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CLOSED AND OPEN SETS IN TOPOLOGIES INDUCED BY LATTICE ORDERED VECTOR GROUPS

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1.1. The lattice ordered vector group G is an l-subgroup of the complete direct sum $\tilde{\Sigma}\{G_x : x \in M\}$ of linearly ordered groups G_x, [1] V, § 8. If for each group G_x ($x \in M$) and each element $a \in G_x$ there is an element $f \in G$ with $f(x) = a$, G is called a subdirect sum of linearly ordered groups $\{G_x : x \in M\}$ or briefly a realization and denoted by $G = (G_x : x \in M)$ or (G, M) only, [7] I, 0.5. If $G \neq 0$, the requirement $G_x \neq 0$ for all $x \in M$ represents an unessential loss of generality. We shall assume it throughout this paper.

Let us define two mappings; $Z : 2^G \to 2^M$ by declaring $Z(P) = \{x \in M : f(x) = 0$ for all $f \in P\}$ ($P \subseteq G$) and $\Psi : 2^M \to 2^G$ as follows: $\Psi(A) = \{f \in G : f(x) = 0$ for all $x \in A\}$ ($A \subseteq M$). The family $\{Z(f) : f \in G\}$ is a base of closed sets of a topology on M; this topology is said to be induced on M by the realization (G, M). The corresponding topological space will be denoted by (M, G), [7] I, 1.5. Denote by $\Gamma(G)$ the complete Boolean algebra of all polars of G. (In [6, 7] we used the term "component" instead of "polar".) The family $\Omega(G) = \{\Psi(A) : A \subseteq M\}$ is a complete lattice under inclusion and $\Gamma(G)$ is a subset of $\Omega(G)$, but not necessarily a sublattice. We also denote by $\mathfrak{N}(M)$, $\mathfrak{M}(M)$ and $\mathfrak{O}(M)$ the lattice (under inclusion) of all closed, regular closed or closed and open ($\equiv$ clopen) subsets of (M, G), respectively. The mappings Z and Ψ are (mutually inverse) antiisomorphisms between the lattices $\Omega(G)$ and $\mathfrak{N}(M)$ as well as between $\Gamma(G)$ and $\mathfrak{M}(M)$, [7] I, 1.15. Of course, under Z and Ψ the corresponding restrictions of so denoted mappings are understood. In this paper we study namely the structure of the lattice, which is the Ψ-image of the lattice $\mathfrak{O}(M)$ of all clopen subsets of (M, G) and the structure of the lattice being the Z-image of the lattice $\Delta(G)$ of all direct factors of G.

1.2. The results of the paper are as follows. If an l-group has a realization, it always has the so-called completely regular realization, which is useful because of its special properties. (The concept is due to P. RIBENBOIM [4]; see definition 2.1). Given an arbitrary realization (G, M) the mapping Ψ transforms the family $\mathfrak{O}(M)$

on the family $\Gamma(G, M)$ of all ambiguous polars of (G, M); a polar K is called ambiguous if $K \subseteq \Psi(x)$ implies $K' \nsubseteq \Psi(x)$ (all $x \in M$). If the space (M, G) is compact, Ψ maps $\mathfrak{O}(M)$ onto the family of principal polars f'' with f completely regular in (G, M) (Theorem 2.3; for the completely regular element see definition 2.1). If the realization (G, M) is completely regular and the space (M, G) compact, Ψ maps $\mathfrak{O}(M)$ onto the family $\Pi(G)$ of all principal polars of G (Corollary 2.3 and Theorem 3.4). If (G, M) under discussion is the Π'-realization (which represents a special type of the completely regular one – see 3.1), the compactness of the space (M, G) is equivalent to the fact that Ψ transforms the family of all compact clopen subsets of (M, G) onto $\Pi(G)$ (Theorem 3.2). If (G, M) is an I-realization (defined in 4.1 by the requirement $\Omega(G) = I(G) =$ family of all l-ideals of G), the mapping Ψ is an antiisomorphism of the algebra $\mathfrak{O}(M)$ onto $\Delta(G)$ of all direct factors of G (Theorem 4.1). Theorem 4.2 gives some equivalent conditions (using the concept of the I-realization) that every polar of G is a direct factor of G (i.e. $\Gamma(G) = \Delta(G)$). One of them requires that the space (M, G) be extremally disconnected. The realization (G, M) is isomorphic to a direct sum of linearly ordered groups if and only if the lattice $\mathfrak{M}(M)$ is a closed sublattice of the lattice $\mathfrak{N}(M)$. By the method used for constructing a representation of an arbitrary Boolean algebra by means of the algebra Δ of all direct factors of an l-group (Theorem 5.2) we are led to consider l-groups of continuous real-valued functions (on a completely regular space M). The l-group G of all such functions is a realization (that is to say a subdirect sum of copies of the group of real numbers) and is the completely regular realization if the space M is extremally disconnected (Theorem 5.3).

2.1. Definition. Let (G, M) be a realization. An element $f \in G$ is said to be *completely regular* in (G, M) if for any $x \in Z(f)$ there exists $g \in G$ so that $x \in M \setminus Z(g) \subseteq Z(f)$.

If every element of G is completely regular in (G, M), the realization (G, M) is said to be *completely regular* [7] II, 3.

Some of the characteristic properties of the completely regular realizations derived in [7] IV, 8.10 can be formulated as characterizations of the completely regular elements, as it is shown in the following theorem. (If it is $\emptyset \neq P \subseteq G$ by P' we mean the polar $\{g \in G : |f| \wedge |g| = 0 \text{ for all } f \in P\}$. For $f \in G$ the symbol f' stands for $\{f\}'$ and P'' for $(P')'$.)

Theorem. *Let (G, M) be a realization and $f \in G$. The following conditions are equivalent:*

(1) *f is completely regular in (G, M);*

(2) *$Z(f)$ is an open set;*

(3) $Z(f') = M \setminus Z(f)$;

(4) *it holds*

(a) $Z(f'') \cap Z(f') = \emptyset$
and one of the following equivalent conditions

(b) $Z(f) = Z(f'')$,

(c) $Z(f) \in \mathfrak{M}(M)$.

Proof. $1 \Rightarrow 2$: Each point $x \in Z(f)$ is an interior point of $Z(f)$ because there exists $g \in G$ with $x \in M \setminus Z(g) \subseteq Z(f)$.

$2 \Rightarrow 4c$ is evident.

$c \Rightarrow b$: Provided $Z(f) \in \mathfrak{M}(M)$ it follows $K = \Psi\, Z(f) \in \Gamma(G)$ and thus $f \in K$; hence $f'' \subseteq K$. From the evident relation $Z(f'') \subseteq Z(f)$ we obtain the reverse inclusion $f'' = \Psi\, Z(f'') \supseteq \Psi\, Z(f) = K$ and so the statement (b) $Z(f'') = Z(K) = Z(f)$, for $K = \Psi\, Z(f) \Rightarrow Z(K) = Z\Psi\, Z(f) = Z(f)$.

$b \Rightarrow c$ is evident.

$2 \Rightarrow 4a$: Recall that $f' = \{g \in G : Z(f) \cup Z(g) = M\}$. Hence we have $Z(f') = \overline{M \setminus Z(f)} = M \setminus Z(f)$ and with regard to 4b it holds $Z(f'') \cap Z(f') = Z(f) \cap Z(f') = Z(f) \cap [M \setminus Z(f)] = \emptyset$.

$4 \Rightarrow 3$: The relations $M = Z(f'') \vee_{\mathfrak{M}} Z(f') = Z(f'') \cup Z(f')$ together with 4a, $\emptyset = Z(f'') \cap Z(f')$, give $Z(f') = M \setminus Z(f'')$ and with respect to 4b we have $Z(f') = M \setminus Z(f)$.

$3 \Rightarrow 1$: For each $x \in Z(f) = M \setminus Z(f')$ there exists $g \in f'$ so that $x \in M \setminus Z(g)$. Also $g \in f' \Rightarrow M \setminus Z(g) \subseteq Z(f)$.

2.2. Lemma. *Let (G, M) be a realization. If A is a clopen compact set in (M, G), then $M \setminus A = Z(f)$ for a completely regular element f in G and $\Psi(M \setminus A) = f''$, $\Psi(A) = f'$.*

Proof. The set $A' = M \setminus A$ is closed, thus $A' = \bigcap\{Z(g) : g \in \Psi(A')\}$. The compact set $A = M \setminus A' = \bigcup\{M \setminus Z(g) : g \in \Psi(A')\}$ is covered by the open sets $M \setminus Z(g)$; hence there exists a finite number of $g_i \in \Psi(A')$ such that $A = \bigcup_i [M \setminus Z(g_i)]$. Then $A' = \bigcap_i Z(g_i) = \bigcap_i Z(|g_i|) = Z(\bigvee_i |g_i|)$. The element $f = \bigvee_i |g_i|$ is completely regular in (G, M), for the set $Z(f) = A'$ is open (2.1). We shall prove $\Psi(A') = f''$. On the one hand $Z(f) \supseteq Z(f'') \Rightarrow \Psi(A') = \Psi\, Z(f) \subseteq \Psi\, Z(f'') = f''$. On the other hand $A' = Z(f) \Rightarrow f \in \Psi(A') \in \Gamma \Rightarrow f'' \subseteq \Psi(A')$. Finally $[\Psi(A)]' = \Psi(A') = f'' \Rightarrow \Psi(A) = f'$.

2.3. If f is an element of the l-group G, the polars f'' and f' are called principal and dual principal, respectively, their families being denoted by $\Pi(G)$ and $\Pi'(G)$.

Theorem. *Let (G, M) be a realization and (M, G) compact. Then Ψ and Z are (mutually inverse) antiisomorphisms between the set $\mathfrak{O}(M)$ of all clopen sets in (M, G) and the set $\overline{\Pi}(G)$ of all principal polars f'', where f is a completely regular element in (G, M). These polars belong to $\Pi(G) \cap \Pi'(G)$.*

Proof. A is clopen $\Rightarrow B = M \setminus A$ is clopen and hence compact $\Rightarrow$ (according to 2.2) $A = M \setminus B = Z(f)$ with f completely regular in (G, M) and $\Psi(A) = \Psi Z(f) = \Psi Z(f'') = f''$. For A is clopen and thus compact, according to 2.2 there exists $g \in G$ with $\Psi(A) = g'$. Therefore $f'' \in \Pi(G) \cap \Pi'(G)$.

The Z-image of the polar f'', where f is a completely regular element in (G, M), is a clopen set because $Z(f'') = Z(f)$ is open by Theorem 2.1.

We have shown $\Psi\mathfrak{O} \subseteq \overline{\Pi}$, $Z\overline{\Pi} \subseteq \mathfrak{O}$. Moreover, $\overline{\Pi} \subseteq \Gamma$, $\mathfrak{O} \subseteq \mathfrak{M}$ is true and by [7] I, 1.15, Ψ and Z are (mutually inverse) antiisomorphisms of the lattices Γ and $\mathfrak{M}$. This completes the proof.

Corollary. *If (G, M) is a completely regular realization and (M, G) compact, then Ψ and Z are (mutually inverse) antiisomorphisms between $\Pi(G)$ and the lattice $\mathfrak{O}(M)$ of all clopen sets in (M, G).*

Proof follows immediately from Theorem 2.3 since every element of the group G is completely regular in (G, M).

2.4. The polars K and K' of an l-grup G are called complementary. K' is the complement of K in the Boolean algebra $\Gamma(G)$.

Definition. Let (G, M) be a realization and $K \in \Gamma(G)$. K will be called an *ambiguous* polar of the realization (G, M) if it holds for any $x \in M : K \subseteq \Psi(x) \Rightarrow K' \nsubseteq \Psi(x)$. The set of all ambiguous polars in (G, M) will be denoted by $\Gamma(G, M)$.

Remarks.

(1) $K \in \Gamma(G, M) \Rightarrow K' \in \Gamma(G, M)$.

(2) The realization (G, M) is completely regular if $\Gamma(G) = \Gamma(G, M)$.

Indeed, $\Psi(x)$ $(x \in M)$ are minimal prime subgroups. Then the complete regularity of (G, M) follows from [7] IV, 8.10.

(3) $\Gamma(G, M)$ contains all direct factors in G.

In fact, if $\Psi(y)$ contains direct factors K and K', it contains $G = K + K'$, too, hence $G_y = 0$, which is a contradiction with our hypothesis of all components G_x of the subdirect sum $(G_x : x \in M)$ being different from 0.

Theorem. *Let (G, M) be a realization. The mappings Ψ and Z are (mutually inverse) antiisomorphisms between $\mathfrak{O}(M)$ and $\Gamma(G, M)$. $\Gamma(G, M)$ is thus a subalgebra of the Boolean algebra $\Gamma(G)$.*

Proof. The condition $K \in \Gamma(G, M)$ can be expressed equivalently as follows: $x \in Z(f)$ for all $f \in K \Rightarrow$ there exists $g \in G$ such that $x \in M \setminus Z(g) \subseteq Z(f)$ for all $f \in K$, or equivalently: $x \in Z(K) \Rightarrow x \in M \setminus Z(g) \subseteq Z(K)$, which means that each point of $Z(K)$ is its interior point. Hence the mapping Z maps $\Gamma(G, M)$ into $\mathfrak{O}(M)$. Now, let $A \in \mathfrak{O}(M)$, $A' = M \setminus A$. Then $\Psi(A) \in \Gamma(G)$, $\Psi(A') = [\Psi(A)]'$. If it is $\Psi(A) \subseteq \Psi(y)$, $\Psi(A') \subseteq \Psi(y)$ for a point $y \in M$, then $A = Z\,\Psi(A) \supseteq Z\,\Psi(y)$, $A' = Z\,\Psi(A') \supseteq Z\,\Psi(y)$ and hence $y \in Z\,\Psi(y) \subseteq A \cap A' = \emptyset$, a contradiction. Thus Ψ maps $\mathfrak{O}(M)$ into $\Gamma(G, M)$. Analogously as in the proof of Theorem 2.3, the proof of the first assertion of the theorem can be completed. The second assertion follows from the fact that $\mathfrak{O}(M)$ is a subalgebra of $\mathfrak{M}(M)$.

Corollary. *Let (G, M) be a realization and (M, G) compact. Then $\Gamma(G, M) = \{f'' : f$ is a completely regular element in $(G, M)\} \subseteq \Pi(G) \cap \Pi'(G)$.*

It follows from Theorems 2.4 and 2.3.

2.5. Theorem. *Let (G, M) be a realization. The following are equivalent:*

(1) *The space (M, G) is extremally disconnected (i.e. closures of open sets are open).*

(2) $\mathfrak{O}(M) = \mathfrak{M}(M)$.

(3) $\Gamma(G, M) = \Gamma(G)$.

(4) *The lattice $\mathfrak{M}(M)$ is a sublattice of the lattice $\mathfrak{N}(M)$.*

(5) *The lattice $\Gamma(G)$ is a sublattice of the lattice $\Omega(G)$.*

(6) *Ψ maps $\mathfrak{O}(M)$ onto $\Gamma(G)$.*

(7) *Z maps $\Gamma(G)$ onto $\mathfrak{O}(M)$.*

Proof. $7 \Rightarrow 2$: Z maps $\Gamma(G)$ onto $\mathfrak{M}(M)$ by [7] I, 1.15 and onto $\mathfrak{O}(M)$ by supposition. Hence 2.

$2 \Rightarrow 6$: Ψ maps $\mathfrak{M}(M)$ onto $\Gamma(G)$ by [7] I, 1.15; thus 6 follows from 2.

$6 \Rightarrow 3$: Ψ maps $\mathfrak{O}(M)$ onto $\Gamma(G, M)$ by Theorem 2.4 and onto $\Gamma(G)$ by supposition. Hence 3.

$3 \Rightarrow 7$: Z maps $\Gamma(G, M)$ onto $\mathfrak{O}(M)$ by Theorem 2.4. Thus 7 follows from 3.

The equivalences $1 \Leftrightarrow 3 \Leftrightarrow 4$ follow from [7] I, 1.21 $(2 \equiv 4 \equiv 3)$, the equivalence $4 \Leftrightarrow 5$ from [7] I, 1.9 and 1.15 (because Ψ maps antiisomorphically $\mathfrak{N}(M)$ onto $\Omega(G)$ and $\mathfrak{M}(M)$ onto $\Gamma(G)$).

Corollary. *If one of the conditions of Theorem 2.5 holds the realization (G, M) is completely regular.*

The assertion follows immediately from the condition 3 (see Remark 2.4 (2)).

2.6. An l-ideal J of an l-group $\mathfrak{G}$ is said to be a prime ideal of $\mathfrak{G}$ if the factor-group $\mathfrak{G}/J$ is linearly ordered under the canonical ordering. The property of being a prime ideal is characterized in the class of l-ideals by the requirement of obtaining at least one of each pair of complementary polars ([7] II, section 2.2). Evidently, (G, M) being a realization, all the l-ideals $\Psi(x)$ are prime.

Theorem. *The realization (G, M) is completely regular if and only if $\Pi(G) \subseteq \subseteq \Gamma(G, M)$ and $Z(f'') = Z(f)$ for all $f \in G$.*

Note. With respect to 2.4 (1) it holds: $\Pi(G) \subseteq \Gamma(G, M) \Leftrightarrow \Pi'(G) \subseteq \Gamma(G, M)$.

Proof. Let the conditions of Theorem 2.6 be fulfilled. Then for $f \in G$ and $x \in M$ it holds: $f \in \Psi(x) \Rightarrow f'' \subseteq \Psi(x) \Rightarrow f' \nsubseteq \Psi(x)$. The condition: $f \in \Psi(x) \Rightarrow f' \nsubseteq \Psi(x)$ for all $f \in G$ is sufficient and necessary for the minimality (referred to the set-theoretic inclusion) of the prime ideal $\Psi(x)$ ([7] III, 7.6). The requirement that $\Psi(x)$ is a minimal prime ideal in G for all $x \in M$ is a sufficient (and necessary) condition of the complete regularity of (G, M) ([7] IV, 8.10).

If the realization (G, M) is completely regular, it holds $Z(f'') = Z(f)$ for all $f \in G$ because of Theorem 2.1. To prove the relation $\Pi(G) \subseteq \Gamma(G, M)$ we use both above mentioned theorems of [7]. The second of them verifies the minimality of the prime ideal $\Psi(x)$ in G for all $x \in M$ and from the first one we obtain for any $f \in G$ and $x \in M$: $f'' \subseteq \Psi(x) \Rightarrow f \in \Psi(x) \Rightarrow f' \nsubseteq \Psi(x)$.

3.1. It is well-known that an l-group $\mathfrak{G}$ is l-isomorphic to a realization if and only if a system $M(\neq \emptyset)$ of prime ideals in G exists with $\bigcap M = 0$. In this case $\mathfrak{G}$ is said to be an r-group and the system M a realizator in $\mathfrak{G}$. The mapping $\alpha : f \in \mathfrak{G} \to \to f(\,) \in \tilde{\Sigma}\{\mathfrak{G}/x : x \in M\}$ of the r-group $\mathfrak{G}$ into the complete direct sum of linearly ordered groups $\{\mathfrak{G}/x : x \in M\}$ defined in the following manner $f(x) = f + x$ (= the class of $\mathfrak{G}$ modulo x containing f), is an l-isomorphism of the r-group $\mathfrak{G}$ onto a subdirect sum of linearly ordered groups $\{\mathfrak{G}/x : x \in M\}$, thus onto a realization (which we shall denote as (G, M)). α is called the canonical l-isomorphism and (G, M) the canonical realization of the r-group $\mathfrak{G}$ corresponding to the realizator M. Every realization which is l-isomorphic to the r-group $\mathfrak{G}$ is said to be a realization of this r-group $\mathfrak{G}$. The requirement $\mathfrak{G}/x \neq 0$ for all $x \in M$ (compare 1.1) is equivalent to the following one: $x \neq G$ for all $x \in M$. If the topology induced by a realization is Hausdorff, it is said to be reduced ([7] II, 3 and IV, 8) or Hausdorff ([4]).

The completely regular realizations of a given r-group $\mathfrak{G}$ play a significant part and among them the so-called Π'-realization ([7] II, 4.16). It is the canonical realization of the given r-group $\mathfrak{G}$ corresponding to the realizator consisting of all minimal prime ideals $\mathfrak{G}$ ([7] II, sect. 4.15 and III, 7.2). Simultaneously, the Π'-realization is an example of the reduced realization.

3.2. Theorem. *Let (G, M) be the Π'-realization of an r-group. The following conditions are equivalent*:

(i) *The space (M, G) is compact.*

(ii) $\Pi'(G) = \Pi(G)$.

(iii) *Ψ and Z are (mutually inverse) antiisomorphisms between the lattice $\Pi(G)$ of all principal polars in G and the lattice $\mathfrak{O}(M)$ of all compact clopen subsets in (M, G).*

Proof. (i) $\Rightarrow$ (iii) follows from Corollary of 2.3 since the Π'-realization is completely regular.

(iii) $\Rightarrow$ (ii): For any $g \in G$ the set $A = Z(g'')$ is clopen and compact in (M, G), hence $A' = M \setminus A = Z(f)$ for some f in G and $\Psi(A') = f''$ holds by Lemma 2.2. Also $\Psi(A) = \Psi\, Z(g'') = g''$ and thus $g' = [\Psi(A)]' = \Psi(A') = f''$. We conclude with $\Pi'(G) \subseteq \Pi(G)$. Now, one gets easily $\Pi'(G) = \Pi(G)$.

(ii) $\Rightarrow$ (i): The space (M, G) is compact by [7] II, 4.19 and 4.18.

3.3. For the purpose of comparing various realizations of the same r-group the following concepts are useful ([7] IV, p. 21).

Let $G = (G_x : x \in M)$ and $H = (H_y : y \in N)$ be realizations. The realization H is said to be *similar* (*equivalent*) to the realization G if there exist an l-isomorphism $\alpha : G$ onto H and a (one-to-one) mapping $\beta : N$ onto M such that it holds: $f(\beta y) = 0 \Leftrightarrow (\alpha f)(y) = 0$ for all $f \in G$ and all $y \in N$. If we require for every l-isomorphism $\alpha : G$ onto H the existence of a mapping β with the above mentioned property, the similarity or equivalence is said to be *strong*. β is always a continuous, open and closed mapping (in case of equivalence β is thus a homeomorphism) of N onto M ([7] IV, 8.2).

Let $G = (G_x : x \in M)$ be a realization. For $\emptyset \neq A \subseteq M$ denote by $G(A)$ the set of restrictions to A of all $f \in G$. $G(A)$ is a realization of the l-group $G/\Psi(A)$ and $(A, G(A))$ is a subspace of the space (M, G) ([7] IV, 8.9). The following theorem holds ([7] IV, 8.12):

Let $G = (G_x : x \in M)$ be the Π'-realization of an l-group $\mathfrak{G}$ and $H = (H_y : y \in N)$ an arbitrary realization of the l-group $\mathfrak{G}$. The realization H is completely regular (completely regular and reduced) if and only if there is a suitable set A dense in M such that the realization H is similar (equivalent) to the realization $G(A)$.

Theorem. *Let (G, M) be the Π'-realization and (H, N) a completely regular (a completely regular and reduced) realization of an l-group $\mathfrak{G}$. If the space (N, H)*

is compact, the realization (H, N) is similar (equivalent) to the realization (G, M), hence the space (M, G) is compact, too. In both cases, similarity or equivalence, the system of all minimal prime ideals in H is equal to $\{\Psi(y) : y \in N\}$; if the realization (H, N) is reduced, it holds $\Psi(y_1) \neq \Psi(y_2)$ whenever $y_1 \neq y_2 (y_1, y_2 \in N)$.

Proof. By the above mentioned theorem ([7] IV, 8.12) there exist an l-isomorphism $\alpha : G$ onto H and a mapping (a one-to-one mapping) $\beta : N$ onto a suitable dense subset A of (M, G) so that $f(\beta y) = 0 \Leftrightarrow (\alpha f)(y) = 0$ for all $f \in G$ and all $y \in N$. The mapping β is continuous, open and closed (a homeomorphism). Since the space N is compact, so is the space A. Because M is Hasudorff, the compact subspace A is closed in M, thus $A = M$. To prove the last assertion let us provide the symbol Ψ with indices G or H to distinguish the mappings of M or N, respectively. $\{\Psi_G(x) : : x \in M\}$ is the system of all minimal prime ideals of G and $\Psi_G(x_1) \neq \Psi_G(x_2)$, whenever $x_1 \neq x_2$. Then $\{\Psi_H(y) : y \in N\}$ is the system of all minimal prime ideals of H, since denoting $\beta y = x$ we have $f \in \Psi_G(x) \Leftrightarrow f(x) = 0 \Leftrightarrow f(\beta y) = 0 \Leftrightarrow (\alpha f)(y) = = 0 \Leftrightarrow \alpha f \in \Psi_H(y)$.

If the realization (H, N) is reduced, β is one-to-one and so $\Psi_H(y_1) \neq \Psi_H(y_2)$ if $y_1 \neq y_2$.

From Theorems 3.2 and 3.3 we obtain the following theorem, which verifies Corollary 2.3 once again.

Theorem. *Let (G, M) be a completely regular realization and (M, G) compact. Then $\Pi'(G) = \Pi(G)$ holds and Ψ and Z are (mutually inverse) antiisomorphisms between the lattices $\Pi(G)$ of all principal polars in G and $\mathfrak{O}(M)$ of all clopen subsets in (M, G).*

Proof. By Theorem 3.3 the Π'-realization of the r-group G induces a compact space and thus the assertion follows by Theorem 3.2.

4.1. Let M be a realizator of an r-group $\mathfrak{G}$ with the following property: $x \neq \mathfrak{G}$ for all $x \in M$ and any l-ideal of $\mathfrak{G}$ is an intersection of elements of a subsystem of M. The canonical realization of the l-group $\mathfrak{G}$ corresponding to this realizator will be called an I-realization ([7] II, sect. 5.5).

Lemma. *The realization $G = (\mathfrak{G}/x : x \in M)$ of an r-group $\mathfrak{G} \neq 0$ is an I-realization if and only if the family $I(G)$ of all l-ideals of G is equal to $\Omega(G)$.*

Proof. Let $G = (\mathfrak{G}/x : x \in M)$ be an I-realization of the r-group $\mathfrak{G}$ corresponding to the realizator M and α the canonical l-isomorphism. Hence

$$\{\alpha(\bigcap_{x \in A} x) : \emptyset \subseteq A \subseteq M\} = \Omega(G),$$

since, for $\emptyset \subseteq A \subseteq M$, $\alpha(\bigcap_{x \in A} x)$ is the set of all $f \in G$ with $\alpha^{-1} f \in x$ $(x \in A)$, thus the set of all $f \in G$ with $f(x) = 0$, $x \in A$. This implies $\alpha(\bigcap_{x\in A} x) = \bigcap_{x\in A} \alpha x = \bigcap_{x\in A} \Psi(x) = \Psi(A) \in \Omega(G)$. Thus $I(G) \subseteq \Omega(G)$. The converse inclusion is obvious.

Let us suppose, conversely, that $I(G) = \Omega(G)$ holds for the realization $G = (\mathfrak{G}/x : x \in M)$ corresponding to the realizator M of the r-group $\mathfrak{G} \neq 0$. As per agreement (see 1.1) $\mathfrak{G}/x \neq 0$ for all $x \in M$, thus $x \neq \mathfrak{G}$. Let J be an l-ideal of $\mathfrak{G}$. Then $\emptyset \subseteq A \subseteq M$ exists such that $\Psi(A) = \alpha J$ (α is the canonical l-isomorphism of $\mathfrak{G}$ onto G). This means that the following holds:

$$f \in J \Leftrightarrow (\alpha f)(x) = 0 \quad \text{for all} \quad x \in A \Leftrightarrow f \in x \quad \text{for all} \quad x \in A\,.$$

It follows $J = \bigcap_{x\in A} x$ and $G = (\mathfrak{G}/x : x \in M)$ is an I-realization.

The preceding lemma justifies us to define the I-realization as a realization (G, M) fulfilling $I(G) = \Omega(G)$.

Let us emphasize that any r-group $\mathfrak{G}(\neq 0)$ has an I-realization. An adequate family of prime ideals is, e.g., the system of values of all $0 \neq a \in \mathfrak{G}$. (A value of the element $0 \neq a \in \mathfrak{G}$ is an l-ideal of $\mathfrak{G}$ that is maximal with respect to not containing a.)

Theorem. *Let (G, M) be an I-realization. Then Ψ and Z are (mutually inverse) antiisomorphisms of the systems $\mathfrak{O}(M)$ of all clopen sets in (M, G) and $\Delta(G)$ of all direct factors of G.*

Proof. By Remark 2.4 (3) the set $\Gamma(G, M)$ contains all direct factors and thus by Theorem 2.4, it suffices to prove $\Psi\mathfrak{O}(M) \subseteq \Delta(G)$. Then we have $\Gamma(G, M) = \Psi\mathfrak{O}(M) \subseteq \Delta(G) \subseteq \Gamma(G, M)$. Let A be a clopen set. Then the set $A' = M \setminus A$ is clopen, too. Since $A, A' \in \mathfrak{N}$, $A \wedge_{\mathfrak{N}} A' = A \cap A' = \emptyset$, $A \vee_{\mathfrak{N}} A' = A \cup A' = M$ holds, it follows $\Psi(A) \vee_{\Omega} \Psi(A') = G$, $\Psi(A) \cap \Psi(A') = \Psi(A) \wedge_{\Omega} \Psi(A') = 0$. With respect to the equality $I(G) = \Omega(G)$ it holds $G = \Psi(A) \vee_I \Psi(A') = \Psi(A) \dotplus \Psi(A')$. Thus it is verified that $\Psi(A)$ is a direct factor of G.

The assertion of the previous theorem can be formulated with regard to Theorem 2.4 as follows: $\Delta(G) = \Gamma(G, M)$.

While the equality $I(G) = \Omega(G)$ characterizes the I-realization, the equality $\Delta(G) = \Gamma(G, M)$ does not. An example will be given in 5.4.

4.2. Theorem. *Let (G, M) be an I-realization. Then every polar in G is a direct factor of G (i.e. $\Gamma(G) = \Delta(G)$), if and only if any of the conditions of Theorem 2.5 is satisfied.*

Proof. It follows from Theorem 4.1 that each polar of G is a direct factor of G if and only if the condition 2.5 (7) holds.

Thus Theorem 5.8 of [7] II is verified once again and, moreover, in an extended version.

4.3. Theorem. *Let (G, M) be an I-realization. Then the l-group G is l-isomorphic to a direct sum of linearly ordered groups if and only if $\bigcap A_\alpha \in \mathfrak{M}(M)$ for arbitrary $\{A_\alpha\} \subseteq \mathfrak{M}(M)$ (i.e. if $\mathfrak{M}(M)$ is a closed sublattice of the lattice $\mathfrak{N}(M)$).*

Proof. By Theorem 7 [6] the assertion of our theorem is equivalent to the fact that the lattice $\Gamma(G)$ is a closed sublattice of the lattice $I(G)$, in our case of the lattice $\Omega(G)$. Since the antiisomorphism Z of $\Omega(G)$ onto $\mathfrak{N}(M)$ maps $\Gamma(G)$ onto $\mathfrak{M}(M)$, $\mathfrak{M}(M)$ is a closed sublattice of $\mathfrak{N}(M)$, which is equivalent to $\bigcap A_\alpha \in \mathfrak{M}(M)$ for arbitrary $\{A_\alpha\} \subseteq \mathfrak{M}(M)$ (because $\mathfrak{M}(M)$ is a closed sub-$\vee$-semilattice of $\mathfrak{N}(M)$ independently of the type of realization).

5.1. The results and methods used in the preceding sections enable us to represent Boolean algebras as algebras of direct factors of l-groups.

The set G of all continuous real-valued functions on a topological space M becomes an l-group under usual pointwise addition and ordering. It can be taken for a subdirect sum G of copies of the additive (in the natural way ordered) group of real numbers, thus for a realization. If the space M is completely regular, this realization G induces the original topology on M [2] Th. 3.2. It will be denoted as usual by (G, M).

Theorem. *Let G be the group of all continuous real-valued functions defined on a completely regular space M. Then it holds for the realization (G, M):*

(1) $\Delta(G) = \Gamma(G, M)$,

(2) $\Gamma(G) = \Delta(G) \Leftrightarrow$ *one of the conditions of Theorem* 2.5 *holds.*

Proof. (1) $\Delta(G) \subseteq \Gamma(G, M)$ by Remark 2.4 (3).

Conversely, let $K \in \Gamma(G, M)$. By Theorem 2.4 $Z(K) \in \mathfrak{O}(M)$. Let f be arbitrary in G. Let f_1, f_2 be functions on M defined as follows: $f_1(x) = 0$, $f_2(x) = f(x)$ for $x \in Z(K)$ and $f_1(x) = f(x)$, $f_2(x) = 0$ otherwise. The functions f_1 and f_2 are continuous on M and $f = f_1 + f_2$. It follows $f_1 \in K$, $f_2 \in K'$ because $Z(f_1) \supseteq Z(K)$, $Z(f_2) \cup Z(K) = M$. Thus $K \in \Delta(G)$.

(2) Let $\Gamma(G) = \Delta(G)$ hold. By Remark 2.4 (3) we have $\Gamma(G) \supseteq \Gamma(G, M) \supseteq \Delta(G)$ and hence the condition 3, Theorem 2.5 is fulfilled. This condition and (1) of our Theorem imply $\Gamma(G) = \Delta(G)$.

5.2. Theorem. *Every Boolean algebra B is isomorphic to the Boolean algebra $\Delta(G)$ of all direct factors of the l-group (G, M) of all continuous real-valued functions on a topological space M. If the algebra B is complete, it holds $\Gamma(G) = \Gamma(G, M) = \Delta(G) = \Pi'(G) = \Pi(G) = \overline{\Pi}(G)$ ($\equiv$ the set of all principal polars f'' with completely regular elements f of (G, M)). Conversely, $\Gamma(G) = \Delta(G)$ implies the completeness of the algebra B.*

Proof. The demanded space is the Stone representation space M of the algebra $\check{B}$ dual to the given Boolean algebra B. The algebra $\check{B}$ is isomorphic to the algebra $\mathfrak{O}(M)$ of all clopen subsets of M ([5] I §8). The space M is Hausdorff, compact (and totally disconnected), thus completely regular, too. The l-group G of all continuous real-valued functions on M is a realization and by Theorem 3.2 [2] (already mentioned above) it induces the original topology on M. To prove the first assertion it suffices to refer to Theorem 5.1 (1) and 2.4 since $\Psi\, \mathfrak{O}(M) = \Gamma(G, M) = \Delta(G)$ and Ψ is an antiisomorphism. To verify the second assertion of the Theorem recall that the Stone representation space of a complete Boolean algebra is extremally disconnected. From Theorem 5.1 and Corollary of Theorem 2.4 it follows: $\Gamma(G) = \Gamma(G, M) = \Delta(G) =$ the set $\bar{\Pi}$ of all principal polars f'' with f completely regular in (G, M). From this we easily verify that $\bar{\Pi} = \Pi(G) = \Pi'(G) = \Delta(G) = \Gamma(G, M) = \Gamma(G)$. The converse is immediate.

K. NEUMANN [3] constructed another representation of Boolean algebras by means of direct factors of l-groups.

5.3. As we have explained in 5.1, the l-group of all continuous real-valued functions on a topological space M forms a realization. We ask under which conditions this realization is completely regular. If the space M is completely regular, the following theorem gives the answer:

Theorem. *Let M be a completely regular and extremally disconnected space. Then the l-group G of all continuous real-valued functions on M is a completely regular realization.*

Proof. In virtue of Theorem 3.2 [2] the realization G induces the original topology on M. Then the Theorem follows from Corollary 2.5.

5.4. By Theorem 5.1 (1) the l-group G of all continuous real-valued functions on a completely regular space M fulfils the condition $\Delta(G) = \Gamma(G, M)$. The equality $\Delta(G) = \Gamma(G, M)$ does not characterize I-realizations since we shall construct a completely regular space M such that (G, M) is not an I-realization.

The l-group G of all continuous real-valued functions on the interval $M = [0, 1)$ is not an I-realization.

Proof. The l-ideal J generated by the function $f(x) = x$ is the set of all $g \in G$ for which a positive integer exists such that $nf \geqq |g|$. Evidently $\Psi(0) \supseteq J$, $\Psi(x) \not\supseteq J$ for $x \neq 0$. If (G, M) were an I-realization, from the condition $I(G) = \Omega(G)$ it would follow $J = \bigcap_{x \in A} \Psi(x)$ for some $A \subseteq M$, thus $J = \Psi(0)$. However, this is not true since the function $e^{1/(1-x)} - e$ belongs to $\Psi(0)$ but does not belong to J.

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