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VERTICES OF SPACE CURVES

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A well-known classical result reads: A closed convex plane curve whose curvature is always positive, i.e., the so called oval, has at least four points at which the curvature assumes its extreme. The first known proof was given by S. D. MUKHOPADHYAYA [16] in 1909 and independently on him by A. KNEESER [12] in 1912. Soon after them, W. BLASCHKE [4], H. MOHRMANN [13] and W. VOGT [22] dealt successively with this problem. We could present a long list of authors who either used different means to prove the classical case or generalized in different ways the above mentioned theorem. Some of these authors will be mentioned later. Among others, works [9], [10] and [11] which have been published lately give an evidence of vitality and appeal of this topic. Discrete forms of the classical case are not less interesting. S. BILINSKI [3] proved that in the cyclical sequence $\alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \ldots, \alpha_n - \alpha_{n-1}, \alpha_1 - \alpha_n$ formed from the angles α_i corresponding to the vertices A_i of a regular convex n-sided polygon either all numbers equal zero or there are at least four sign changes. A. D. ALEXANDROV [1] (Chap. VI, § 1) arrived at another form the importance of which is corroborated by the fact that it is possible to derive the classical case from it. A. MOÓR [14], [15] dealt with an analogue of the four-vertex theorem for closed space curves. He found a certain invariant which on a space curve has a similar property as the curvature of an oval curve; but Moór's requirement that a certain integral over a closed curve should equal zero is, from the point of view of geometry, not quite lucid when applied to the general case. Perhaps the best known proof of the classical case was given by G. HERGLOTZ and published by W. Blaschke in [7] (§24) (see also [6], 1924, pp. 15–17). W. Blaschke mentions also that H. GERICKE proved in 1937 the existence of closed space curves having only two vertices; however, W. Blaschke does not give any reference to the literature and H. Gericke in his well known work [8] does not deal with the spatial problem.

The aim of this paper is to study the extremes of curvatures k_τ ($\tau = 1, 2, \ldots, n - 1$) of a closed curve of the p-th class (generally $p \geqq n + 2$) with Frenet's moving n-hedron $\mathbf{t}, \mathbf{e}_1, \ldots, \mathbf{e}_{n-2}, \mathbf{n}$ in the n-dimensional Euclidean space E_n ($n \geqq 3$). Let the radius-vector of a current point of the curve be denoted by $\mathbf{x}$. As the parameter we shall choose either the arc length s of the given curve or the arc length α (in the case of integration variable σ) of the spherical image $\mathscr{C}^*$ of tangent vectors $\mathbf{t}$ of the given curve; we shall denote by a the length of $\mathscr{C}^*$. Primes denote the derivatives with respect either to s or to α according to the argument. For the sake of simplicity, the set $\mathscr{N} = \{1, 2, \ldots, n\}$ is introduced. For each $i \in \mathscr{N}$, $\mathbf{I}_i$ is the i-th coordinate vector, t_i the i-th coordinate of the vector $\mathbf{t}$ and ω_i the deviation of vectors $\mathbf{I}_i$, $\mathbf{t}$. The radius of flexion is r and, finally, $\varkappa_j = k_j/k_1$ $(j = 2, \ldots, n - 1)$.

General assumptions:

(a) The curve reproduces Frenet's n-hedron; this means that for each $\alpha \in \langle 0, a \rangle$ it is $\mathbf{t}(\alpha + a) = \mathbf{t}(\alpha)$, $\mathbf{n}(\alpha + a) = \mathbf{n}(\alpha)$, $\mathbf{e}_\nu(\alpha + a) = \mathbf{e}_\nu(\alpha)$ $(\nu = 1, \ldots, n - 2)$.
(b) For each value of the parameter, the determinant $[\mathbf{t}, \mathbf{e}_1, \ldots, \mathbf{e}_{n-2}, \mathbf{n}] = 1$.
(c) The curvatures k_τ are always positive on the given curve.
(d) When dealing with extremes of curvature k_τ we suppose that on the given curve it is $k''_\tau \neq 0$ whenever $k'_\tau = 0$.

An arbitrary closed of the p-th class with properties (a)–(d) will be denoted by $\mathscr{C}$. A curve $\mathscr{C}$ is said to be a curve $\mathscr{C}_s$ if $\mathbf{t}(\alpha + \frac{1}{2}a) = -\mathbf{t}(\alpha)$ for each $\alpha \in \langle 0, \frac{1}{2}a \rangle$. In accordance with Z. Nádeník [18] (p. 447), points $\mathbf{x}(\alpha)$, $\mathbf{x}(\alpha + \frac{1}{2}a)$ of a curve $\mathscr{C}_s$ are called *opposite points*. It will be shown later that curves $\mathscr{C}_s$ can exist only in spaces of even dimensions.

Any point of a curve $\mathscr{C}$ at which $k'_1 = 0$ and $k''_1 \neq 0$ is called *a vertex*. As $r' = 0$ if and only if $k'_1 = 0$ and for points with $r' = 0$ it is $r'' = -k''_1/k_1^2$, we can equivalently define *a vertex* of a curve $\mathscr{C}$ to be a point at which $r' = 0$ and $r'' \neq 0$; it will be useful to keep this in mind especially when applying the assumption (d) mentioned above. Any point of a curve $\mathscr{C}$ at which $k'_j = 0$ and $k''_j \neq 0$ $(j = 2, \ldots, n - 1)$ is called *a vertex of the j-th type*.

Any hyperplane as well as any hypersphere divide the space into two parts. A point will be said to be *a point of intersection* of a curve $\mathscr{C}$ with a hyperplane or a hypersphere if it has the following property; its each neighbourhood contains inner points of both parts of the space which belong to the curve. If a point of intersection exists, the hyperplane (hypersphere) is said to intersect the curve.

I.

We shall now introduce theorems dealing with vertices on a general curve $\mathscr{C}$.

W. Blaschke [5] (pp. 116 and 160–161) interprets the radius of flexion as a positive mass and deduces the four-vertex theorem on the basis of the closedness of an oval curve using its centre of gravity. Also L. Bieberbach [2] (pp. 23–27) proves the classical case from the closedness of an oval, proceeding then to the more general convex plane curve. Conditions of closedness of a curve $\mathscr{C}$ have the form

$$(1) \qquad \int_{\mathscr{C}^*} r(\sigma)\, \mathbf{t}(\sigma)\, \mathrm{d}\sigma = \mathbf{0}\,,$$

which formally coincides with conditions of closedness of an oval curve (see [5], p. 116); the proof of conditions (1) is given on the page 79. These conditions are used similarly to W. Blaschke in [5]. Hence it results:

Theorem I (see No 1): *Let $\varphi(\sigma)$ be a positive twice differentiable function defined on C^* such that $\varphi''(\sigma) \neq 0$ whenever $\varphi'(\sigma) = 0$ and at least for one $i \in \mathcal{N}$ let*

$$\int_{\mathscr{C}^*} \varphi(\sigma)\, t_i(\sigma)\, \mathrm{d}\sigma = 0\,. \tag{2}$$

Let us measure the arc on $\mathscr{C}^*$ from one of the bisecting points between the points with the maximum and minimum of $\varphi(\sigma)$. Let for each $\sigma \in \langle 0, \frac{1}{2}a\rangle$ either $\omega_i(\sigma) \leqq \frac{1}{2}\pi$, $\omega_i(\sigma) + \omega_i(-\sigma) \leqq \pi$ or $\omega_i(\sigma) \geqq \frac{1}{2}\pi$, $\omega_i(\sigma) + \omega_i(-\sigma) \geqq \pi$ hold. Then $\varphi(\sigma)$ has at least four extremes on $\mathscr{C}^*$.

Theorem II (see No 2): *Let us measure the arc on $\mathscr{C}^*$ from one of the bisecting points between the points with the maximum and minimum of $r(\sigma)$. Assume that at least for one $i \in \mathcal{N}$ and for each $\sigma \in \langle 0, \frac{1}{2}a\rangle$ either $\omega_i(\sigma) \leqq \frac{1}{2}\pi$, $\omega_i(\sigma) + \omega_i(-\sigma) \leqq \pi$ or $\omega_i(\sigma) \geqq \frac{1}{2}\pi$, $\omega_i(\sigma) + \omega_i(-\sigma) \geqq \pi$ holds. Then the curve $\mathscr{C}$ has at least four vertices.*

The connecting line of two points of an oval curve has no other common point of intersection with it. This is one of the two crucial points of Herglotz's proof (see [6], 1924, pp. 15–17). When applying Herglotz's proof to the space case the existence of a hyperplane with a property analogous to that of the connecting line between two points of an oval curve must be required.

Theorem III (see No 3). *Suppose that $\int_{\mathscr{C}^*} \mathbf{t}(\sigma)\, \mathrm{d}\sigma = \mathbf{0}$ and that there exists a hyperplane intersecting the curve $\mathscr{C}$ only in its vertices, m being their number. Then $\mathscr{C}$ has at least $m + 2$ vertices.*

An oval curve with a constant sum of radii of flexion at opposite points is called a curve of constant breadth. It has at least six vertices (see [21], p. 38 or [20]). The following theorem which will be needed in the sequel deals with vertices on a curve $\mathscr{C}$ with a constant sum of radii of flexion at opposite points:

Theorem IV (see No 4): *The number of vertices on a curve $\mathscr{C}$ with constant function $\mathrm{R} = r(\alpha) + r(\alpha + \frac{1}{2}a)$ always equals $4l - 2$ (l positive integer). If the function $\tilde{\psi}(\alpha) = r(\alpha) - \frac{1}{2}\mathrm{R}$ has m zero points in the interval $\langle 0, \frac{1}{2}a)$, then the curve $\mathscr{C}$ has at least $2m$ vertices for m odd and at least $2m + 2$ vertices for m even; the value of at least m maxima and m minima is respectively higher and lower than $\frac{1}{2}R$.*

II.

Some curves $\mathscr{C}_s$ are a close spatial analogue of the oval curve, therefore it is natural that we can obtain more detailed results concerning its extremes than in the general case.

Theorem V (see No 5).

a) *Suppose that there exists a hyperplane intersecting a curve $\mathscr{C}_s$ at m points, all of them being vertices of the curve. Then $\mathscr{C}_s$ has at least $m + 2$ vertices.*

b) *Suppose that there exists a hyperplane intersecting a curve $\mathscr{C}_s$ at m points, all of them being vertices of the j-th type of the curve. Then $\mathscr{C}_s$ has at least $m + 2$ vertices of the j-th type.*

c) *Let a curve $\mathscr{C}_s$ be a curve with constant sum of radii of flexion at opposite points. Suppose that there exists either a hyperplane or a hypersphere intersecting it at $4\,l - 2$ points, all of them being vertices of the curve. Then the curve $\mathscr{C}_s$ has at least $4l + 2$ vertices.*

On an oval curve there exist at least three points with the flexion equal to that at the opposite points. This property of the oval curve was derived by P. VINCENSINI [21] (pp. 35–38) from Segre's theorem [19]. The existence of opposite points with equal flexion was proved in another way by W. SÜSS [20]. Hence W. Süss deduces the four-vertex theorem giving simultaneously the lower (upper) bound for maxima (minima). Süss's method proved to be advantageous also for curves $\mathscr{C}_s$ in spaces of higher even dimensions yielding the following theorem.

Theorem VI (see No 6). *Let $\varphi(\sigma)$ be a twice differentiable function defined on $\mathscr{C}_s^*$ such that $\varphi''(\sigma) \neq 0$ whenever $\varphi'(\sigma) = 0$. Supposing that*

$$\int_{\mathscr{C}^*_s} \varphi(\sigma)\, \mathbf{t}(\sigma)\, \mathrm{d}\sigma = \mathbf{0} \tag{3}$$

and that there exists a hyperplaee passing throught the centre of the unit hypersphere and intersecting $\mathscr{C}_s^*$ only at two points at which the function $\psi(\sigma) = \varphi(\sigma) - \varphi(\sigma + \frac{1}{2}a)$ changes the sign, then the value of $\varphi(\sigma)$ in at least three points of the curve $\mathscr{C}_s$ is the same as at the opposite points and $\varphi(\sigma)$ has extremes at least at four points.

Further theorem is a consequence which is easy to establish.

Theorem VII (see No 7):

a) *Suppose that there exists a hyperplane passing through the centre of the unit hypersphere and intersecting $\mathscr{C}_s^*$ only at two points at which the difference $r(\sigma) - r(\sigma + \frac{1}{2}a)$ changes the sign. Then the curve $\mathscr{C}_s$ has at least four vertices and at least three points with the flexion equal to that at the opposite points.*

b) *Let, apart from the assumption mentioned in a), the sum* $\mathrm{R} = r(\alpha) + r(\alpha + \frac{1}{2}a)$ *be constant on* $\mathscr{C}_s$. *Then* $\mathscr{C}_s$ *has at least six vertices; at least three maxima are greater and at least three minima lower than* $\frac{1}{2}\mathrm{R}$.

As it was proved by W. Süss [20], each oval curve has at least two maxima of the radius of flexion greater and at least two minima lower than $L/2\pi$, where L is the length of the curve. On centrally symmetrical curves $\mathscr{C}$, the existence of at least four extremes of each curvature k_τ is evident. Let L be the length of the curve $\mathscr{C}$. We denote $r_j(\sigma) = k_j^{-1}(\sigma)$ and $L_j = \int_0^a r_j(\sigma)\,d\sigma$ $(j = 2, \ldots, n-1)$. Then the centrally symmetrical curve $\mathscr{C}$ has at least two maxima of $r(\sigma)$ or $r_j(\sigma)$ greater and at least two minima lower than L/a or L_j/a, respectively; on a general curve $\mathscr{C}$ such an estimate can be established only for the absolute maximum and minimum of $r(\sigma)$ or $r_j(\sigma)$.

The last theorem concerning extremes of curvatures on a curve $\mathscr{C}_s$, which will be presented now, has a more general character:

Theorem VIII (see No 8).

Let $\Phi(\alpha)$ *be a twice differentiable function formed by a finite number of elementary functional operations from the functions* $\varkappa_j(\alpha)$ $(j = 2, \ldots, n-1)$ *with the following property:* $\Phi''(\alpha) \neq 0$ *whenever* $\Phi'(\alpha) = 0$. *Then* $\Phi(\alpha)$ *always has at least four local extremes on the curve* $\mathscr{C}_s$.

III.

Now we shall consider some special curves $\mathscr{C}$ of the n-th class from spaces of an even dimension. In order to prove the existence of vertices on them we shall use certain integral equalities valid for them. The same process was chosen by G. Herglotz in the classical case and also by A. Moór [15] when studying extremes of his invariant on space curves. We shall be interested only in vertices (not in vertices of the j-th type). Therefore we reduced the starting assumption about the class of the curve $\mathscr{C}$ so that the existence of Frenet's n-hedron at each point of the curve be guaranteed.

Theorem IX (see No 9): *Let — in a space of an even dimension — exist a hyperplane or a hypersphere intersecting a curve* $\mathscr{C}$ *with constant ratios of curvature* $k_{2i}(s)/k_{2i+1}(s)$ $(i = 1, \ldots, \frac{1}{2}(n-2))$ *only at vertices. Provided that the number of these points of intersection is m, the curve* $\mathscr{C}$ *has at least* $m + 2$ *vertices.*

In a space of an even dimension the curve $\mathbf{x} = \mathbf{x}(s)$ having constant ratios of curvatures $k_{2i}(s)/k_{2i+1}(s)$ $(i = 1, \ldots, \frac{1}{2}(n-2))$ is, as defined by E. Čech (see [17], pp. 57–58), Bertrand's curve with the only parametric system of conjugate lines

$$(4) \qquad {}^1\mathbf{x}({}^1s) = \mathbf{x}(s) + \lambda \left\{ \mathbf{e}_1(s) + \sum_{j=1}^{(n-4)/2} \prod_{\nu=1}^{j} \frac{k_{2\nu}(s)}{k_{2\nu+1}(s)}\, \mathbf{e}_{2j+1}(s) + \right. \left. + \prod_{\nu=1}^{(n-2)/2} \frac{k_{2\nu}(s)}{k_{2\nu+1}(s)}\, \mathbf{n}(s) \right\},$$

where λ is a parameter different from zero (when the flexion is constant then also $\lambda \neq r(s)$). For curves from E_4, E. Čech proved the above assertion and it is given by Z. Nádeník in [17] (p. 62); Čech's result is easily transferable into spaces of a higher even dimension. Therefore we shall give only the basic steps of it: Differentiating successively (4) we see that ${}^1s = \varepsilon \int \{1 - k_1(s)\}\, ds$. Further, ${}^1\mathbf{t} = \varepsilon \mathbf{t}$, ${}^1\mathbf{e}_\nu = \varepsilon \mathbf{e}_\nu$ ($\nu = 1, \ldots, n - 2$) and ${}^1\boldsymbol{n} = \varepsilon \boldsymbol{n}$, where $\varepsilon = 1$ or $\varepsilon = -1$. Obviously Theorem IX deals only with Bertrand's curves (4) which lead to closed curves $\mathscr{C}$.

Proofs of the above theorems follow.

1. It is evident that a curve $\mathscr{C}$ is closed if and only if $\mathbf{x}(\alpha + a) = \mathbf{x}(\alpha)$ for each $\alpha \in \langle 0, a \rangle$; as $\int_{\mathscr{C}^*} r(\sigma)\, \mathbf{t}(\sigma)\, d\sigma = \int_\alpha^{\alpha+a} \mathbf{x}'(\sigma)\, d\sigma = [\mathbf{x}(\sigma)]_{\sigma=\alpha}^{\sigma=\alpha+a} = \mathbf{x}(\alpha + a) - \mathbf{x}(\alpha)$ the curve $\mathscr{C}$ is closed if and only if (1) holds. Now we shall proceed to the proof of Theorem I: The existence of two points at which $\varphi(\sigma)$ has an extreme is guaranteed; let the minimum be attained at the point Q, the maximum at the point P on $\mathscr{C}^*$. The bisecting points M, N of arcs $\overset{\frown}{PQ}$ correspond, according to the assumption, to the values of parameter $0, \frac{1}{2}a$. We shall arrange the integral (2) so that the integration interval is $\langle 0, \frac{1}{2}a \rangle$:

$$(1,1) \qquad \int_{\mathscr{C}^*} \varphi(\sigma)\, t_i(\sigma)\, d\sigma = \int_0^{a/2} \{\varphi(\sigma)\, t_i(\sigma) + \varphi(-\sigma)\, t_i(-\sigma)\}\, d\sigma = 0\,.$$

Suppose that $\varphi(\sigma)$ has no extreme on $\mathscr{C}^*$ except the points P, Q. When passing from Q to P the value of $\varphi(\sigma)$ increases on both arcs $\overset{\frown}{QP}$; therefore in the interval $(0, \frac{1}{2}a)$ the inequality

$$(1,2) \qquad \varphi(\sigma) > \varphi(-\sigma)$$

holds. Further $t_i(\sigma) = \boldsymbol{l}_i\, \mathbf{t}(\sigma) = \cos \omega_i(\sigma)$; if $\omega_i(\sigma) + \omega_i(-\sigma) \leqq \pi$, then $\cos \omega_i(\sigma) + \cos \omega_i(-\sigma) \geqq 0$ and consequently

$$(1,3) \qquad t_i(\sigma) \geqq -t_i(-\sigma)\,.$$

As $\omega_i(\sigma) \leqq \frac{1}{2}\pi$, it is $t_i(\sigma) \geqq 0$; the equality cannot hold in the whole interval of integration since it would contradict the starting assumption (c). Multiplying (1,3) and (1,2) we conclude that $\varphi(\sigma)\, t_i(\sigma) + \varphi(-\sigma)\, t_i(-\sigma) \geqq 0$ where the equality does not hold identically; hence we have a contradiction with the equality (1,1). Thus the function $\varphi(\sigma)$ has two further extremes. The alternative case can be proved analogously.

2. The proof follows directly from (1) and Theorem I; the extreme of $r(\sigma)$ on $\mathscr{C}^*$ means a vertex on the curve $\mathscr{C}$ at the corresponding point $\mathbf{x}(\sigma)$.

3. Let $\boldsymbol{v}$ be an arbitrary fixed vector, v_0 an arbitrary constant. As $\int_{\mathscr{C}^*} \mathbf{t}(\sigma)\, d\sigma = \mathbf{0}$, therefore $\int_{\mathscr{C}^*} \mathbf{v} \,.\, \mathbf{t}(\sigma)\, \mathrm{d}\sigma = 0$ and further

$$\int_{\mathscr{C}^*} \mathbf{v} \,.\, \mathbf{t}(\sigma)\, \mathrm{d}\sigma = \int_{\mathscr{C}} \mathbf{v} \,.\, \mathbf{t}(s)\, k_1(s)\, \mathrm{d}s = - \int_{\mathscr{C}} \mathbf{v} \,.\, \mathbf{x}(s)\, k_1'(s)\, \mathrm{d}s = 0 \,.$$

Then the equality

$$\int_{\mathscr{C}} \{\mathbf{v} \,.\, \mathbf{x}(s) + v_0\}\, k_1'(s)\, \mathrm{d}s = 0 \tag{3,1}$$

also holds. The hyperplane $\mathbf{v} \,.\, \mathbf{x} + v_0 = 0$ from the assumption of the theorem determines two half spaces in which it holds $\mathbf{v} \,.\, \mathbf{x}(s) + v_0 \leqq 0$, $\mathbf{v} \,.\, \mathbf{x}(s) + v_0 \geqq 0$ respectively. Suppose that $\mathscr{C}$ has no other vertex; then in half space it is also $k_1'(s) \geqq 0$ and in the second half space it's $k_1'(s) \leqq 0$; this means that the integrand in (3,1) which is not identically equal to zero, does not change the sign. But this contradicts the equality (3,1). The existence of at least two other vertices has been proved.

4. When the radius of flexion $r(\alpha)$ of the curve $\mathscr{C}$ attains its maximum for the value α_0 of the parameter, it necessarily follows that it has its minimum for $\alpha_0 + \frac{1}{2}a$. If the number of extremes in the interval $\langle 0, \frac{1}{2}a)$ were even, the first extreme being, say, the maximum, then the last extreme would be a minimum. Therefore the first extreme in the interval $\langle \frac{1}{2}a, a)$ would have to be a minimum; this is not possible because extremes alternate. Thus it is impossible for the number of extremes on the considered curve $\mathscr{C}$ to be a multiple of four. Further, $\tilde{\psi}(\alpha + \frac{1}{2}a) = -\tilde{\psi}(\alpha)$, therefore $\tilde{\psi}(\alpha)$ has in the interval $\langle 0, a)$ necessarily $2m$ roots. Between each two adjacent roots there is at least one extreme; for the maximum $r(\alpha) > \frac{1}{2}R$, for the minimum $r(\alpha) < \frac{1}{2}R$. The difference between the number of vertices for m even and odd is the consequence of the characteristic number $4l - 2$.

5. Before starting with the proof of the theorem, it is necessary to introduce some basic properties of curves $\mathscr{C}_s$. If we take the arc length α as the parameter of the spherical image $\mathscr{C}_s^*$ of tangent vectors, then Frenet's formulae have the form

$$\begin{aligned}
\mathbf{t}'(\alpha) &= \mathbf{e}_1(\alpha)\,,\\
\mathbf{e}_1'(\alpha) &= -\mathbf{t}(\alpha) + \varkappa_2(\alpha)\, \mathbf{e}_2(\alpha)\,,\\
\mathbf{e}_\mu'(\alpha) &= -\varkappa_\mu(\alpha)\, \mathbf{e}_{\mu-1}(\alpha) + \varkappa_{\mu+1}(\alpha)\, \mathbf{e}_{\mu+1}(\alpha)\,, \quad (\mu = 2, \ldots, n-3)\\
\mathbf{e}_{n-2}'(\alpha) &= -\varkappa_{n-2}(\alpha)\, \mathbf{e}_{n-3}(\alpha) + \varkappa_{n-1}(\alpha)\, \mathbf{n}(\alpha)\,,\\
\mathbf{n}'(\alpha) &= -\varkappa_{n-1}(\alpha)\, \mathbf{e}_{n-2}(\alpha)\,.
\end{aligned} \tag{5,1}$$

From the definition of the curve $\mathscr{C}_s$ the symmetry of $\mathscr{C}_s^*$ with respect to the centre of a unit hypersphere follows; the same also holds for all spherical images of normals of the curve $\mathscr{C}_s$. From the first two formulae (5,1) we successively find that $\mathbf{e}_1(\alpha + \frac{1}{2}a) = -\mathbf{e}_1(\alpha)$, $\mathbf{e}_2(\alpha + \frac{1}{2}a) = -\mathbf{e}_2(\alpha)$ and $\varkappa_2(\alpha + \frac{1}{2}a) = \varkappa_2(\alpha)$ on $\mathscr{C}_s$; proceeding recursively we arrive at the conclusion that generally for each α the following equalities hold for $\mathscr{C}_s$:

$$(5,2)\qquad \mathbf{n}(\alpha + \tfrac{1}{2}a) = -\mathbf{n}(\alpha)\,,\quad \mathbf{e}_\nu(\alpha + \tfrac{1}{2}a) = -\mathbf{e}_\nu(\alpha)\quad (\nu = 1, \dots, n-2)\,,$$

$$(5,3)\qquad \varkappa_j(\alpha + \tfrac{1}{2}a) = \varkappa_j(\alpha)\quad (j = 2, \dots, n-1)\,.$$

As it has been mentioned, curves $\mathscr{C}_s$ can exist only in spaces of even dimensions. Indeed, in the space of an odd dimension the starting assumption (b) can not be fulfilled; if the value of the corresponding determinant at the point $\mathbf{x}(\alpha)$ equals 1, then, as it follows from (5,2), its value at the opposite point $\mathbf{x}(\alpha + \frac{1}{2}a)$ equals -1.

Now we shall prove Theorem V.

a) This part is a simple consequence of Theorem III, since on the curve $\mathscr{C}_s$, $\int_{\mathscr{C}^*_s} \mathbf{t}(\sigma)\,\mathrm{d}\sigma = \mathbf{0}$.

b) According to (5,3), the following equalities hold on $\mathscr{C}_s$ for each fixed vector $\mathbf{v}$: $\mathbf{v}\,.\int_{\mathscr{C}^*_s} \varkappa_j(\sigma)\,\mathbf{t}(\sigma)\,\mathrm{d}\sigma = 0$ $(j = 2, \dots, n-1)$. But we have $\int_{\mathscr{C}_s^*} \varkappa_j(\sigma)\,\mathbf{v}\,.\,\mathbf{t}(\sigma)\,\mathrm{d}\sigma = -\int_{\mathscr{C}_s} \mathbf{v}\,.\,\mathbf{x}(s)\,k_j'(s)\,\mathrm{d}s$ and therefore the equalities

$$(5,4)\qquad \int_{\mathscr{C}_s} \{\mathbf{v}\,.\,\mathbf{x}(s) + v_0\}\,k_j'(s)\,\mathrm{d}s = 0$$

hold, where v_0 is an arbitrary constant. The equalities (5,4) are entirely analogous to (3,1); in the same way as in the proof of Theorem III the proof can be given by contradiction.

c) In the case of a hyperplane this part is a consequence of Theorems III and IV. Suppose that there exists a hypersphere having the required properties. Differentiating the evident equality $\int_{\mathscr{C}_s^*}\{\mathbf{x}(\sigma) - \mathbf{x}(\sigma + \frac{1}{2}a)\}\,.\,\{\mathbf{x}(\sigma) - \mathbf{x}(\sigma + \frac{1}{2}a)\}'\,\mathrm{d}\sigma = 0$ we obtain the equality $\int_{\mathscr{C}_s^*}\{\mathbf{x}(\sigma) - \mathbf{x}(\sigma + \frac{1}{2}a)\}\,.\,\mathbf{t}(\sigma)\,\mathrm{d}\sigma = 0$ which implies

$$\int_{\mathscr{C}_s^*} \mathbf{x}(\sigma)\,.\,\mathbf{t}(\sigma)\,\mathrm{d}\sigma = \int_{\mathscr{C}_s^*} \mathbf{x}(\sigma + \tfrac{1}{2}a)\,.\,\mathbf{t}(\sigma)\,\mathrm{d}\sigma = \int_{\mathscr{C}_s^*} \mathbf{x}(\sigma)\,.\,\mathbf{t}(\sigma + \tfrac{1}{2}a)\,\mathrm{d}\sigma =$$
$$= -\int_{\mathscr{C}_s^*} \mathbf{x}(\sigma)\,.\,\mathbf{t}(\sigma)\,\mathrm{d}\sigma\,.$$

Therefore $\int_{\mathscr{C}_s^*} \mathbf{x}(\sigma)\,.\,\mathbf{t}(\sigma)\,\mathrm{d}\sigma = 0$ and integrating over the curve $\mathscr{C}_s$ we have $\int_{\mathscr{C}_s} \mathbf{x}(s)\,.\,\mathbf{x}(s)\,k_1'(s)\,\mathrm{d}s = 0$. But for an arbitrary vector $\mathbf{v}$ and an arbitrary constant v_0, (3,1) also holds on $\mathscr{C}_s$; hence we may write

$$(5,5)\qquad \int_{\mathscr{C}_s} \{\mathbf{x}(s)\,.\,\mathbf{x}(s) + \mathbf{v}\,.\,\mathbf{x}(s) + v_0\}\,k_1'(s)\,\mathrm{d}s = 0\,.$$

Now let us only sketch the rest briefly: The hypersphere with the equation $\mathbf{x}\,.\,\mathbf{x} + \mathbf{v}\,.\,\mathbf{x} + v_0 = 0$ divides the space into two parts; on the inner arcs of the curve $\mathscr{C}_s$

$\mathbf{x}(s) \,.\, \mathbf{x}(s) + \mathbf{v} \,.\, \mathbf{x}(s) + v_0 \leqq 0$, on outer ones, $\mathbf{x}(s) \,.\, \mathbf{x}(s) + \mathbf{v} \,.\, \mathbf{x}(s) + v_0 \geqq 0$. If the curve $\mathscr{C}_s$ had not further vertices, it would be $k_1'(s) \geqq 0$ on the inner arcs and $k_1'(s) \leqq 0$ on the outer ones (or vice versa). The integrand in (5,5), which does not identically equal zero, would not change its sign; but then the equality (5,5) could not hold. The number of vertices is therefore necessarily greater at least by two; from Theorem IV it follows that it increases at least by four.

6. We proceed analogously as W. Süss [20]. If we arrange (3) so that the integration interval is $\langle 0, \frac{1}{2}a \rangle$ then

$$(6,1) \qquad \int_0^{a/2} \psi(\sigma)\, \mathbf{t}(\sigma)\, \mathrm{d}\sigma = \mathbf{0}$$

and it holds that

$$(6,2) \qquad \psi(\sigma + \tfrac{1}{2}a) = -\psi(\sigma)\,.$$

If the trivial case when identically $\varphi(\sigma) = \varphi(\sigma + \frac{1}{2}a)$ is excluded from our consideration, then according to (6,2) the periodic function $\psi(\sigma)$ changes its sign at least at two points in the interval $\langle 0, a)$. To these zero points of the function $\psi(\sigma)$ at least one couple of points on $\mathscr{C}_s^*$ corresponds which are symmetrically conjugate with respect to the origin, with the same value $\varphi(\sigma)$. If the required hyperplane exists and if the arc on $\mathscr{C}_s^*$ is measured from one of the points of intersection, then $\psi(\sigma) = \psi(\sigma + \frac{1}{2}a) = 0$. Let us denote by $\mathbf{v}$ the vector which is perpendicular to the required hyperplane. We scalarly multiply (3) by vector $\mathbf{v}$; scalar product $\mathbf{v} \,.\, \mathbf{t}(\sigma)$ does not change the sign in the interval $\langle 0, \frac{1}{2}a \rangle$ and is not identically equal to zero (the contrary would be in contradiction with the starting assumption (c)). According to (6,2) $\psi(\sigma)$ has in the right neighbourhood of the point zero the same sign as in the left neighbourhood of the point $\frac{1}{2}a$. Hence the equality (6,1) can hold only if $\psi(\sigma)$ changes its sign at least at two inner points of the interval $\langle 0, \frac{1}{2}a \rangle$. Therefore there exist at least three points on $\mathscr{C}_s^*$ where $\varphi(\sigma)$ has the same value as at the points symmetrical with respect to the origin. If $\varphi(\sigma)$ had only one maximum on $\mathscr{C}_s^*$ (and therefore only one minimum), one of these extremes would have to be in the interval $(0, \frac{1}{2}a)$, the other in the interval $(\frac{1}{2}a, a)$; but in this case the function $\psi(\sigma)$ would have the same sign in the whole interval $(0, \frac{1}{2}a)$. We already know that this cannot occur. It is sufficient now to pass from $\mathscr{C}_s^*$ to the curve $\mathscr{C}_s$ and the proof is complete.

7. a) The proof follows from conditions of closedness (1) and from Theorem VI.

b) It follows from (1) and from the proof of Theorem VI that the function $\psi(\sigma) = r(\sigma) - r(\sigma + \frac{1}{2}a)$ has at least three zero points in the interval $\langle 0, \frac{1}{2}a)$; therefore also the function $\tilde{\psi}(\sigma) = \frac{1}{2}\psi(\sigma) = r(\sigma) - \frac{1}{2}R$ has at least three zero points in the interval $\langle 0, \frac{1}{2}a)$. The assertion follows then directly from Theorem IV.

8. According to (5,3) $\Phi(\alpha + \frac{1}{2}a) = \Phi(\alpha)$ holds; four extremes are therefore an obvious consequence of the periodicity of $\Phi(\alpha)$.

9. When choosing the natural parameter s, Frenet's formulae have the well known form:

$$
\begin{aligned}
(9,1) \qquad \mathbf{t}_1'(s) &= \phantom{-k_1(s)\, \mathbf{t}(s) +{}} k_1(s)\, \mathbf{e}_1(s)\,, \\
\mathbf{e}_1'(s) &= -k_1(s)\, \mathbf{t}(s) + k_2(s)\, \mathbf{e}_2(s)\,, \\
\mathbf{e}_\mu'(s) &= -k_\mu(s)\, \mathbf{e}_{\mu-1}(s) + k_{\mu+1}(s)\, \mathbf{e}_{\mu+1}(s) \qquad (\mu = 2, \dots, n-3)\,, \\
\mathbf{e}_{n-2}(s) &= -k_{n-2}(s)\, \mathbf{e}_{n-3}(s) + k_{n-1}(s)\, \mathbf{n}(s)\,, \\
\mathbf{n}'(s) &= -k_{n-1}(s)\, \mathbf{e}_{n-2}(s)\,.
\end{aligned}
$$

First we shall prove that, on a curve $\mathscr{C}_s$ with constant ratios of curvatures $k_{2i}(s)/k_{2i+1}(s)$ $(i = 1, \dots, \frac{1}{2}(n-2))$, equalities (3,1) and (5,5) hold for each fixed vector $\mathbf{v}$ and each constant v_0. Let us denote $J_1 = \int_{\mathscr{C}} v_0\, k_1'(s)\, \mathrm{d}s$, $J_2 = \int_{\mathscr{C}} \mathbf{v} \,.\, \mathbf{x}(s) \,.\, k_1'(s)\, \mathrm{d}s$, $J_3 = \int_{\mathscr{C}} \mathbf{x}(s) \,.\, \mathbf{x}(s)\, k_1'(s)\, \mathrm{d}s$. Evidently $J_1 = 0$; we shall show that also $J_2 = J_3 = 0$. Integrating by parts and substituing according to (9,1) for $\mathbf{t}(s)$ and $\mathbf{e}_2(s)$ we obtain successively

$$
\begin{aligned}
J_2 &= \int_{\mathscr{C}} \mathbf{v} \,.\, \mathbf{x}(s)\, k_1'(s)\, \mathrm{d}s = -\int_{\mathscr{C}} \mathbf{v} \,.\, \mathbf{t}(s)\, k_1(s)\, \mathrm{d}s = \\
&= \int_{\mathscr{C}} \mathbf{v} \,.\, \mathbf{e}_1'(s)\, \mathrm{d}s - \int_{\mathscr{C}} \mathbf{v} \,.\, \mathbf{e}_2(s)\, k_2(s)\, \mathrm{d}s = \\
&= \int_{\mathscr{C}} \mathbf{v} \,.\, \mathbf{e}_1'(s)\, \mathrm{d}s + \frac{k_2(s)}{k_3(s)} \int_{\mathscr{C}} \mathbf{v} \,.\, \mathbf{e}_3'(s)\, \mathrm{d}s - \frac{k_2(s)}{k_3(s)} \int_{\mathscr{C}} \mathbf{v} \,.\, \mathbf{e}_4(s)\, k_4(s)\, \mathrm{d}s\,.
\end{aligned}
$$

The first two integrals in the last expression for J_2 evidently equal zero; hence

$$
(9,2) \qquad J_2 = -\frac{k_2(s)}{k_3(s)} \int_{\mathscr{C}} \mathbf{v} \,.\, \mathbf{e}_4(s)\, k_4(s)\, \mathrm{d}s\,.
$$

As we suppose that the ratios of curvatures are constant it recurrently follows from (9,1) $(j = 1, \dots, \frac{1}{2}(n-6))$: If

$$
(9,3) \qquad J_2 = -\prod_{\nu=1}^{j} \frac{k_{2\nu}(s)}{k_{2\nu+1}(s)} \int_{\mathscr{C}} \mathbf{v} \,.\, \mathbf{e}_{2j+2}(s)\, k_{2j+2}(s)\, \mathrm{d}s\,,
$$

then also

$$
(9,4) \qquad J_2 = -\prod_{\nu=1}^{j+1} \frac{k_{2\nu}(s)}{k_{2\nu+1}(s)} \int_{\mathscr{C}} \mathbf{v} \,.\, \mathbf{e}_{2j+4}(s)\, k_{2j+4}(s)\, \mathrm{d}s\,.
$$

This assertion is based on the following argument: If we substitute $\mathbf{e}_{2j+2}(s)$ in (9,3) by the corresponding expression from (9,1) we obtain two integrals one of which equals zero while the other has the form (9,4). (9,3) is fulfilled for $j = 1$ according to (9,2); hence proceeding recursively for $j = \frac{1}{2}(n - 6)$ it follows that

$$J_2 = -\prod_{\nu=1}^{(n-4)/2} \frac{k_{2\nu}(s)}{k_{2\nu+1}(s)} \int_{\mathscr{C}} \mathbf{v} \cdot \mathbf{e}_{n-2}(s)\, k_{n-2}(s)\, \mathrm{d}s\,.$$

When substituting here for $\mathbf{e}_{n-2}(s)$ the corresponding expression from the last equation (9,1) we directly obtain that $J_2 = 0$. Thus the equality (3,1) is proved.

To prove the equality (5,5) it is sufficient to show that $J_3 = 0$; integration by parts and substitution for $\mathbf{t}(s)$ and $\mathbf{e}_2(s)$ from (9,1) yield

$$J_3 = -2\int_{\mathscr{C}} \mathbf{t}(s) \cdot \mathbf{e}_1(s)\, \mathrm{d}s - 2\frac{k_2(s)}{k_3(s)} \int_{\mathscr{C}} \mathbf{t}(s) \cdot \mathbf{e}_3(s)\, \mathrm{d}s - $$
$$- 2\frac{k_2(s)}{k_3(s)} \int_{\mathscr{C}} \mathbf{x}(s) \cdot \mathbf{e}_4(s)\, k_4(s)\, \mathrm{d}s\,.$$

As the first two integrals in J_3 equal zero owing to the orthogonality of the corresponding vectors, it holds that

$$J_3 = -2\frac{k_2(s)}{k_3(s)} \int_{\mathscr{C}} \mathbf{x}(s) \cdot \mathbf{e}_4(s)\, k_4(s)\, \mathrm{d}s\,. \tag{9,5}$$

Similarly as in the case of J_2, the following recurrent relation ($j = 1, \ldots, \frac{1}{2}(n - 6)$) can be verified by using (9,1): if

$$J_3 = -2\prod_{\nu=1}^{j} \frac{k_{2\nu}(s)}{k_{2\nu+1}(s)} \int_{\mathscr{C}} \mathbf{x}(s) \cdot \mathbf{e}_{2j+2}(s)\, k_{2j+2}(s)\, \mathrm{d}s\,,$$

then also

$$J_3 = -2\prod_{\nu=1}^{j+1} \frac{k_{2\nu}(s)}{k_{2\nu+1}(s)} \int_{\mathscr{C}} \mathbf{x}(s) \cdot \mathbf{e}_{2j+4}(s)\, k_{2j+4}(s)\, \mathrm{d}s\,.$$

According to (9,5) the recurrent relation is valid for $j = 1$; we can then write for $j = \frac{1}{2}(n - 6)$:

$$J_3 = -2\prod_{\nu=1}^{(n-4)/2} \frac{k_{2\nu}(s)}{k_{2\nu+1}(s)} \int_{\mathscr{C}} \mathbf{x}(s) \cdot \mathbf{e}_{n-2}(s)\, k_{n-2}(s)\, \mathrm{d}s\,.$$

When substituting here for $\mathbf{e}_{n-2}(s)$ the corresponding expression from the last equation (9,1) and integrating by parts, we obtain the required equality $J_3 = 0$.

Thus equalities (3,1) and (5,5) hold on any curve $\mathscr{C}$ which has the given properties; how to use these equalities to verify Theorem IX can be seen from the proofs of Theorem III and of part c) of Theorem V.

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