Syed A. Huq

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A NOTE ON MUTANTS IN SEMIGROUPS

SYED A. HUQ, Canberra

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Dedicated to the memory of Professor HANNA NEUMANN.

1. INTRODUCTION

The aim of this work is to confirm the following conjecture of JIN BAI KIM [3].

Conjecture. *Any semigroup S has no decomposition $S = \bigcup M_i$ into a finite number of disjoint mutants M_i $(i = 1, 2, \ldots, n)$ of S.*

We acknowledge thanks to R. M. BRYANT who suggested that the crucial step of the proof must be some sort of use of Van der Waerden's theorem without which no proof could have been completed.

2. PROOF OF THE CONJECTURE

Definition. Let M be a subset of a semigroup S, then M is a *mutant* of S if and only if $M . M \subset S \setminus M$.

It is clear that the mutant of a semigroup S, do not contain any idempotent of S and the union of two mutants need not be a mutant.

Also if $T \subseteq S$, we define $E(T) = \{e \in T, e^2 = e\}$ and $A \setminus B = (a \in A : a \notin B)$.

Following Kim [3], we have to assume have to S is infinite and $E(S) = \emptyset$.

We notice that for any $s \in S$, $\{s, s^2, \ldots\}$ are all distinct. Hence the semigroup contains a set N, isomorphic to the set of positive integers Z^+.

If S admits a decomposition as $S = \bigcup_i M_i$ $(i = 1, 2, \ldots, n)$ of disjoint mutants M_i, so will do the set N, as $N = \bigcup_i N_i^*$; $N_i^* = N \cap M_i$ are the disjoint mutant components of N.

Hence it is sufficient to prove that the semigroup of positive integers Z^+, has no decomposition as $Z^+ = \bigcup M_i$ $(i = 1, 2, \ldots, n)$, the union of finite number of disjoint mutants M_i. To prove this we appeal to the following theorem [4], [5].

Van der Waerden's Theorem. *Let k and l be two arbitrary natural numbers. Then there exists a natural number $h(l, k)$ such that, if an arbitrary segment of length $h(l, k)$ of the sequence of natural numbers is divided in any manner into k classes (some of which may be empty), then an arithmetic progression of length l, appears in at least one of these classes.*

and prove the

Theorem. *For each positive integer n, there exists a positive integer $l(n)$ such that if*

$\{1,2, \ldots, l(n)\} = M_1 \dot{\cup} M_2 \ldots \dot{\cup} M_n$, is a decomposition into disjoint subsets M_i for $i = 1, 2, \ldots, n$, then at least one M_i is not a mutant.

This confirms our result in particular; since if the set of positive integers is decomposable as a union of disjoint mutants, so will be the subset $\{1, 2, \ldots, l(n)\}$.

Proof. For $n = 1$, choose $l(1) = 2$, then $M_1 = \{1, 2\}$ is not a mutant, since $1 + 1 \in M_1$.

Next assume $n > 1$, and that the assertion is true for $n - 1$.

Choose $h = h(l(n - 1), n)$ in Van der Waerden's theorem and assume $\{1, 2, \ldots \ldots, h\} = M_1 \dot{\cup} M_2 \ldots \dot{\cup} M_n$ is a decomposition into disjoint subsets.

By the theorem, one of the M_i say M_1 has an arithmetic progression of length $l(n - 1)$.

Hence M_1 contains $a, a + d, \ldots, a + l(n - 1) d$. If M_i are all mutants, M_1 must be disjoint with the set $\{d, 2d, \ldots, l(n - 1) d\}$ i.e., $D = \{d, 2d, \ldots, l(n - 1) d\} \subseteq \subseteq M_2 \dot{\cup} M_3 \ldots \dot{\cup} M_n$ i.e., $\{d, 2d, \ldots, l(n - 1) d\} = K_2 \dot{\cup} K_3 \ldots \dot{\cup} K_n$ where K_i are mutants, being $M_i \cap D = K_i$ i.e., $\{1, 2, \ldots, l(n - 1)\} = L_2 \dot{\cup} L_3 \ldots \dot{\cup} L_n$ where L_i $(i = 2, 3, \ldots, n)$ are disjoint mutants — a contradiction to the fact that the theorem is true for $n - 1$.

Further studies on mutants, like open ones in topological semigroups and (m, n) mutants in semigroups due to ISEKI [1] — will be left for the future.

References

[1] *K. Iseki:* On mutant sets in semigroups, Proc. Japan Acad. *38* (1962), p. 478—479.

[2] *A. A. Mullin:* Properties of mutants, Bull. Amer. Math. Soc. *67* (1961), p. 82.

[3] *J. B. Kim:* Mutants in semigroups, Czech. Math. Journal *19 (94)*, (1969), p. 86—90.

[4] *A. Y. Khinchin:* Three pearls of number theory, Garylock Press, Rochester, New York, (1962).

[5] *B. L. Van der Waerden:* Beweis einer Baudetschen Vermutang Nieuw. Arch. Wiskunde, *15* (1927), p. 212—216.

Author's address: Department of Mathematics, Institute of Advanced Studies, Australian National University, Canberra, ACT 2600, Australia.