a set M let us construct the following transfinite sequence:

$$W_0 = X \times \{0\},$$
$$W_1 = W_0 \cup ((W_0F - \bigcup_{i\in I} X_iFv_iF) \times M \times \{1\}),$$
$$W_{\alpha+1} = W_\alpha \cup ((W_\alpha F - \bigcup_{\beta\in\alpha} W_\beta F) \times M \times \{\alpha + 1\}) \text{ for } \alpha \geqq 1,$$
$$W_\alpha = \bigcup_{\beta\in\alpha} W_\beta \text{ for a limit ordinal } \alpha.$$

$X, \{v_i, i \in I\}$ and M will be called parameters. We shall say that the sequence $\{W_\alpha, \alpha \in On\}$ stop iff there exists $\alpha \in On$ with $W_\alpha = W_{\alpha+1}$.

III.4. Lemma. *Let F be a functor preserving inclusions such that for every cardinal α there exists a cardinal β with $|\beta F| \leqq \beta$ and $\beta > \alpha$ and if γ is a cardinal with $\operatorname{conf}\beta \leqq \gamma \leqq \beta$ then γ is not an unattainable cardinal of F. Then every sequence* III.3. *with arbitrary parameters stops.*

Proof. Let M, X and $\{v_i, i \in I\}$ be parameters. For $\alpha = \max(|X|, |M|)$ let β be a cardinal with all the assumed properties. We shall show by the transfinite induction that for every $\delta \in On$ with $\delta \leqq \beta$ we have $|W_\delta| \leqq \beta$. As $|W_0| \leqq \alpha < \beta$ and $|\beta F| \leqq \beta$ we have $|W_1| \leqq \beta$. Let $\delta \in On$, $1 < \delta \leqq \beta$ such that for every $\xi \in \delta$, $|W_\xi| \leqq \beta$.

(a) If δ is a limit ordinal, then $|W_\delta| = |\bigcup_{\xi\in\delta} W_\xi| \leqq \beta$.

(b) If $\delta = \xi + 1$ then $|W_{\xi+1}| = |W_\xi \cup ((W_\xi F - \bigcup_{\eta\in\xi} W_\eta F) \times M \times \{\xi + 1\})| \leqq \beta$.

We shall show that $\bigcup_{\delta\in\beta} W_\delta F = W_\beta F$ and thus $W_\beta = W_{\beta+1}$: for any $x \in W_\beta F$, $\|\mathscr{I}_{x,W_\beta}\|$ is an unattainable cardinal of F and $\|\mathscr{I}_{x,W_\beta}\| \leqq \beta$, thus $\|\mathscr{I}_{x,W_\beta}\| > \operatorname{conf}\beta$. So there exists $Y \subseteq W_\beta$ such that $|Y| < \operatorname{conf}\beta$ and $x \in YF$. There must exist $\delta < \beta$ with $Y \subseteq W_\delta$ and thus $x \in W_\delta F$.

III.5. Corollary. *Let F be a small functor preserving inclusions, then every sequence* III.3. *with arbitrary parameters stops.*

Proof. For every α we shall obtain β by setting $\beta = \max(\alpha, \sup A_F)'$, where $'$ denotes "the succesor of a cardinal".

III.7. Proposition (by V. Koubek) (**GCH**). *Let F be a functor preserving inclusions which is not excessive. Then every sequence* III.3. *with arbitrary parameters stops.*

Proof. Presume that there exist parameters $M, X, \{v_i, i \in I\}$ such that the sequence $\{W_\delta, \delta \in On\}$ constructed in III.3. does not stop, then by III.5. there exists a cardinal α such that every cardinal $\beta > \alpha$ is singular provided it is not an unattainable cardinal of F and $|\beta F| \leqq \beta$. As F is not excessive there exists a cardinal $\beta > \max(\alpha, |M|, |X|, \aleph_0)$ with $|\beta F| \leqq \beta$. By II.15. (**GCH** is assumed) β is not an unattainable cardinal of F. Thus β is singular. As $W_\beta \neq W_{\beta+1}$ there exists $x \in W_\beta F - \bigcup_{\gamma \in \beta} W_\gamma F$. For every $V \in \mathscr{I}_{x,W_\beta}$ and for every $\gamma \in \beta$ we have $V \nsubseteq W_\gamma$ and therefore $W \cap W_\delta \neq 0$ for every $\delta \in \beta$. We can prove by the transfinite induction in the same way as in III.4. that $|W_\beta| \leqq \beta$. As $\|\mathscr{I}_{x,W_\beta}\|$ is an unattainable cardinal of F (see Lemma II.13.) we have $\|\mathscr{I}_{x,W_\beta}\| < \beta$. Thus there exists $U \in \mathscr{I}_{x,W_\beta}$ with $|U| < \beta$. We shall show by the transfinite induction that there is a mapping $f : W_\beta \to \beta$ such that f/U is a monomorphism and $\sup W_\beta f = \beta$. As β is a singular cardinal and $|U| < \beta$, there exists a sequence $\{Y_\xi, \xi \in \operatorname{conf} \beta\}$ such that $\beta = \bigcup_{\xi \in \operatorname{conf} \beta} Y_\xi$ and for every $\xi \in \operatorname{conf} \beta$ we have $|Y_\xi| > \beta$ and $Y_\xi \subseteq Y_{\xi+1}$ and for limit ordinal ξ, $Y_\xi = \bigcup_{\eta \in \xi} Y_\eta$ and $|Y_{\xi+1} - Y_\xi| \geqq |U|$ and $|Y_1| \geqq |U|$. For $\xi \in \operatorname{conf} \beta$ denote $T_\xi = \bigcup_{\gamma \in Y_\xi} W_\gamma$. As $|U| \leqq |Y_1|$ there is a mapping $f_1 : T_1 \to Y_1$ such that f_1/U is a monomorphism. If $\xi \in \operatorname{conf} \beta$ and for every $\eta \in \xi$ such $f_\eta : T_\eta \to Y_\eta$ are defined that $f_{\eta+1}/U_\eta = f_\eta$ and $(V_{\eta+1}) f_{\eta+1} \cap (Y_{\eta+1} - Y_\eta) \neq 0$ and $f_\eta/(T_\eta \cap U)$ are monomorphisms, then we can define f_ξ as follows: if ξ is a limit ordinal, then $f_\xi = \bigcup_{\eta \in \xi} f_\eta$ and if $\xi = \eta + 1$ then as $|U| \leqq |Y_{\eta+1} - Y_\eta|$ there exists a mapping $f_{\eta+1} : T_{\eta+1} \to Y_{\eta+1}$ such that $f_{\eta+1}/(T_{\eta+1} - T_\eta) \cap U$ is a monomorphism and $f_{\eta+1}/T_\eta = f_\eta$. Thus for every $V \in \mathscr{I}_{x,X}$, $\sup Vf = \beta$. If we suppose **GCH** then for $Z = \bigcup_{\delta \in \beta} 2^\delta$ we have $|Z| = \beta$ because β is a singular cardinal and then $\beta = \aleph_\lambda$ where λ is a limit ordinal. We shall show that $|ZF| > \beta$, which is a contradiction. Define a mapping $\varphi : 2^\beta \to ZF$ as follows: for $h : \beta \to 2$ put $h\varphi = x fF\mu_h F$ where μ_h is a mapping $\beta \to 2$ defined for $\delta \in \beta$ as $\delta\mu_h = h/\delta$. Show that φ is a monomorphism: presume that there are $h_1, h_2 \in 2^\beta$ with $h_1 \neq h_2$ and $x(f\mu_{h_1}) F = x(f\mu_{h_2}) F = z$. Then for every $S \in \mathscr{I}_{xfF,\beta}$, $S\mu_{h_1} \cap S\mu_{h_2} \in \mathscr{I}_{z,Z}$ (see II.12.). For $\delta = \min\{\delta' \in \beta, \delta' h_1 = \delta' h_2\}$ we have $(S \cap \delta)\, \mu_{h_1} = S\mu_{h_1} \cap S\mu_{h_2}$. Since μ_{h_1} is obviously a monomorphism, by Lemma II.16. it follows that $\mathscr{I}_{z,Z} = (\mathscr{I}_{xfF,\beta})\, \mu_{h_1} F$; hence $(S \cap \delta) \in \mathscr{I}_{xfF,\beta}$. But for every $S \in \mathscr{I}_{xfF,\beta}$, $\sup S = \beta$ because f/U is a monomorphism and then by Lemma II.16. $\mathscr{I}_{xfF,\beta} = (\mathscr{I}_{x,X})\, fF$ and thus for every $S \in \mathscr{I}_{xfF,\beta}$ there exists $V \in \mathscr{I}_{x,X}$ with $Vf \subseteq S$.

III.8. Lemma. *Let F be an excessive functor preserving inclusions, let $\alpha \geqq \max(\aleph_0, |1F|)$ be a cardinal such that $|X| \geqq \alpha$ implies $|XF| > |X|$. Then the sequence* III.3. *does not stop provided parameters X, M, $\{v_i, i \in I\}$ fulfil one of the following conditions*:

(a) $\alpha \leqq |X|$, $|I| \leqq |X|$ *and for every* $i \in I$, $|(\mathrm{R}_i v_i) F| \leqq |X|$, $M \neq 0$;

(b) $\alpha \leqq |X|$, $I = 2$, $(X_0 v_0) F \cup (X_1 v_1) F \neq XF$ and $|X| = |X - (X_0 v_0 \cap X_1 v_1)|$ *and* $M \neq 0$.

Proof. First we shall show that $|XF - \bigcup_{i\in I} (X_i v_i) F| \geqq |X|$. If (a) is fulfilled, then as $|(X_i v_i) F| \leqq |X|$ and $|I| \leqq |X|$ we have $|\bigcup_{i\in I} (X_i v_i) F| \leqq |X|$. As $|X| \geqq \alpha$, $|XF| > |X|$ and thus $|XF - \bigcup_{i\in I} (X_i v_i) F| = |XF| > |X|$. If (b) is fulfilled than clearly either $|X_1 v_1 - X_0 v_0| = |X|$ or $|X_0 v_0 - X_1 v_1| = |X|$ for $|X| \geqq \aleph_0$. Assume that $|x_1 v_1 - - X_0 v_0| = |X|$. Let $\bigcup_{r\in R} Y_r = X_1 v_1 - X_0 v_0$ be a disjoint decomposition with $|R| = |X|$ and $|Y_r| = |X|$ for every $r \in R$. Then there exist isomorphisms $g_r : (X_1 v_1 - X_0 v_0) \to \to Y_r$. For every $r \in R$ define a mapping $f_r : X \to X$ as follows: $xf_r = xg_r$ if $x \notin X_0 v_0$, $xf_r = x$ if $x \in X_0 v_0$. There exists $\xi \in XF - ((X_0 v_0) F \cup (X_1 v_1) F)$. Clearly for every $U \in \mathscr{J}_{\xi,X}$, $U \cap (X - X_0 v_0) \neq 0$ and $U \cap (X - X_1 v_1) \neq 0$. Verify that for every $r \in R$, $\xi f_r F \in XF - ((X_0 v_0) F \cup (X_1 v_1) F)$: if e.g. $\xi f_r F \in (X_0 v_0) F$, in other words $X_0 v_0 \in \mathscr{J}_{\xi f_r F, X}$, then there would exist $U \in \mathscr{J}_{\xi,X}$ with $Uf_r \subseteq X_0 v_0$, for by Lemma II.16. $(\mathscr{J}_{\xi,X}) fF = \mathscr{J}_{\xi f_r F, X}$. But for every $U \in \mathscr{J}_{\xi,X}$, $U \cap (X_1 v_1 - X_0 v_0) \neq 0$ and therefore also $Uf_r \cap (X - X_0 v_0) \neq 0$. If $r_1, r_2 \in R$ and $r_1 \neq r_2$ then $\xi f_{r_1} F \neq \xi f_{r_2} F$ for if $\xi f_{r_1} F = \xi f_{r_2} F$ then $\xi f_{r_1} F \in (X_0 v_0 \cup Y_{r_1}) F \cap (X_0 v_0 \cup Y_{r_2}) F = (X_0 v_0 \cup (Y_{r_1} \cap \cap Y_{r_2})) F = (X_0 v_0) F$, which would be a contradiction. Thus $|XF - ((X_0 v_0) F \cup \cup (X_1 v_1) F)| \geqq |R| = |X|$. Further we shall show by the transfinite induction for both cases (a) and (b) together that for every $\delta \in On$, $|W_{\delta+1} F - W_\delta F| > |X|$: This is true for $\delta = 0$ since $|W_1 F - W_0 F| = |(W_0 \cup ((XF - \bigcup_{i\in I} (X_i v_i) F \times M \times \{1\})) F - - W_0 F| \geqq ||(XF - \bigcup_{i\in I} X_i v_i F) F| - |1F|| \geqq |XF| > |X|$ for $|XF| > |X| \geqq \alpha \geqq |1F|$ and $(XF - \bigcup_{i\in I} X_i v_i F \times M \times \{1\}) \cap W_0 = 0$ and hence by II.9. $|(XF - \bigcup_{i\in I} (X_i v_i) F \times \times M \times \{1\}) F \cap W_0 F| \leqq |1F|$. Let δ be an ordinal such that for every $\gamma \in \delta$, $|W_{\gamma+1} F - W_\gamma F| > |X|$. Let δ be limit. We proved that $|W_1 F - W_0 F| \geqq |W_0 F|$, therefore, as $|W_\delta| \geqq |W_1 F - W_0 F|$, we have $|W_\delta| \geqq |W_0 F|$. A mapping $f_m : \bigcup_{0 \neq \beta \in \delta} W_\beta F \to \to W_\delta$ where $m \in M$, defined by $xf_m = (x, m, \delta)$ where $\delta = \min \{\delta', x \in W_\delta F\}$, is a monomorphism, therefore $|W_\delta| \geqq |\bigcup_{0 \neq \beta \in \delta} W_\beta F|$ and thus $|W_\delta F| > |W_\delta| \geqq |\bigcup_{0 \neq \beta \in \delta} W_\beta F|$. $\cdot\, |W_{\delta+1} - W_\delta F| \geqq ||(W_\delta F - \bigcup_{\beta\in\delta} W_\beta F) F| - |1F|| \geqq ||W_\delta FF| - |1F|| > |X|$. If $\delta = = \beta + 1$ then $|W_{\beta+2} F - W_{\beta+1} F| \geqq ||W_{\beta+1} F - W_\beta F| - |1F|| \geqq ||XF| - |1F|| > > |X|$. Thus the sequence $\{W_\delta, \delta \in On\}$ does not stop.

IV.

IV.1. Lemma. *The only functors assigning to every set either the empty set or a one-point set are C_0, C_1 and C_{01}.*

Proof. This is obvious since if $G \neq C_0$ then $XG \neq 0$ for every $X \neq 0$.

IV.2. Proposition. *$A(F, G)$ has sums or finite sums preserved by $\Box$ if and only if one of the following conditions is satisfied*:

(a) *F preserves sums or finite sums, respectively*;

(b) *$OF = 0$ and $G = C_{M1}$*;

(c) *$OF \neq 0$ and either $G = C_1$ or $G = C_{01}$.*

Proof. The sufficiency is easy to see. To prove the necessity assume that neither (b) nor (c) is satisfied, that is in other words that $G \neq C_{M1}$ for every set M. Then by IV.1. there exists $X \neq 0$ with $|XG| \geqq 2$ and thus we have $a, b \in XG$ with $a \neq b$. By II.8. Note, it suffices to verify that F is separating and preserves unions. First suppose that F is not separating. Let $u \in 1F$ be a distinguished point of F. Consider algebras $(X, k_a), (X, k_b)$, where k_a, k_b are constant mappings. Let $\langle X \vee X, i_1, i_2 \rangle$ be the sum of $\{X, X\}$ in ***Set***. Then there exists no operation $\omega : (X \vee X) F \to \to (X \vee X) G$ such that $\langle (X \vee X, \omega), i_1, i_2 \rangle$ is the sum of $\{(X, k_a), (X, k_b)\}$ in $A(F, G)$. Indeed if the contrary were true then $a i_1 G = b\, i_2 G$. The equality holds since $u_X\, i_1 F = u_X\, i_2 F = u_{X \vee X}$ and so $a i_1 G = u_X k_a\, i_1 G = u_X\, i_1 F = u_X\, i_2 F = = u_X k_b\, i_2 G = b i_2 G$. Since $X i_1 \cap X i_2 = 0$ and $a\, i_1 G \in (X i_1)\, G$ and $a i_2 G \in (X i_2)\, G$ we have $a\, i_1 G = a i_2 G = v_{X \vee X}$, where v is a distinguished point of G. Consequently $a = v_X = b$ as $i_1 G$ and $i_2 G$ are monomorphirms. This is a contradiction.

Second suppose that F is separating but does not preserve e.g. finite unions. Then there exist by II.8. disjoint sets X and Y such that $(X \cup Y)\, F - (XF \cup YF) \neq 0$ and $|(X \cup Y)\, G| \geqq 2$. Let $a, b \in (X \cup Y)\, G$ and $a \neq b$. Then, considering any two algebras (X, ω_X) and (Y, ω_Y) with underlying sets X and Y, we obtain two different direct bounds $\langle (X \cup Y, \omega_1), j_X, j_Y \rangle$ and $\langle (X \cup Y), \omega_2), j_X, j_Y \rangle$ where j_X, j_Y are inclusions $X \to X \cup Y$, $Y \to X \cup Y$, respectively. Define $\omega_1 / XF = \omega_2 / XF = \omega_X$ and $\omega_1 / YF = \omega_2 / YF = \omega_Y$, $\omega_1 / (X \cup Y)\, F - (XF \cup YF) = a$ and $\omega_2 / (X \cup Y)\, F - (XF \cup \cup YF) = b$. But obviously if $\langle (X \cup Y, \omega), j_X, j_Y \rangle$ is the sum of $\{(X, \omega_X), (Y, \omega_Y)\}$ then ω is the only operation such that $\langle (X \cup Y, \omega), j_X, j_Y \rangle$ is a direct bound.

V.

We emphasise again that we consider categories $A(F, G)$ where F preserves non-trivial inclusions, $G \neq C_0$ and in this section also $G \neq C_{M1}$ as this simple case is described in IV. We remark that $C_{11} = C_1 \simeq Q_0$.

V.1. Theorem. *Let F be not separating and let $A(F, G)$ have finite sums, then G is connected and preserves counions.*

Proof. Recalling III.1., we shall show that G satisfies III.1.(d). For arbitrary $X, Y, x_0 \in XG$ and $y_0 \in YG$ consider algebras (X, k_{x_0}) and (Y, k_{y_0}), where $k_{x_0} : XF \to \to XG$, $k_{y_0} : YF \to YG$ are constant mappings to x_0 and y_0, respectively. Then there exists their sum $\langle (S, \omega_S), v_X, v_Y \rangle$. Let $\langle X \vee Y, i_X, i_Y \rangle$ be the sum in ***Set***; we have $f : X \vee Y \to S$ with $i_X f = v_X$ and $i_Y f = v_Y$. This f is the mapping desired in III.I.(d). Actually, as F is not separating, the existence of a distinguished point $u \in 1F$ guarantees that $x_0(i_X f)\, G = y_0(i_Y f)\, G$, as $x_0(i_X f)\, G = u_X k_{x_0}(v_X)\, G = u_X v_X F \omega_S = u_S \omega_S = u_Y(v_Y F)\, \omega_S = u_Y k_{y_0} v_Y G = y_0(i_Y f)\, G$. If $g : X \vee X \to Z$ with $x_0(i_X g)\, G = y_0(i_Y g)\, G = z_0$, then $\langle (Z, k_{z_0}), i_X g, i_Y g \rangle$ forms a direct bound and hence $f \leqq g$ which completes the proof.

V.2. Theorem. *Let F be not separating and let F preserve unions or finite unions then $A(F, G)$ has sums or finite sums, respectively, if and only if G preserves counions and is connected.*

Proof. The necessity is a consequence of V.1. To prove the sufficiency consider any system of algebras $\{(X_\alpha, \omega_\alpha), \alpha \in A\}$. Denote by R the class of all mappings r with domain $\bigvee_{\alpha \in A} X_\alpha$ such that if B is the range of r then there exists an operation $\omega_B : BF \to BG$ such that $\langle (B, \omega_B), \{i_{X_\alpha} r, \alpha \in A\} \rangle$ forms a direct bound of $\{(X_\alpha, \omega_\alpha), \alpha \in A\}$. By III.1.(a) G is connected, which implies (via the fact that $(1, \omega_1)$, where ω_1 is the only mapping $1F \to 1G$, is the terminal object of the category $A(F, G)$) that $R \neq 0$. Obviously there exists a mapping f with the domain $\bigvee_{\alpha \in A} X_\alpha$, such that for $x, y \in \bigvee_{\alpha \in A} X_\alpha$, $xf = yf$ iff $xr = yr$ for every $r \in R$. Obviously, there exists a set of surjections $R' \subseteq R$, such that $f = \bigcup^* R'$. Denoting $S = Im\, f$, we shall show that there is an operation $\omega_S : SF \to SG$ such that $\langle (S, \omega_S), \{i_{X_\alpha} f, \alpha \in A\} \rangle$ is a sum of $\{(X_\alpha, \omega_\alpha), \alpha \in A\}$. As F preserves unions and f is onto S, in order to describe an operation ω_S we need only to define the values of ω_S in $x(i_{X_\alpha} f)\, F$ for $x \in X_\alpha F$, $\alpha \in A$. We put $x(i_{X_\alpha} f)\, F \omega_S = x \omega_\alpha (i_{X_\alpha} f)\, G$. This definition is correct since as soon as $x(i_{X_\alpha} f)\, F = x'(i_{X_{\alpha'}} f)\, F$, where $\alpha, \alpha' \in A$ and $x \in X_\alpha F$, $x' \in X_{\alpha'} F$, then $x(i_{X_\alpha} r)\, F = x'(i_{X_{\alpha'}} r)\, F$ for every $r \in R$, as evidently $f \leqq r$ implies $fF \leqq rF$. Hence $x \omega_\alpha (i_{X_\alpha} r)\, G = x' \omega_{\alpha'} (i_{X_{\alpha'}} r)\, G$ and, as G preserves counions also $x \omega_\alpha (i_{X_\alpha} f)\, G = x'(i_{X_{\alpha'}} f)\, G = x'(i_{X_{\alpha'}} f)\, G$. It is easily seen that $\langle (S, \omega_S), \{i_{X_\alpha} f, \alpha \in A\} \rangle$ forms a direct bound of $\{(X_\alpha, \omega_\alpha), \alpha \in A\}$. It is a sum since each direct bound of $\{(X_\alpha, \omega_\alpha), \alpha \in A\}$ can be represented as $\langle (B, \omega_B), \{i_{X_\alpha} r, \alpha \in A\} \rangle$ for some $r \in R$. Thus $f \leqq r$, in other words there exists φ with $f\varphi = r$ and so $i_{X_\alpha} f \varphi = i_{X_\alpha} r$ for every $\alpha \in A$. As f is onto S, such φ is unique. By an easy calculation we obtain that φ is a morphism, which completes the proof.

V.3. Proposition. *Let F not preserve unions or finite unions and let the category $A(F, G)$ have sums or finite sums, respectively. Then there exists a set M such that G is a factorfunctor of Q_M.*

Proof. In this argument it is essential that we make use of the following fact: Let G be a functor such that there is a set M and $m \in MG$ such that for every Y with $|Y| \geqq |M|$ and for every $y \in Y$ there is $\varphi : M \to Y$ with $m\varphi G = y$. Then G is a factorfunctor of Q_M. This holds as the transformation $\varepsilon : Q_M \to G$ such that $(1_M)\,\varepsilon^M = m$, existing in virtue of the lemma of Yoneda, is clearly an epitransformation.

So, let F not preserve unions, then it is easily seen that there exists a disjoint system of sets $\{X_i, i \in I\}$ such that for every $i \in I$, $|X_i| \geqq \aleph_0$ and $(\bigcup_{i\in I} X_i)\,F - \bigcup_{i\in I} X_i F \neq 0$. In case that F does not preserve finite unions we can suppose that $I = 2$.

(a) Turning first to F separating, choose arbitrarily operations $\omega_i : X_i F \to X_i G$ and denote by $\langle (S, \omega_S), \{\mu_i, i \in I\}\rangle$ the sum of $\{(X_i, \omega_i), i \in I\}$. Then for every Y with $|Y| \geqq |\bigcup_{i\in I} X_i|$ and $y \in YG$ consider a direct bound $\langle (Y, \omega_Y), \{\nu_i, i \in I\}\rangle$ such that for every $i \in I$, $\nu_i = i_{X_i}\psi$, where ψ is a monomorphism $\bigcup_{i\in I} X_i \to Y$ and for $x \in X_i F$ holds $x\nu_i F\omega_Y = x\omega_i\nu_i G$ while for $x \in YF - \bigcup_{i\in I} (X_i\nu_i)\,F$ we have $x\omega_Y = y$.

(b) If on the other hand F is not separating then the existence of sums guarantees that G is connected. Since if G were not connected, then by II.6. we could write $G = G_1 \vee G_2$. However, any two algebras (X, ω_X) and (Y, ω_Y) such that $u_X\omega_X \in XG_1$ and $u_Y\omega_Y \in YG_2$, where u is a distinguished point of F, have not even a direct bound. Hence, for arbitrarily chosen $x_i \in X_i$ we have $\{x_i\}\,G = \{a_i\}$. Put $\omega_i = k_{a_i}$ and denote by $\langle (S, \omega_S), \{\mu_i, i \in I\}\rangle$ the sum of $\{(X_i, \omega_i), i \in I\}$. Then for every Y with $|Y| \geqq |\bigcup_{i\in I} X_i|$ and for every $y \in YG$ consider a direct bound $\langle (Y, \omega_Y), \{\nu_i, i \in I\}\rangle$ such that for every $i \in I$, $\nu_i = i_{X_i}p\psi$, where p is the projection of $\bigcup_{i\in I} X_i$ on $\bigcup_{i\in I} X_i/\sim$, where $\sim$ is an equivalence defined by $a \sim b$ iff there exist $i, j \in I$ with $a = x_i$ and $b = x_j$, and ψ is a monomorphism $\bigcup_{i\in I} X_i/\sim\ \to Y$. An operation ω_Y is defined as follows: for $x \in X_i F$ it is $x\nu_i F\omega_Y = x\omega_i\nu_i G$ and for $x \in YF - \bigcup_{i\in I} (X_i\nu_i)\,F$, $x\omega_Y = y$.

It is easily seen that in both cases (a) and (b) there exists $s \in (\bigcup_{i\in I} X_i\mu_i)\,F - \bigcup_{i\in I} (X_i\mu_i)\,F$. According to the above argument, it suffices to prove that for every Y with $|Y| \geqq |\bigcup_{i\in I} X_i|$ and for every $y \in YG$ there exists $\varphi : S \to Y$ with $(s\omega_S)\,\varphi G = y$. We know that there exists $\varphi : S \to Y$ with $\mu_i\varphi = \nu_i$ for every $i \in I$. In both cases (a) and (b) it is easy to see that $\varphi/\bigcup_{i\in I} X_i\mu_i$ is a monomorphism and hence so is $\varphi F/\bigcup_{i\in I}(X_i\nu_i)\,F$. Therefore, if $s\varphi F \in (X_i\nu_i)\,F = (X_i\nu_i)\,F\varphi F$, then necessarily $s \in (X_i\mu_i)\,F$ which contradicts our assumption. Thus $s\varphi F \in YF - \bigcup_{i\in I} (X_i\nu_i)\,F$ and hence $y = s\varphi F\omega_Y = s\omega_S\varphi G$, which completes the proof.

V.4. Theorem. *Let F not preserve unions or finite unions and let $A(F, G)$ have sums or finite sums, respectively, then there exists a set N with $G \cong Q_N$.*

Proof. As a consequence of V.3. there exists a set M such that G is a factorfunctor of Q_M. If F is not separating we may then apply V.1. and hence we obtain the result by III.2.

Turning now to F separating we claim that for every X and for every $x \in XG$ it holds $\bigcap_{U \in \mathscr{J}_{x,X}} U \in \mathscr{J}_{x,X}$. Otherwise, in virtue of the fact that $\mathscr{J}_{x,X}$ is a filter, we obtain that for every $P \in \mathscr{J}_{x,X}$ there exists $p \in P$ with $P - \{p\} \in \mathscr{J}_{x,X}$. Further consider a disjoint collection of sets $\{X_\alpha, \alpha \in A\}$ such that $(\bigcup_{\alpha \in A} X_\alpha)F - \bigcup_{\alpha \in A} X_\alpha F \neq 0$ and for every $\alpha \in A$, $X \cap X_\alpha = 0$ and choose arbitrarily operations $\omega_\alpha : X_\alpha F \to X_\alpha G$. Denote by $\langle (S, \omega_S), \{\mu_\alpha, \alpha \in A\} \rangle$ the sum of $\{(X_\alpha, \omega_\alpha), \alpha \in A\}$. Without any loss of generality assume that also G preserves non-trivial inclusions. Put $Y = X \cup (\bigcup_{\alpha \in A} X_\alpha)$ and define $\omega_Y : YF \to YG$ as follows: $\omega_Y / X_\alpha F = \omega_\alpha$ and $\omega_Y / (\bigcup_{\alpha \in A} X_\alpha) F - \bigcup_{\alpha \in A} X_\alpha F = k_x$ and $\omega_Y / YF - (\bigcup_{\alpha \in A} X_\alpha) F = k_y$, where $y \in (\bigcup_{\alpha \in A} X_\alpha) G$ is arbitrary. Clearly $\langle (Y, \omega_Y), \{j_\alpha, \alpha \in A\} \rangle$, where $j_\alpha : X_\alpha \to Y$ are inclusions, forms a direct bound of $\{(X_\alpha, \omega_\alpha), \alpha \in A\}$, and thus there exists unique $f : (S, \omega_S) \to (Y, \omega_Y)$ with $\mu_\alpha f = j_\alpha$ for every $\alpha \in A$. Evidently $\bigcup_{\alpha \in A} X_\alpha \subseteq Sf$ and so $(\bigcup_{\alpha \in A} X_\alpha) F \subseteq (SF) fF$. Therefore there exists $s \in SF$ with $s fF \in (\bigcup_{\alpha \in A} X_\alpha) F - \bigcup_{\alpha \in A} X_\alpha F$ and thus, since $x = s fF \omega_Y = s\omega_S fG \in (Sf)\, G$, it follows that $Sf \in \mathscr{J}_{x,Y}$. As $X \subseteq Y$ we obtain applying Lemma II.16. that $\mathscr{J}_{x,Y} = \{U \subseteq Y, U \cap X \in \mathscr{J}_{x,X}\}$. Now consider $g : Y \to Y$ with $ag = b$ and $bg = a$ and $cg = c$ for every $c \in Y$ with $c \neq a, b$. Verify that g is a morphism $(Y, \omega_Y) \to (Y, \omega_Y)$. If $t \in X_\alpha F$ for some $\alpha \in A$ we have $t\omega_Y gG = t\omega_Y = t\, gF\omega_Y$, since $g/X_\alpha = 1_{X_\alpha}$, and consequently $gG/X_\alpha G = 1_{X_\alpha G}$ and also $gF/X_\alpha F = 1_{X_\alpha F}$, and since $(X_\alpha F)\, \omega_Y \subseteq X_\alpha G$. If $t \in (\bigcup_{\alpha \in A} X_\alpha) F - \bigcup_{\alpha \in A} X_\alpha F$, then in view of $g/\bigcup_A X_\alpha = 1_{\bigcup_{\alpha \in A} X_\alpha}$, it suffices to check that $x\, gG = x$, for $t\omega_Y\, gG = x\, gG$ and $t\, gF\omega_Y = x$. Actually, since $g/(Sf - \{a, b\}) = 1_{Sf - \{a,b\}}$ and $x \in (Sf - \{a, b\})\, G$, it follows that $x\, gG = x$. If $t \in YF - (\bigcup_{\alpha \in A} X_\alpha) F$, then also $t\, gF \in YF - (\bigcup_{\alpha \in A} X_\alpha) F$ for g is a monomorphism, hence gF is a monomorphism and $(\bigcup_{\alpha \in A} X_\alpha) F\, gF = \bigcup_{\alpha \in A} X_\alpha F$. Thus $t\omega_Y\, gG = y\, gG$ and $t\, gF\omega_Y = y$. Since $y\, gG = y$ ($y \in (\bigcup_{\alpha \in A} X_\alpha)\, G$ and $g/\bigcup_{\alpha \in A} X_\alpha = 1_{\bigcup_{\alpha \in A} X_\alpha}$), it follows that $t\omega_Y\, gG = t\, gF\omega_Y$. Thus we have a morphism $g : (Y, \omega_Y) \to (Y, \omega_Y)$ with $fg \neq f$, for $a, b \in Sf$, and as, moreover, $\mu_\alpha fg = \mu_\alpha f$, we get a contradiction with the uniqueness of f. Therefore we may conclude that for every X and $x \in XG$ it holds $\bigcap_{U \in \mathscr{J}_{x,X}} U \in \mathscr{J}_{x,X}$.

Now, let $\varepsilon : Q_M \to G$ be an epitransformation, the existence of which is guaranteed by V.3. Denoting $m = (1_M)\, \varepsilon^M$ we have $m \in NG$ where $N = \bigcap_{U \in \mathscr{J}_{m,M}} U$. Then, in virtue of the lemma of Yoneda we have a transformation $\tau : Q_N \to G$ such that $(1_N)\, \tau^N = m$. It is easy to verify that since ε is an epitransformation, so is τ. Let us show that τ is also a monotransformation. Assume the contrary. Clearly, it remains to examine Q_N with $|N| \geqq 2$, since it is easily seen that the only factorfunctor of Q_0 is C_1, and the only factorfunctors of $Q_1 \simeq I$ are C_1 and I. Thus we obtain $\varphi \neq \psi$ such that φ, ψ :

$: N \to N$ and $m\varphi G = \varphi\tau^N = \psi\tau^N = m\psi G$, since for any $\varphi_1 \neq \psi_1$, $\varphi_1, \psi_1 : N \to X$ with $\varphi_1\tau^X = \psi_1\tau^X$, in virtue of the fact that 2 is a cogenerator in ***Set***, we have $\xi : X \to$ $\to N$ such that $\varphi_1\xi \neq \psi_1\xi$ and clearly $(\varphi_1\xi)\,\tau^N = (\psi_1\xi)\,\tau^N$. Denoting $Z = N \cup \bigcup_{\alpha\in A} X_\alpha$ (for the sake of simplicity we suppose that $N \cap \bigcup_{\alpha\in A} X_\alpha = 0$), define operations $\omega_1, \omega_2 :$ $: ZF - ZG$ as follows: $\omega_1/X_\alpha F = \omega_2/X_\alpha F = \omega_\alpha$ for every $\alpha \in A$, $\omega_1/ZF -$ $- (\bigcup_{\alpha\in A} X_\alpha)\,F = \omega_2/ZF - (\bigcup_{\alpha\in A} X_\alpha)\,F = k_y$ for some $y \in (\bigcup_{\alpha\in A} X_\alpha)\,G$, and $\omega_1/(\bigcup_{\alpha\in A} X_\alpha)\,F -$ $- \bigcup_{\alpha\in A} X_\alpha F = k_m$ (as $m \in NG$, it follows that $m \in ZG$), and $\omega_2/(\bigcup_{\alpha\in A} X_\alpha)\,F - \bigcup_{\alpha\in A} X_\alpha F =$ $= k_{m\varphi F}$. Let $\bar\varphi, \bar\psi : Z \to Z$ be such that $\bar\varphi/N = \varphi$, $\bar\varphi/\bigcup_{\alpha\in A} X_\alpha = 1_{\bigcup_{\alpha\in A} X_\alpha}$ and $\bar\psi/N = \psi$, $\bar\psi/\bigcup_{\alpha\in A} X_\alpha = 1_{\bigcup_{\alpha\in A} X_\alpha}$. Evidently $\bar\varphi, \bar\psi$ are morphisms $(Z, \omega_1) \to (Z, \omega_2)$ and $\bar\varphi \neq \bar\psi$. Denoting by $j_\alpha : X_\alpha \to Z$ the inclusions we obtain that obviously $\langle(Z, \omega_1), \{j_\alpha, \alpha \in A\}\rangle$ and $\langle(Z, \omega_2), \{j_\alpha, \alpha \in A\}\rangle$ form direct bounds of $\{(X_\alpha, \omega_\alpha), \alpha \in A\}$ and $j_\alpha\bar\varphi = j_\alpha =$ $= j_\alpha\bar\psi$ for every $\alpha \in A$. As $\langle(S, \omega_S), \{\mu_\alpha, \alpha \in A\}\rangle$ is a sum we have the unique morphism $h : (S, \omega_S) \to (Z, \omega_1)$ with $\mu_\alpha h\bar\varphi = \mu_\alpha h\bar\psi = j_\alpha$, it follows that $h\bar\varphi = h\bar\psi$. As there exists $v \in (\bigcup_{\alpha\in A} X_\alpha)\,F - \bigcup_{\alpha\in A} X_\alpha F$ we have $m = v\,hF\omega_1 = v\omega_S\,hF \in (Sh)\,F$, in other words $Sh \in \mathscr{I}_{m,Z}$. From the above, however, we have $N = \bigcap_{U\in\mathscr{I}_{m,Z}} U$ and hence $N \subseteq Sh$. This together with $\bar\varphi/N \neq \bar\psi/N$ and $h\bar\varphi = h\bar\psi$ yields a contradiction. Therefore $G \simeq Q_N$ holds.

V.5. Theorem (GCH). *Let F not preserve unions or finite unions, and $G = Q_M$, then $A(F, G)$ has sums or finite sums, respectively, if and only if F is not excessive.*

Proof. To prove the sufficiency consider an arbitrary system of algebras $\{(X_i, \omega_i),$ $i \in I\}$.

(a) First, if F is not separating we obtain in the same way as in V.2. the epimorphism $f : \bigvee_{i\in I} X_i \to S$. Whenever $\bigcup_{i\in I} (X_i i_{X_i} f)\,F = SF$ we may construct an operation $\omega_S :$ $: SF \to SG$ in the same way as in V.2. establishing the sum $\langle(S, \omega_S), \{i_{X_i}f, i \in I\}\rangle$ of $\{(X_i, \omega_i), i \in I\}$. Otherwise put $W_0 = S \times \{0\}$ and denoting $v_i = i_{X_i}f\psi$, where $\psi : S \to W_0$ is such that $x\psi = (x, 0)$ for every $x \in S$, define a partial operation ω on W_0F as follows: if $x \in X_i$ for some $i \in I$ put $xv_iF\omega = x\omega_i v_i Q_M$. The same discussion as in V.2. ensures, via the fact that Q_M is connected and preserves counions, that ω is correctly defined.

(b) Let F be separating. In case that $\bigcup_{i\in I} X_i i_{X_i} F = (\bigvee_{i\in I} X_i)\,F$ we obtain the sum preserved by the forgetful functor (see IV.2.). Otherwise, putting $S = \bigvee_{i\in I} X_i$ and $v_i = i_{X_i}\psi$, where $\psi : S \to S \times \{0\} = W_0$ is the same as above, define a partial operation ω as follows: for $x \in X_i F$ and $i \in I$ put $x(v_i F)\,\omega = x\omega_i v_i Q_M$.

In both cases (a) and (b) let us construct the transfinite sequence $\{W_\alpha, \alpha \in On\}$ (see III.3.) with parameters S, M and $\{v_i, i \in I\}$. Lemma III.7. guarantees that there exists $\alpha \in On$ with $W_\alpha = W_{\alpha+1}$, in other words, $W_\alpha F = \bigcup_{\beta\in\alpha} W_\beta F$. Denoting $W = W_\alpha$, let us complete the definition of the operation $\omega : WF \to WQ_M$. Let $x \in W_\alpha F$ and $\beta = \min\{\beta', x \in W_{\beta'}F\}$. If $\beta = 0$ and $x \in \bigcup_{i\in I} (X_i v_i) F$ then $x\omega$ is already defined in (a) or (b). Otherwise define $x\omega = f_x$, where $f_x : M \to W$ is such that $mf_x = (x, m, \beta + 1)$ for $m \in M$. Verify that $\langle (W, \omega), \{v_i, i \in I\}\rangle$ is the sum of $\{(X_i, \omega_i), i \in I\}$. Clearly it is a direct bound. Considering another direct bound $\langle (A, \omega_A), \{\mu_i, i \in I\}\rangle$ define a mapping $\varphi : W \to A$ by the transfinite induction: on W_0, define $xv_i\varphi = x\mu_i$ for $x \in X_i$, $i \in I$. Even if F is not separating, it is correct (see the discussion in V.2.). Now assume that φ is defined on W_β, then for $y \in W_{\beta+1} - W_\beta$, i.e. $y = (x, m, \beta + 1)$ for some $x \in W_\beta F - \bigcup_{\gamma\in\beta} W_\gamma F$ and $m \in M$, define $y\varphi = (m)\, x\varphi F\omega_A$. It is easy to verify that φ is morphism $(W, \omega) \to (A, \omega_A)$ and that φ is the unique morphism with $v_i\varphi = \mu_i$ for every $i \in I$, since it must satisfy $x\varphi F\omega_A = (x\omega)\, \varphi Q_M = f_x\varphi$ for every $x \in WF - \bigcup_{i\in I} (X_i v_i) F$.

To prove the necessity suppose that F is excessive, i.e. there exists $\lambda \in Cn$, $\lambda \geqq \max\{\aleph_0, |1F|\}$ such that $|X| \geqq \lambda$ implies $|XF| > |X|$.

If F does not preserve unions, then clearly there exists a disjoint system of sets $\{X_i, i \in I\}$ such that $(\bigcup_{i\in I} X_i) F \neq \bigcup_{i\in I} X_i F$ and for every $i \in I$, $|X_i| = \gamma$ and $|I| \geqq \max\{|\gamma F|, \lambda\}$.

If F does not preserve finite unions, it is easy to see that there exist sets X_0 and X_1 with $X_0 \cap X_1 = 0$ and $|X_0| = |X_1| \geqq \lambda$ and $(X_0 \cup X_1) F \neq X_0 F \cup X_1 F$.

For every $i \in I$ (in the finite case we mean $I = 2$) choose arbitrarily $x_i \in X_i$ and consider an operation $\omega_i : X_i F \to X_i Q_M$ such that for every $x \in X_i F$, $x\omega_i = k_{x_i} : M \to X_i$ (according to our convention $M \neq 0$). We shall prove that $\{(X_i, \omega_i), i \in I\}$ has not a sum.

If F is separating put $S = \bigcup_{i\in I} X_i$ and $v_i = i_{X_i}$.

If F is not separating, put $S = \bigcup_{i\in I} X_i/\sim$, where $x \sim y$ iff there exist $i, j \in I$ with $x = x_i$ and $y = x_j$, and $v_i = i_{X_i}p$, where p is the projection of $\bigcup_{i\in I} X_i$ on $\bigcup_{i\in I} X_i/\sim$.

Denote by ψ the mapping $S \to W_0$ such that $x\psi = (x, 0)$ for every $x \in S$. Now, it is easy to verify that the parameters M, S, $\{v_i\psi, i \in I\}$ satisfy either condition (a) or (b) (in the finite case) of Lemma III.8. and thus the sequence $\{W_\alpha, \alpha \in On\}$ (see III.3.) with parameters M, S, $\{v_i\psi, i \in I\}$ does not stop. We claim that for every $\alpha \in On$ there exists an operation $\omega_\alpha : W_\alpha F \to W_\alpha Q_M$ such that $\langle (W_\alpha, \omega_\alpha), \{v_i\psi, i \in I\}\rangle$ forms a direct bound. Indeed, for $x \in X_i F$ with $i \in I$ put $x(v_i\psi)\, F\omega_\alpha = x\omega_i(v_i\psi)\, Q_M$ and for $x \in W_0 F - \bigcup_{i\in I} (X_i v_i \psi) F$ or $x \in W_\beta F$ for some β with $0 \neq \beta \in \alpha$ define $x\omega_\alpha = f_x : M \to W_\alpha$ such that for every $m \in M$, $mf_x = (x, m, \beta + 1)$ where $\beta = \min\{\beta', x \in W_{\beta'}F\}$.

Elsewhere ω_α can be defined arbitrarily. ω_α is defined correctly even if F is not separating, since $x(v_i\psi) F = x\omega_i(v_i\psi) Q_M = k_{x_i}(v_i\psi) Q_M = k_{x_i}v_i\psi Q_M = k_{x_j}v_j\psi Q_M$ for every $i, j \in I$. Assume that $\{(X_i, \omega_i), i \in I\}$ has the sum $\langle(S, \omega_S), \{\mu_i, i \in I\}\rangle$. Then for every $\alpha \in On$ there exists unique $\varphi_\alpha : S \to W_\alpha$ with $\mu_i\varphi_\alpha = v_i\psi$ for every $i \in I$. To show that for every $\alpha \in On$ it is $|S\varphi_\alpha| \geqq \alpha$, prove first the following:

Let $f : (S, \omega_S) \to (W_\alpha, \omega_\alpha)$ be a morphism, $\alpha \in On$ and $A \subseteq Sf$, then for every $a \in AF$ and for every $m \in M$ it is $(m)\, a\omega_\alpha \in Sf$.

This is true, since in virtue of the lemma of Yoneda we have a transformation $\tau : Q_A \to F$ with $(1_A)\, \tau^A = a$. Denoting by $i_A : A \to W_\alpha$ the inclusion, we have $(i_A)\, \tau^{W_\alpha} = a$. As $A \subseteq Sf$, it follows that there exists $g \in SQ_A$ with $(g) fQ_A = i_A$, and since $a = (g) fQ_A\tau^{W_\alpha} = g\tau^S fF$ and $g\tau^S \in SF$, it follows that $(g\tau^S)\, \omega_S fQ_M = g\tau^S fF\omega_\alpha = a\omega_\alpha$ and thus for every $m \in M$ it is $(m)\, a\omega_\alpha = ((m)\, g\tau^S\omega_S) f$, in other words $(m)\, a\omega_\alpha \in Sf$.

Further, prove by the transfinite induction that for every $\alpha \in On$ and for every $\beta \in \alpha$ it is $W_\beta \subseteq S\varphi_\alpha$. Evidently for every $\alpha \in On$ we have $W_0 \subseteq S\varphi_\alpha$. Let $\beta \in \alpha$ and for every $\delta \in \beta$ let $W_\delta \subseteq S\varphi_\alpha$, then if β is a limit ordinal it follows that $W_\beta = \bigcup_{\delta\in\beta} W_\delta \subseteq S\varphi_\alpha$ and if $\beta = \delta + 1$, then $W_{\delta+1} = W_\delta \cup ((W_\delta F - \bigcup_{\gamma\in\delta} W_\gamma F) \times M \times \{\delta + 1\})$. From the above, for every $x \in W_\delta F - \bigcup_{\gamma\in\delta} W_\gamma F$ and $m \in M$ we have $m(x\omega_\alpha) = (x, m, \delta + 1) \in S\varphi_\alpha$ and hence $W_{\delta+1} \subseteq S\varphi_\alpha$.

Therefore for every $\alpha \in On$ we have $|S| \geqq |S\varphi_\alpha| \geqq \alpha$, which yields a contradiction and completes the proof.

Remark. If we do not suppose the generalized continuum hypothesis we can reformulate Theorem V.5. by substituting the property of F to be excessive by the following condition $\mathbf{R_M}$: We shall say that a functor F and a set M satisfy condition $\mathbf{R_M}$ iff for every set X and every collectionwise epimorphic system of mappings $\{v_i : X_i \to X, i \in I\}$ the sequence III.3. with parameters $M, X, \{v_i, i \in I\}$ will stop. We can prove by the same reasoning as in V.5. the following theorem:

Let F not preserve unions or finite unions and $G = Q_M$, then $A(F, G)$ has sums or finite sums, respectively, if and only if F and M satisfy $\mathbf{R_M}$.

Following III.6, every small functor and every set M satisfy $\mathbf{R_M}$, but there are also big functors satisfying $\mathbf{R_M}$ for every M, e.g. those from (4) cited above. Thus the characteristics of the property $\mathbf{R_M}$ without the assumption of generalized continuum hypothesis remains open.

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