Ján Jakubík

On σ-complete lattice ordered groups

Czechoslovak Mathematical Journal, Vol. 23 (1973), No. 1, 164–174

Persistent URL: http://dml.cz/dmlcz/101154

Czechoslovak Mathematical Journal, 23 (98) 1973, Praha

ON σ-COMPLETE LATTICE ORDERED GROUPS

JÁN JAKUBÍK, Košice

(Received May 12, 1972)

INTRODUCTION

An l-group G is said to be σ-complete if each bounded countable subset of G has the supremum and the infimum. The concept of a singular l-group was used by CONRAD and MCALLISTER [4]. The importance of singular l-groups is emphasized by the fact that each complete l-group is a direct product of a singular l-group and a vector lattice. ROTKOVIČ [15] examined σ-complete l-groups without semilinear elements. An l-group does not contain semilinear elements if and only if it is singular (Lemma 2.5.1).

An l-group G is called (conditionally) orthogonally complete if each (bounded) disjoint subset of G has the supremum. Analogously we can define orthogonal completennes of Boolean algebras. Orthogonally complete l-groups and vector lattices were studied in several papers (cf., e.g., PINSKER [12], BERNAU [1], CONRAD [3], JAKUBÍK [6]). It is well-known that an orthogonally complete Boolean algebra must be complete (SMITH - TARSKI [17]). On the other hand, simple examples show that an orthogonally complete l-group need not be complete. VEKSLER and GEJLER [19] have found necessary and sufficient conditions for a conditionally orthogonally complete vector lattice to be complete. In §2 we show that if a singular l-group is conditionally orthogonally complete and σ-complete, then it is complete.

Let α be an infinite cardinal. WEINBERG [20] proved that if G is the additive l-group consisting of all continuous real-valued functions defined on a Hausdorff completelly regular topological space (with the natural partial order) then G satisfies the following condition:

$(*)$ *If G is $(\alpha,2)$-distributive, then it is (α, α)-distributive.*

By using the decomposition of a complete l-group G into a direct product of a singular l-group and a vector lattice it was proved in [7] that each complete l-group G fulfils $(*)$. In §3 we prove that each archimedean l-group G with the decomposition property satisfies $(*)$. Lattice ordered groups with the decomposition property were studied by BERNAU [1]; for the case of vector lattices cf. VEKSLER and GEJLER [19]. RABINOVIČ [13], [14] examined the analogous notion of lattices with the decomposition property. Each lattice ordered group that is σ-complete and conditionally orthogonally complete has the decomposition property; therefore such an l-group fulfils $(*)$. The problem (proposed by Weinberg [20]) wheather $(*)$ holds for each l-group remains still open.

1. BASIC NOTIONS

For the standard notions concerning lattices and lattice ordered groups cf. BIRKHOFF [2] and FUCHS [5]. We denote lattice operations by $\wedge$ and $\vee$, the group operation is denoted by $+$ (though it need not be commutative). Let G be an l-group, $\emptyset \neq X \subset G$. We put

$$X^\delta = \{y \in G : |y| \wedge |x| = 0 \text{ for each } x \in X\} .$$

The set X^δ is said to be a polar of G. Each polar is a closed convex l-subgroup of G. Let $K^0(G)$ be the set of all polars of G; this system is partially ordered by the inclusion. $K^0(G)$ is a complete Boolean algebra and for each subset $\emptyset \neq \{A_i\} \subset K^0(G)$ the meet $\bigwedge A_i$ in $K^0(G)$ coincides with $\bigcap A_i$ (ŠIK [18]). For $g \in G$ we denote $\{g\}^{\delta\delta} = [g]$.

Let A, B be convex l-subgroups of G such that $A \cap B = \{0\}$ and $A + B = G$. Then each element $g \in G$ can be written uniquelly as $x = a + b$ with $a \in A$, $b \in B$; the elements a, b are components of g in A or B, respectively. It is easy to verify that each operation $\circ \in \{\wedge, \vee, +\}$ in G is performed componentwise. The l-group G is said to be a direct product of its l-subgroups A, B; in symbols $G = A \otimes B$. The l-groups A, B are direct factors of G. The component of x in A will be denoted by $x(A)$. In the case $A = [g]$ for some $g \in G$ we write $x(A) = x[g]$.

An l-group G is said to have the decomposition property if $G = X^\delta \otimes X^{\delta\delta}$ for each $\emptyset \neq X \subset G$ (cf. JAMESON [10]; another terminology is used by BERNAU [1]).

Let $\{G_i\}$ $(i \in I)$ be a system of l-groups and let ΠG_i be their direct product. Let H be an l-subgroup of ΠG_i such that for each $i \in I$ and each $g_i \in G_i$ there exists $h \in H$ with the property $h(i) = g_i$, $h(j) = 0$ for each $j \in I$, $j \neq i$. Then H is said to be a completely subdirect product of l-groups G_i.

Let $\{H_i\}_{i \in I}$ be a system of l-subgroups of an l-group G such that each H_i is a direct factor of G. Assume that the mapping $\varphi(g) = (\ldots, g(H_i), \ldots)_{i \in I}$ is an isomorphism of G into ΠH_i such that $\varphi(G)$ is a completely by subdirect product of l-groups H_i. Then G is called a completely by subdirect product of its l-subgroups H_i.

Elements $x, y \in G$ are called disjoint if $|x| \wedge |y| = 0$. A system $X \subset G^+$ is said to be disjoint if any two distinct elements of X are disjoint.

An element $0 < e \in G$ is a weak unit of G if $e \wedge |x| = 0$ implies $x = 0$ for each $x \in G$. A system $\{A_i\}$ $(i \in I)$ of convex l-subgroups of G is disjoint if for any pair i, j of distinct elements of I and each $a_i \in A_i$, $a_j \in A_j$ we have $|a_i| \wedge |a_j| = 0$.

Let L be a lattice and let α, β be cardinals. Let T, S be sets satisfying card $T \leqq \alpha$, card $S \leqq \beta$. L is said to be $(\wedge, \vee) - (\alpha, \beta)$-distributive, if the equation

$$\text{(d)} \qquad \bigwedge_{t\in T} \bigvee_{s\in S} x_{t,s} = \bigvee_{\varphi\in S^T} \bigwedge_{t\in T} x_{t,\varphi(t)}$$

holds in L identically whenever all joins and meets standing in (d) exist in L. The $(\vee, \wedge) - (\alpha, \beta)$-distributivity is defined dually. If L satisfies both these laws then it is called (α, β)-distributive.

Let B be a Boolean algebra and let $X(B)$ be the Boolean space of B. We denote by $F(B)$ the system of all integer valued functions f on $X(B)$ such that for each integer n, the set $\{x \in X(B) : f(x) = n\}$ is clopen in $X(B)$. Then $F(B)$ (with the natural partial order) is an additive lattice ordered group.

2. SINGULAR l-GROUPS

Let G be an l-group. An element $0 < s \in G$ is called singular if $s \wedge (s - x) = 0$ for each $x \in G$, $0 \leqq x \leqq s$ (Conrad - McAllister [4]). Also, s is singular if and only if the interval $[0, s]$ is a Boolean algebra [7]. G is said to be singular if for each $0 < g \in G$ there is a singular element $s \in G$ such that $0 < s \leqq g$.

The following two propositions are known (cf. Birkhoff [2], Chap. XIV, Thm. 17 and Jameson [10], Proposition 2.5.6).

2.1. *Each σ-complete l-group is archimedean and commutative.*

2.2. *Let G be a σ-complete l-group, $0 < a \in G$. Then $G = \{a\}^\delta \otimes \{a\}^{\delta\delta}$.*

2.3. *Let $G \neq \{0\}$ be a σ-complete l-group and let $\{x_i\}$ be a maximal disjoint system of strictly positive elements of G, $H_i = [x_i]$. Then G is a complete subdirect product of l-subgroups H_i.*

Proof. $\{H_i\}_{i\in I}$ is a maximal disjoint system of convex l-subgroups $\neq \{0\}$ of G and according to 2.2 each H_i is a direct factor of G. Hence the mapping

$$\varphi : x \to (\ldots, x(H_i), \ldots)_{i\in I}$$

is a homomorphism of G into ΠH_i. Let $y \in \varphi^{-1}(0)$, $y \geqq 0$. Then $y(H_i) = 0$, thus $y \wedge x_i = 0$ for each $i \in I$. This implies $y = 0$. Therefore $\varphi^{-1}(0) = \{0\}$ and so φ is an isomorphism of G into H_i. Let $i \in I$, $h_i \in H_i$. Then $h_i(H_i) = h_i$ and $h_i(H_j) = 0$ for each $j \in I$, $j \neq i$. Hence $\varphi(G)$ is a completely subdirect product of l-groups H_i.

We denote by $S(G)$ the system of all singular elements of G.

2.4. *Let G be a singular l-group and let $\{x_i\}$ be a maximal disjoint system of $S(G)$. Then $\{x_i\}$ is a maximal disjoint system of G.*

Proof. Let $0 \leqq y \in G$ be disjoint with each x_i. If $0 < y$, then there is $s \in S(G)$ with $0 < s \leqq y$ and so the element s is disjoint with each x_i, a contradiction. Therefore $y = 0$.

Obviously for each $0 < g \in G$, the element g is a weak unit of $[g]$.

2.5. *Let G be a σ-complete singular l-group. Then G is a completely subdirect product of l-groups H_i $(i \in I)$ where each H_i is a σ-complete singular l-group with a weak unit e_i such that e_i is singular.*

The proof follows from 2.3, 2.4 and from the fact that each direct factor of a singular and σ-complete l-group is singular and σ-complete.

An element $x \neq 0$ of an l-group G is called semilinear (Rotkovič [15]) if for each $x' \in G$ with $0 < x' \leqq |x|$ there exists $y \in G$ such that

$$0 < 2y \leqq x' .$$

2.5.1. *Let G be an l-group. The following conditions are equivalent:*

(a) *G is singular.*

(b) *G does not contain semilinear elements.*

Proof. Let G be singular, $0 \neq x \in G$. Then there is a singular element $0 \neq x' \in G$ with $x' \leqq |x|$. Let $y, z \in G$, $0 < y \leqq x'$, $x' = y + z$. We have $x' = y \vee z$, $y \wedge z = 0$, therefore $2y \wedge z = 0$ and hence by using distributivity of G,

$$x' \wedge 2y = y ,$$

thus $2y$ non $\leqq x'$. This shows that G has no semilinear elements. Conversely, assume that (b) is valid. Hence for each $0 \neq x \in G$ there exists $x' \in G$, $0 < x' \leqq |x|$ such that for each $0 < y \in G$ we have $2y$ non $\leqq x'$. We show that the element x' is singular.

Let $z, t \in G^+$, $z + t = x'$. Denote $z \wedge t = u$ and let $u + z_1 = z$, $u + t_1 = t$. Then we have $u, z_1, t_1 \in G^+$ and

$$2u \leqq u + z_1 + u + t_1 = x' ,$$

thus $u = 0$ and hence $z \wedge t = 0$, $z + t = z \vee t$. Therefore $z \wedge (x' - z) = 0$ for each $z \in [0, x']$. The element x' is singular and G is a singular l-group.

By using 2.5.1, the proposition 2.5 can be deduced also from [15], Thm. 5.

If G is an archimedean l-group, then we denote by $G^\wedge$ the Dedekind completion of G. We may assume that G is a closed l-subgroup of $G^\wedge$ and that each element $0 < x \in G^\wedge$ is the least upper bound of a subset of G^+.

2.6. *Let H be an archimedean l-group with a weak unit e such that e is singular in H. Then e is singular in $H^\wedge$.*

Proof. We denote by $[0, e]$ the interval of $H^\wedge$ with the endpoints 0 and e. Since each element of $[0, e]$ is a supremum of some subset of $[0, e] \cap H$ it follows that $[0, e]$ is the Dedekind completion of the lattice $[0, e] \cap H$. According to the assumption the lattice $[0, e] \cap H$ is a Boolean algebra and therefore its Dedekind completion $[0, e]$ is a Boolean algebra as well; thus e is a singular element of $H^\wedge$.

Let H be as in 2.6. Let H_1 be the orthogonal completion of $H^\wedge$. Thus H_1 is a complete l-group that is orthogonally complete, $H^\wedge$ is a closed convex l-subgroup of H_1 and for each $0 < h_1 \in H_1$ there is a disjoint subset $\{x_j\}$ $(j \in J)$ of $H^\wedge$ such that $h_j = \bigvee x_j$. (Cf. [6].) From this and from 2.6 it follows that e is a singular element of H_1 and that e is a weak unit of H_1. Therefore the l-group H_1 is singular.

The following assertion was proved in [8].

2.7. *Let $H \neq \{0\}$ be an l-group that is singular, complete and orthogonally complete. Assume that H has a weak unit e such that e is a singular element of H. Let $0 \leqq h \in H$. Then h can be uniquelly represented in the form $h = \bigvee ne_n^*$ $(n = 1, 2, \ldots)$ such that $e_{n_1}^* \wedge e_{n_2}^* = 0$ for $n_1 \neq n_2$ and $\bigvee e_n^* = e^* \leqq e$. If $0 = h' = \bigvee ne'_n$ is another such representation for $h' \in H$, then $h \leqq h'$ if and only if $e^* \leqq e' = \bigvee e'_n$ and $e_i^* \wedge e'_j > 0 \Rightarrow i \leqq j$.*

Let $0 \leqq h \in H$, $0 \leqq h' \in H$. Under the same denotations as above put $e_0^* = e - e^*$, $e'_0 = e - e'$. Since $[0, e]$ is a Boolean algebra we infer that $e = \bigvee e_n^* = \bigvee e'_n$ $(n = 0, 1, 2, \ldots)$ and $h = \bigvee ne_n^*$, $h' = \bigvee ne'_n$ $(n = 0, 1, 2, \ldots)$. Then we have:

2.7.1. $h \leqq h'$ if and only if $e_n^* \leqq \bigvee e'_i$ $(i \geqq n)$ for each $n \geqq 1$.

Proof. Let $h \leqq h'$, $n \geqq 1$. Then $e^* \leqq e'$ and $e_n^* \wedge e'_j = 0$ for $1 \leqq j < n$, thus from

$$e_n^* \leqq e' = (e'_1 \vee \ldots \vee e'_{n-1}) \vee (\bigvee_{j \geqq n} e'_j)$$

we obtain that $e_n^* \leqq \bigvee_{j \geqq n} e'_j$, $n = 1, 2, \ldots$

Conversely, assume that $e_n^* \leqq \bigvee_{j \geqq n} e'_j$ for each $n \geqq 1$. Then $\bigvee_{n \geqq 1} e_n^* \leqq \bigvee_{n \geqq 1} e'_n$ and $e_n^* \wedge e'_j = 0$ for $j = 1, 2, \ldots, n - 1$. Therefore $h \leqq h'$.

For a Boolean algebra B let $F(B)$ have the same meaning as in §1.

2.8. *Let $\{0\} \neq H$ be an archimedean l-group with a weak unit e that is singular in H. Let $B = [0, e]$, $F = F(B)$. Then H is isomorphic with an l-subgroup of F.*

Proof. Let H_1 be as above. The l-group H_1 is orthogonally complete and also complete; H is a closed l-subgroup of H_1. According to 2.7.1 each $0 \leqq h \in H$ can be uniquelly represented in the form $h = \bigvee ne_n^*$ $(n = 0, 1, 2, \ldots)$, $e_n^* \in H_1$, $\bigvee e_n^* = e$, $e_{n_1}^* \wedge e_{n_2}^* = 0$ for $n_1 \neq n_2$. From the construction of the elements e_n^* described in [8] and from the fact that H is a closed l-subgroup of H_1 it follows that each e_n^* belongs to H and hence $e_n^* \in B$. Let $\bar{e}_n$ be the subset of the Boolean space $X(B)$ of the Boolean algebra B that corresponds to the element $e_n^* \in B$. Then $\bar{e}_n$ is a clopen subset of $X(B)$ and $\bar{e}_{n_1} \cap \bar{e}_{n_2} = \emptyset$ for $n_1 \neq n_2$. Consider the function $f \in F$ such that $f(x) = n$ whenever $x \in \bar{e}_n$ $(n = 0, 1, 2, \ldots)$. Then the mapping $h \to f$ is an isomorphism of the lattice ordered semigroup H^+ into F^+. From this we obtain that there exists an isomorphism of the l-group H into F.

From the method of the above proof we simultaneously obtain the following generalization of 2.7:

2.9. *Let $H \neq \{0\}$ be an l-group that is singular, archimedean and conditionally orthogonally complete. Assume that H has a weak unit e such that e is a singular element of H. Then the assertion of* 2.7 *is valid for H.*

2.10. *Let G be an l-group that is a completely subdirect product of l-subgroups H_i $(i \in I)$. Assume that G is conditionally orthogonally complete and that each H_i is a complete l-group. Then G is a complete l-group.*

Proof. Let $g_j \in G$ $(j \in J)$, $g \in G$, $0 \leqq g_j \leqq g$ for each $j \in J$. Then

$$0 \leqq g_j(H_i) \leqq g(H_i)$$

for each $j \in J$ and each $i \in I$. Since H_i is a complete l-group, there exists

$$\bigvee_{j \in J} g_j(H_i) = \bar{g}_i$$

in H_i. We have $\bar{g}_i \leqq g(H_i) \leqq g$. Since the system $\{\bar{g}_i\}$ $(i \in I)$ is disjoint and G is conditionally orthogonally complete, $\bigvee \bar{g}_i = x$ exists in G. Then $x(H_i) = \bar{g}_i \geqq$ $\geqq g_j(H_i)$ for each $i \in I$ and each $j \in J$, thus $x \geqq g_j$ for each $j \in J$. Let $y \in G$, $g_j \leqq y$ for each $j \in J$. Hence $g_j(H_i) \leqq y(H_i)$ for each $i \in I$ and each $j \in J$. Therefore $x(H_i) =$ $= \bar{g}_i \leqq y(H_i)$ for each $i \in I$ and this implies $x \leqq y$. Thus $x = \bigvee_{j \in J} g_j$. This shows that G is a complete l-group.

2.11. Theorem. *Let H be an l-group that is conditionally orthogonally complete and archimedean. Assume that H has a weak unit e such that e is a singular element of H. Then H is a complete l-group.*

Proof. Because the weak unit e is singular, the l-group H is singular. Since H is conditionally orthogonally complete, the Boolean algebra $B = [0, e]$ is orthogonally complete. Hence B is complete (Smith - Tarski [17]; cf. also Sikorski [15], Thm. 20.1). Let $g, g_k \in H^+$ $(k \in K)$, $g_k \leqq g$ for each $k \in K$. According to 2.9 the elements g, g_k can be represented in the form described in 2.7; let

$$g = \bigvee n e'_n, \quad g_k = \bigvee n e_n(k) \quad (n = 0, 1, 2, \ldots)$$

be such representations. All elements e'_n, $e_n(k)$ belong to the complete Boolean algebra B. From $g_k \leqq g$ it follows $e_n(k) \leqq \bigvee_{i \geqq n} e'_i$ for each $k \in K$ and each $n \geqq 1$.

We define by induction elements $e_n \in B$ $(n = 0, 1, 2, \ldots)$ as follows. We put

$$e_0 = \bigvee_{k \in K} e_0(k) .$$

Assume that $e_0, \ldots, e_n$ are defined and the system $\{e_0, \ldots, e_n\}$ is disjoint. Denote $f_n = e_0 \vee \ldots \vee e_n$ and let g'_n be the complement of f_n in B. We put

$$e_{n+1} = g'_n \wedge \left(\bigvee_{k \in K} e_{n+1}(k)\right) .$$

Then the system $\{e_n\}$ $(n = 1, 2, \ldots)$ is disjoint and hence the system $\{ne_n\}$ $(n = = 1, 2, \ldots)$ is disjoint as well. Let $k \in K$ be fixed. We will verify that

$$ne_n \leqq g_k \quad (n = 1, 2, \ldots) .$$

We have to show that

$$e_n \leqq \bigvee_{i \geqq n} e_i(k) .$$

Thus it suffices to prove that

$$e_n \wedge e_t(k) = 0 \tag{1}$$

for each $t < n$. From $e = e_0 \vee e_1 \vee \ldots \vee e_{n-1} \vee g'_{n-1}$ we obtain $e_n(k) = (e_n(k) \wedge \wedge e_0) \vee (e_n(k) \wedge e_1) \vee \ldots \vee (e_n(k) \wedge e_{n-1}) \vee (e_n(k) \wedge g'_{n-1}) \leqq e_0 \vee \ldots \vee e_{n-1} \vee \vee (\bigvee_{j \in K} e_n(j) \wedge g'_{n-1}) = e_0 \vee \ldots \vee e_{n-1} \vee e_n$ and therefore

$$e_n(k) \wedge e_{n+j} = 0 \quad \text{for} \quad j \geqq 1 .$$

Thus the relation (1) is proved. Hence the system $\{ne_n\}$ $(n = 0, 1, 2, \ldots)$ is bounded and so according to the assumtpion there exists the element

$$h = \bigvee ne_n \quad (n = 0, 1, 2, \ldots)$$

in H and $h \leqq g_k$ for each $k \in K$.

Let $0 < h' \in H$, $h' \leqq g_k$ for each $k \in K$. The element h' can be represented in the form $h' = \bigvee ne''_n$ $(n = 0, 1, \ldots)$ where the system $\{e''_n\}$ is disjoint and $\bigvee e''_n = e$. From $h' \leqq g_k$ we obtain

$$e''_n \wedge e_m(k) = 0$$

for each $n \geqq 1$, $m < n$, $k \in K$ and therefore

$$e''_n \wedge e_m \leqq e''_n \wedge \left(\bigvee_{k \in K} e_m(k)\right) = 0$$

for each $n \geqq 1$ and each $m < n$. This implies that $h' \leqq h$. We have proved that $h = = \bigwedge g_k$ $(k \in K)$. From this it follows that H is complete.

2.12. Theorem. *Let G be a singular l-group. Then the following conditions are equivalent*:

(i) *G is complete.*

(ii) *G is σ-complete and conditionally orthogonally complete.*

Proof. Obviously (i) $\Rightarrow$ (ii). From 2.5, 2.10 and 2.11 it follows that (ii) $\Rightarrow$ (i).

2.13. *Let G be a vector lattice. Then the conditions* (i) *and* (ii) *from* 2.12 *are equivalent.*

This follows from [19], Thm. 3 and 4.

It remains as an open question whether the assertion of Thm. 2.12 holds for each l-group G.

3. THE (α, β)-DISTRIBUTIVITY

In this section we prove that if G is an archimedean l-group with the decomposition property that is $(\alpha, 2)$-distributive, then it is (α, α)-distributive and the Dedekind completion $G^\wedge$ of G is also (α, α)-distributive. In particular, an orthogonally complete and σ-complete l-group that is $(\alpha, 2)$-distributive must be (α, α)-distributive.

3.1. *Let G be an archimedean l-group. Then the mapping $A \to A \cap G$ $(A \in K^0(G^\wedge))$ is an isomorphism of the Boolean algebra $K^0(G^\wedge)$ onto $K^0(G)$.*

Proof. Let $A \in K^0(G^\wedge)$. Then it is easy to verify that $A \cap G \in K^0(G)$ and the mapping $\varphi : A \to A \cap G$ is monotone. Let $B \in K^0(G)$ and let X be the set of all elements $x \in G$ with $|x| \wedge |b| = 0$ for each $b \in B$. Further let $\psi(B) = A_1$ be the set of all elements of $G^\wedge$ that are disjoint to each element of X. Then $A_1 \in K^0(G^\wedge)$ and $\varphi(A_1) = B$; hence φ is onto.

Let $A \in K^0(G^\wedge)$, $\varphi(A) = B$, and let X, A_1 be as above, $0 \leqq a \in A$. There exists a system $\{g_i\} \subset G^+$ such that $\bigvee g_i = a$. Then $\{g_i\} \subset A$, thus $g_i \in B$; therefore $g_i \wedge |x| = 0$ for each $x \in X$. Since G is infinitely distributive, we obtain $a \wedge |x| = 0$ and therefore $a \in A_1$. From this it follows $A \subset A_1$. Conversely, let $0 \leqq a_1 \in A_1$. Again, there is a system $\{g'_i\} \subset G^+$ such that $\bigvee g'_i = a_1$. We have $\{g'_i\} \subset B \subset A$ and since A is a closed sublattice of $G^\wedge$, we obtain $a_1 \in A$. Therefore $A_1 \subset A$. Thus $A_1 = A$, hence φ is a monomorphism. Because the mapping ψ is monotone and $\psi = \varphi^{-1}$, φ is an isomorphism.

3.2. *Let G be an l-group with the decomposition property, $A, B \in K^0(G)$ and let C be the supremum of $\{A, B\}$ in $K^0(G)$, $0 \leqq g \in C$. Then there exist $a \in A^+$, $b \in B^+$ such that $g = a + b$.*

Proof. This follows from the fact that the supremum in the lattice of direct factors is the sum ([18], Thm. 1).

3.3. Theorem. *Let G be an archimedean l-group with the decomposition property that is $(\alpha, 2)$-distributive. Then the l-group $G^\wedge$ is (α, α)-distributive.*

Proof. Assume that $G^\wedge$ is not (α, α)-distributive. For any complete l-group H, the Boolean algebra $K^0(H)$ is (α, α)-distributive if and only if H is (α, α)-distributive [9]. Hence the Boolean algebra $K^0(G^\wedge)$ is not (α, α)-distributive. Thus (cf. [11], [17]) $K^0(G^\wedge)$ is not $(\alpha, 2)$-distributive. According to 3.1, the Boolean algebra $K^0(G)$ is not $(\alpha, 2)$-distributive. Then there exists a system $\{X_{t,s}\} \subset K^0(G)$ $(t \in T, s \in S$, card $T \leqq \alpha$, $S = \{1, 2\})$ such that

$$\bigwedge_{t\in T} \bigvee_{s\in S} X_{t,s} = X\,, \quad \bigvee_{\varphi\in S^T} \bigwedge_{t\in T} X_{t,\varphi(t)} = Y$$

and $X \neq Y$. Hence Y is a proper subset of X. Let $Y_{t,s} = (X_{t,s} \vee Y) \wedge X$. Since $K^0(G)$ is infinitely distributive, we have

$$\bigwedge_{t\in T} \bigvee_{s\in S} Y_{t,s} = X\,, \quad \bigvee_{\varphi\in S^T} \bigwedge_{t\in T} Y_{t,\varphi(t)} = Y\,.$$

Further, since $Y_{t,s} \in [Y, X]$, we obtain

$$Y_{t,1} \vee Y_{t,2} = X \quad \text{for each} \quad t \in T\,,$$

$$\bigwedge_{t\in T} Y_{t,\varphi(t)} = Y \quad \text{for each} \quad \varphi \in \{1, 2\}^T\,.$$

Let A be the relative complement of Y in the interval $[\{0\}, X]$ of $K^0(G)$. The mapping $\psi : Z \to A \wedge Z$ $(Z \in [Y, X])$ is an isomorphism of $[Y, X]$ onto $[\{0\}, A]$. Put $A_{t,s} = \psi(Y_{t,s})$. Then

$$A_{t,1} \vee A_{t,2} = A \neq \{0\} \quad \text{for each} \quad t \in T\,, \tag{2}$$

$$\bigwedge_{t\in T} A_{t,\varphi(t)} = \{0\} \quad \text{for each} \quad \varphi \in \{1, 2\}^T\,. \tag{3}$$

There exists $0 < a \in A$. According to (2) and 3.2 the element a can be written in the form

$$a_{t,1} \vee a_{t,2} = a \quad \text{for each} \quad t \in T\,, \tag{4}$$

where $0 \leqq a_{t,1} \in A_{t,1}$, $0 \leqq a_{t,2} \in A_{t,2}$. From (3) we obtain $\bigcap A_{t,\varphi(t)} = \{0\}$ and therefore

$$\bigwedge_{t\in T} a_{t,\varphi(t)} = 0 \quad \text{for each} \quad \varphi \in \{1, 2\}^T\,. \tag{5}$$

From (4) and (5) it follows that the l-group G is not $(\alpha, 2)$-distributive, which is a contradiction.

Since G is a closed l-subgroup of $G^\wedge$, we obtain from 3.3 immediately:

3.4. Corollary. *Let G be an archimedean l-group with the decomposition property that is $(\alpha, 2)$-distributive. Then G is (α, α)-distributive.*

Since each complete l-group is an archimedean l-group with the decomposition property, we have:

3.5. Corollary. ([7], Thm. 3.9.) *If a complete l-group G is* $(\alpha, 2)$*-distributive, then it is* (α, α)*-distributive.*

3.6. *Let G be a σ-complete and conditionally orthogonally complete l-group. Then G is an l-group with the decomposition property.*

Proof. Let $X \subset G^+$. From the Axiom of Choice it follows that there exists a system $\{y_i\}$ $(i \in I)$, $0 \leqq y_i$ such that (i) $y_{i_1} \wedge y_{i_2} = 0$ for any pair of distinct elements $i_1, i_2 \in I$, (ii) $y_i \wedge |x| = 0$ for each $i \in I$ and each $x \in X$, and (iii) if $0 < y \in X^\delta$, then $y \wedge y_i > 0$ for some $i \in I$. Let $0 \leqq z \in G$. According to 2.2 for each $i \in I$ there exists $z[y_i]$. Clearly $z[y_i] \leqq z$ and the system $\{z[y_i]\}$ $(i \in I)$ is disjoint. By the assumption, the join $\bigvee_{i \in I} z[y_i] = t$ exists in G. Then $z - t = z_0 \geqq 0$. We have $t[y_i] \leqq z[y_i]$ and $z[y_i] \leqq t$, thus

$$z[y_i] = z[y_i][y_i] \leqq t[y_i] ;$$

therefore $z[y_i] = t[y_i]$ and hence $z_0[y_i] = 0$ for each $i \in I$. From this it follows that $z_0 \in X^{\delta\delta}$. We have proved that each $z \in G^+$ can be written in the form $z = z_0 + t$ with $0 \leqq z_0 \in X^{\delta\delta}$, $0 \leqq t \in X^\delta$. Therefore $G = X^{\delta\delta} \otimes X^\delta$.

From 3.4 and 3.6 we obtain:

3.7. Theorem. *Let G be a σ-complete and conditionally orthogonally complete l-group. If G is* $(\alpha, 2)$*-distributive, then it is* (α, α)*-distributive.*

References

[1] *S. J. Bernau:* Orthocompletion in lattice groups. Proc. London Math. Soc. *16* (1966), 107—130.

[2] *G. Birkhoff:* Lattice theory, 2nd edition, Providence 1948.

[3] *P. Conrad:* The lateral completion of a lattice ordered group. Proc. London Math. Soc. *19* (1969), 444—480.

[4] *P. Conrad, D. McAllister:* The completion of a lattice ordered group. J. Austral. Math. Soc. *9* (1969), 182—208.

[5] *Л. Фукс:* Частично упорядоченные алгебраические системы, Москва 1965.

[6] *Я. Якубик:* Представления и расширения *l*-групп. Чех. мат. ж. *13* (1963), 267—283.

[7] *J. Jakubík:* Distributivity in lattice ordered groups. Czech. Math. J. *22* (1972), 108—125.

[8] *J. Jakubík:* Cantor-Bernstein theorem for lattice ordered groups. Czech. Math. J. *22* (1972), 159—175.

[9] *M. Jakubíková:* Abgeschlossene vollständige *l*-Untergruppen der Verbandsgruppen. Matem. časopis *23* (1973), 55—63.

[10] *G. Jameson:* Ordered linear spaces. Lecture Notes in Mathematics 141, Springer Verlag, Berlin 1970.

[11] *R. S. Pierce:* Distributivity in Boolean algebras. Pacif. J. Math. *7* (1957), 983—992.

[12] *А. Г. Пинскер:* Расширение полуупорядоченных групп и пространств. Уч. зап. Ленинград. гос. пед. инст. им. Герцена *86* (1949), 235—284.

[13] *М. Г. Рабинович:* Вполне разложимые структуры, Сибир. мат. ж. *10* (1969), 920—939.

[14] *М. Г. Рабинович:* О пополнении одного класса структур. Сибир. мат. ж. (1970), 585—596.

[15] *Г. Й. Роткович:* О полуупорядоченных группах. Уч. зап. Ленинград. гос. пед. инст. им. Герцена *404* (1971), 439—451.

[16] *R. Sikorski:* Boolean algebras. Berlin 1964.

[17] *E. C. Smith, A. Tarski:* Higher degrees of distributivity and completeness in Boolean algebras. Trans. Amer. Math. Soc. *84* (1957), 230—257.

[18] *Ф. Шик:* К теории структурно упорядоченных групп. Чех. мат. ж. *6* (1956), 1—25.

[19] *А. И. Векслер, В. А. Гейлер:* О порядковой и дизюнктной полноте линейных полуупорядоченных пространств. Сибир. Мат. ж. *13* (1972), 43—51.

[20] *E. C. Weinberg:* Higher degrees of distributivity in lattices of continuous functions. Trans. Amer. Math. Soc. *104* (1962), 334—346.

Author's address: 040 00 Košice, Zbrojnícka 7, ČSSR (Vysoká škola technická)