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INTEGRALS OF THE CAUCHY TYPE

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Introduction. The present paper is a continuation of [2] where we investigated the behavior of the real part $P_K F(z)$ of the integral

(1) $$I_K F(z) = \int_K \frac{F(\xi)}{\xi - z} \, d\xi$$

assuming that K is a simple oriented path of finite length in the complex plane R^2 and F belongs to a certain class of continuous real-valued functions defined on K. Certain geometric quantities analogous to those employed in [2] can also be used for deriving necessary and sufficient conditions guaranteeing the existence of angular limits of the double layer logarithmic potential

(2) $$W_K F(z) = \operatorname{Im} I_K F(z)$$

which has many applications (see [1], [7]). Let us fix a point $\eta \in K$, a bounded lower-semicontinuous function $Q \geqq 0$ and consider the class $\Omega_Q(\eta)$ of all continuous real-valued functions F satisfying

(3) $$F(\xi) - F(\eta) = o(Q(\xi)) \quad \text{as} \quad \xi \to \eta .$$

We shall be engaged with necessary and sufficient conditions on K ensuring the existence of angular limits of $W_K F$ at η for any $F \in \Omega_Q(\eta)$.

As in [2], we form the sum

$$U_K^Q(\varrho, \eta) = \sum_{\xi} Q(\xi) , \quad \xi \in \{\zeta \in K; \ |\zeta - \eta| = \varrho\} ,$$

counting, with the weight $Q(\xi)$, the points ξ in the intersection of K and the circumference of center η and radius ϱ. Since $U_K^Q(\varrho, \eta)$ is a Lebesgue measurable function of the variable $\varrho > 0$ we may put for any $r > 0$

$$U_{Kr}^Q(\eta) = \int_0^r U_K^Q(\varrho, \eta) \, d\varrho .$$

For $\gamma \in \langle 0, 2\pi \rangle$ we consider also the intersections ξ of K with the half-line $S^\gamma(\eta) = \{\eta + \varrho e^{i\gamma}; \varrho > 0\}$ and introduce the sum

$$V_K^Q(\gamma, \eta) = \sum_\xi Q(\xi), \quad \xi \in K \cap S^\gamma(\eta),$$

counting them with the weight $Q(\xi)$. Then $V_K^Q(\gamma, \eta)$ is a Lebesgue measurable function of the variable $\gamma \in \langle 0, 2\pi \rangle$, so that the definition

$$V_K^Q(\eta) = \int_0^{2\pi} V_K^Q(\gamma, \eta)\, d\gamma$$

is justified.

With this notation we may formulate the condition

$$V_K^Q(\eta) + \sup_{r>0} r^{-1} U_{K,r}^Q(\eta) < \infty \tag{4}$$

which appears to be necessary and sufficient for the existence of angular limits of $W_K F$ at η for any $F \in \Omega_Q(\eta)$. The following more precise assertion is a typical corollary of main results on $W_K F$ established below (some of these results have been announced without proofs in [3], [6]).

Theorem 1. *Let $S \subset R^2 \setminus K$ be a connected set whose closure meets K at η only and suppose that the contingent*[1]) *of S at η together with its reflection in η is disjoint from the contingent of K at η. If*

$$\limsup_{z \to \eta, z \in S} W_K F(z) < \infty$$

for any $F \in \Omega_Q(\eta)$, then (4) holds. Conversely, assume (4) and suppose that $\Theta(r) \geqq 0$ is bounded continuous non-decreasing function of the variable $r \geqq 0$, $\sup_{r \geqq 0} \Theta(r) > 0$. If F is a (real-valued) bounded Baire function on K satisfying

$$F(\xi) - F(\eta) = O(\Theta(|\xi - \eta|)\, Q(\xi) \quad as \quad \xi \to \eta, \tag{5}$$

then the integral

$$W_K^0 F(\eta) = \int_K [F(\xi) - F(\eta)] \operatorname{Im} \frac{d\xi}{\xi - \eta}$$

converges and if $\Delta \arg [\xi - z; \xi \in K]$ denotes the increment of the argument of

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[1]) Let us recall that the contingent of a set $M \subset R^2$ at $\eta \in R^2$ consists of all the half-lines $\{\eta + \varrho\vartheta; \varrho > 0\}$ such that there is a sequence $z_n \in M \setminus \{\eta\}$ tending to η with $\lim_\eta (z_n - \eta) / |z_n - \eta| = \vartheta$ (see [9], § 2 in chap. X).

$\xi - z$ as ξ describes K, then for $z \in S$ the following estimate holds

(6) $$W_K F(z) - W_K^0 F(\eta) - F(\eta) \Delta \arg [\xi - z;\ \xi \in K] = = O(|z - \eta| \int_{|z-\eta|}^{\infty} r^{-2} \Theta(r) \,\mathrm{d}r) \quad as \quad z \to \eta\,.$$

If F satisfies (5) with O replaced by o, then the right-hand side in (6) may be replaced by

(7) $$o\left(|z - \eta| \int_{|z-\eta|}^{\infty} r^{-2} \Theta(r) \,\mathrm{d}r\right) + O(|z - \eta|)\,.$$

It is interesting to observe the analogy between the above condition (4) and the condition (5) considered in [2] in connection with $P_K F$. Combining results on $P_K F$ with those concerning $W_K F$ one may naturally obtain theorems on

$$I_K F = P_K F + \mathrm{i} W_K F\,.$$

In particular, writing (in accordance with [2])

$$U_K^Q(\eta) = \int_0^{\infty} \varrho^{-1} U_K^Q(\varrho, \eta) \,\mathrm{d}\varrho\,,$$

we get that

(8) $$U_K^Q(\eta) + V_K^Q(\eta) < \infty$$

is necessary and sufficient for the existence of angular limits of

$$I_K^0 F(z) = I_K F(z) - F(\eta) \int_K \frac{\mathrm{d}\xi}{\xi - z}$$

at η for any $F \in \Omega_Q(\eta)$. More precisely:

Theorem 2. *If, under the above assumption on S,*

$$\limsup_{z \to \eta, z \in S} |I_K^0 F(z)| < \infty$$

for any $F \in \Omega_Q(\eta)$, then (8) holds. Conversely, if (8) is valid and F is an arbitrary bounded complex-valued Baire function on K satisfying (5), where $\Theta(.)$ has the meaning described above, then the integral

$$I_K^0 F(\eta) = \int_K \frac{F(\xi) - F(\eta)}{\xi - \eta} \,\mathrm{d}\xi$$

converges and the following estimate holds

$$(9)\quad I_K^0 F(z) - I_K^0 F(\eta) = O\left(|z - \eta| \int_{|z-\eta|}^{\infty} r^{-2}\, \Theta(r)\, \mathrm{d}r\right) \quad as \quad z \to \eta\,, \quad z \in S\,;$$

if (5) *holds with O replaced by o, then the right-hand side in* (9) *may be replaced by* (7).

If λ denotes the Hausdorff linear measure (= length as defined in [9], chap. II, § 8), then (8) is equivalent with

$$\int_K \frac{Q(\xi)}{|\xi - \eta|}\, \mathrm{d}\lambda(\xi) < \infty\,.$$

Notation. If g is a real- or complex-valued function defined on an interval $J \subset R^1$ and $G \subset J$ is open in J, then $\operatorname{var} g(G)$ will denote the variation of g on G which is defined as usual (compare section 1 in [2]). Letting, by the standard procedure,

$$\operatorname{var} g(M) = \inf \{\operatorname{var} g(G);\ G \text{ open in } J,\ G \supset M\}$$

we extend $\operatorname{var} g$ to a Carathéodory outer measure defined for all $M \subset J$ which in turn gives rise to a measure on the system of all $\operatorname{var} g$-measurable sets. The symbol $\int_M F \,\mathrm{dvar}\, g$ will always mean the integral of an extended-real-valued function F over $M \subset J$ with respect to this measure $\operatorname{var} g$. If various parameters enter in the definition of g and the variable must be explicitly noted in order to avoid confusion, then we shall replace the symbols $\operatorname{var} g(M)$, $\int_M F \,\mathrm{dvar}\, g$ by the symbols like

$$\operatorname{var}_t [g(t); M]\,, \int_M F(t)\, \mathrm{dvar}_t\, g(t)\,,$$

respectively.

The following simple lemma will be useful below.

1. Lemma. *If g is a continuous real-valued function of locally finite variation on an interval J, then for any lower-semicontinuous function $F \geqq 0$ on J*

$$(10)\qquad \int_J F \,\mathrm{dvar}\, g = \int_J F \,\mathrm{dvar}\, e^{ig}\,.$$

Proof. This is an easy consequence of theorem 3.15 in [10].

Notation. In what follows ψ will always denote a continuous complex-valued function of bounded variation on $\langle a, b\rangle$, $q \geqq 0$ will be a fixed bounded lower-semicontinuous function on $\langle a, b\rangle$ and we put for $z \in R^2$ and $\varrho > 0$ in accordance with [2]

$$u_\psi^q(\varrho, z) = \sum_t q(t)\,, \quad |\psi(t) - z| = \varrho\,,$$

where the sum is extended over all $t \in \langle a, b \rangle$ for which $\psi(t)$ lies on the circumference of center z and radius ϱ. Let us recall (see lemma 7 in [2]) that $u^q_\psi(\varrho, z)$ is a Lebesgue measurable function of the variable $\varrho > 0$, so that we may define for any $r > 0$

$$u^q_{\psi r}(z) = \int_0^r u^q_\psi(\varrho, z)\, d\varrho .$$

Let us remark that this quantity has its origin in the well-known Banach's theorem on variation of a continuous function. Writing

$$G_r(z) = \{t \in \langle a, b \rangle;\ |\psi(t) - z| < r\} \tag{11}$$

and employing lemma 4 in [2] (where we set $J = \langle a, b \rangle$, $f(t) = |\psi(t) - z|$ and $F =$ the characteristic function of $G_r(z)$ multiplied by q) we get

$$u^q_{\psi r}(z) = \int_{G_r(z)} q(t)\, \mathrm{dvar}_t\, |\psi(t) - z| . \tag{12}$$

Further put for $\gamma \in \langle 0, 2\pi \rangle$

$$v^q_\psi(\gamma, z) = \sum_t q(t)\,, \quad \psi(t) \neq z\,, \quad \frac{\psi(t) - z}{|\psi(t) - z|} = e^{i\gamma}\,,$$

the last sum being extended over all $t \in \langle a, b \rangle$ for which $\psi(t)$ lies on the half-line $S^\gamma(z) = \{z + \varrho e^{i\gamma};\ \varrho > 0\}$.

A simplification of the reasoning used for the proof of lemma 7 in [2] yields the following

2. Lemma. *For fixed $z \in R^2$, $v^q_\psi(\gamma, z)$ is a Lebesgue measurable function of the variable $\gamma \in \langle 0, 2\pi \rangle$. The integral*

$$v^q_\psi(z) = \int_0^{2\pi} v^q_\psi(\gamma, z)\, d\gamma$$

is a lower-semicontinuous function of the variable $z \in R^2$. If J runs over the system $\mathscr{S}(z)$ of all components of $\{t \in \langle a, b \rangle;\ \psi(t) \neq z\}$ and, for $J \in \mathscr{S}(z)$, $\vartheta_z(t; J)$ is a continuous argument of $\psi(t) - z$ on J, then

$$v^q_\psi(z) = \sum_J \int_J q(t)\, \mathrm{dvar}_t\, \vartheta_z(t; J)\,, \quad J \in \mathscr{S}(z)\,. \tag{13}$$

Proof. Employing lemma 7 in [2] in the same way as in the proof of the relation (9) in [2] we get that $v^q_\psi(\gamma, z)$ is a Lebesgue measurable function of the variable $\gamma \in \langle 0, 2\pi \rangle$ and (13) holds. The lower-semicontinuity of $v^q_\psi(z)$ follows from (13); to see this it is sufficient to let $\varrho = \infty$ and replace F by q in the proof of lemma 7 in [2], where the lower-semicontinuity of $v^q_{\psi\varrho}(z)$ is deduced from the corresponding relation (9).

3. Notation. If f is a real-valued bounded Baire function on $\langle a, b\rangle$ we define for $z \in R^2 \setminus \psi(\langle a, b\rangle)$

$$w_\psi f(z) = \operatorname{Im} \int_a^b \frac{f(t)}{\psi(t) - z} \, d\psi(t) .$$

As in [2], we denote for $S \subset R^2$ and $\eta \in R^2$ by

$$S \odot \eta = S \cup \{2\eta - \xi;\ \xi \in S\}$$

the union of S and its reflection in η. The contingent (see theorem 1 in introduction) of S at η will be denoted by $\operatorname{contg}(\eta, S)$.

4. Theorem. *Let $S \subset R^2 \setminus \psi(\langle a, b\rangle)$ be a connected set whose closure meets $\psi(\langle a, b\rangle)$ at η only. If*

$$\limsup_{z \to \eta, z \in S} |w_\psi f(z)| < \infty \tag{14}$$

for each continuous function f on $\langle a, b\rangle$ satisfying

$$f(t) = o(q(t)) \quad as \quad \psi(t) \to \eta , \tag{15}$$

then

$$v_\psi^q(\eta) < \infty . \tag{16}$$

If, besides that, the contingent of $\psi(\langle a, b\rangle)$ at η is disjoint from the contingent of $S \odot \eta$ at η, then

$$\sup_{r>0} r^{-1} u_{\psi r}^q(\eta) < \infty . \tag{17}$$

Proof. In accordance with the proof of theorem 9 in [2] we shall denote by $\mathscr{C}_q$ the Banach space of all continuous functions f on $\langle a, b\rangle$ satisfying (15) as well as

$$|f| \leqq c_f q \tag{18}$$

for suitable constant c_f; the norm $\|f\|$ is defined as the least $c_f \geqq 0$ fulfilling (18). For $f \in \mathscr{C}_q$ and $z \in S$ we have

$$w_\psi f(z) = \int_a^b f(t) \, d\vartheta_z(t) ,$$

where $\vartheta_z(t)$ is a continuous argument of $\psi(t) - z$ on $\langle a, b\rangle$. For fixed $z \in S$,

$$f \mapsto w_\psi f(z)$$

is a linear functional on $\mathscr{C}_q$ whose norm equals

$$\int_a^b q(t) \operatorname{dvar}_t \vartheta_z(t) = v_\psi^q(z) \tag{19}$$

(see (13)). Using (14) and employing the principle of uniform boundedness we conclude in the same way as in the proof of theorem 9 in [2] that

$$\sup_{z \in S} v_\psi^q(z) < \infty . \tag{20}$$

In view of the lower-semicontinuity of $v_\psi^q(.)$, this implies (16). In order to complete the proof it is now sufficient ot establish the following

5. Proposition. *Let $S \subset R^2 \setminus \psi(\langle a, b\rangle)$ be a connected set whose closure meets $\psi(\langle a, b\rangle)$ at a single point η such that*

$$\operatorname{contg}(\eta, S \odot \eta) \cap \operatorname{contg}(\eta, \psi(\langle a, b\rangle)) = \emptyset . \tag{21}$$

Then (20) *implies* (17).

Proof. Let $\mathscr{S}(\eta)$ be the system of all components of $\{t \in \langle a, b\rangle; \psi(t) \neq \eta\}$. Given $J \in \mathscr{S}(\eta)$ and $z \in S$ denote by $\vartheta_\eta(t; J)$ and $\omega_z(t; J)$ a continuous argument of $\psi(t) - \eta$ and $(\psi(t) - \eta)/(z - \eta)$ on J, respectively. Further denote by $\varphi_z(t)$ a continuous argument of $(\psi(t) - z)/(z - \eta)$ on $\langle a, b\rangle$. In view of (21) there is an $R > 0$ such that for $z \in S$ and $t \in J \in \mathscr{S}(\eta)$ the following implication holds:

$$(|z - \eta| < R,\ |\psi(t) - \eta| < R) \Rightarrow |\sin \omega_z(t; J)| \geqq R . \tag{22}$$

We may suppose that R has been chosen small enough to secure $R = |z_0 - \eta|$ for suitable $z_0 \in S$. Let $r \in (0, R)$ and choose a $z \in S$ with $|z - \eta| = r$ (which is possible, because S is connected and η is in the closure of S). Put

$$\varrho_\xi(t) = |\psi(t) - \xi| , \quad \xi \in R^2 , \quad t \in \langle a, b\rangle , \tag{23}$$

fix $J \in \mathscr{S}(\eta)$ and consider $t \in J$ such that $\varrho_\eta(t) < r$. Then

$$\varrho_z(t) \leqq r + \varrho_\eta(t) < 2r ,$$

whence

$$\frac{|\sin [\varphi_z(t) - \omega_z(t; J)]|}{|\sin \omega_z(t; J)|} = \frac{r}{\varrho_z(t)} > \frac{1}{2} .$$

We have thus by (22)

$$|\sin [\varphi_z(t) - \omega_z(t; J)]| > \tfrac{1}{2}R . \tag{24}$$

Defining

$$J_r = \{t \in J;\ \varrho_\eta(t) < r\}$$

and making use of the equality

$$r^{-1}\,\varrho_\eta(t) = \frac{\sin\varphi_z(t)}{\sin\left[\varphi_z(t) - \omega_z(t; J)\right]}$$

we obtain by lemma 3 in [2]

$$(25)\qquad r^{-1}\int_{J_r} q \,\mathrm{dvar}\,\varrho_\eta \leqq A_1 + A_2\,,$$

where

$$A_1 = \int_{J_r} q(t)\,\left|\sin^{-1}\left[\varphi_z(t) - \omega_z(t; J)\right]\right| \mathrm{dvar}_t \sin\varphi_z(t)\,,$$

$$A_2 = \int_{J_r} q(t)\,\mathrm{dvar}_t \sin^{-1}\left[\varphi_z(t) - \omega_z(t; J)\right].$$

Using (24) and lemma 2 in [2] we get

$$A_1 \leqq 2R^{-1}\int_{J_r} q(t)\left|\cos\varphi_z(t)\right| \mathrm{dvar}_t\,\varphi_z(t) \leqq 2R^{-1}\int_{J_r} q(t)\,\mathrm{dvar}_t\,\varphi_z(t) = $$
$$= 2R^{-1}\int_{J_r} q(t)\,\mathrm{dvar}_t\,\vartheta_z(t)\,,$$

because $\varphi_z(t)$ may differ only by an additive constant from the continuous argument $\vartheta_z(t)$ of $\psi(t) - z$.

Similarly,

$$A_2 \leqq \int_{J_r} q(t)\left|\sin^{-2}\left[\varphi_z(t) - \omega_z(t; J)\right]\right| \mathrm{dvar}_t \left[\varphi_z(t) - \omega_z(t; J)\right] \leqq$$
$$\leqq 4R^{-2}\left[\int_{J_r} q(t)\,\mathrm{dvar}_t\,\varphi_z(t) + \int_{J_r} q(t)\,\mathrm{dvar}_t\,\omega_z(t; J)\right] =$$
$$= 4R^{-2}\left[\int_{J_r} q(t)\,\mathrm{dvar}_t\,\vartheta_z(t) + \int_{J_r} q(t)\,\mathrm{dvar}_t\,\vartheta_\eta(t; J)\right],$$

because $\omega_z(t; J) - \vartheta_\eta(t; J)$ is constant on J. Combining these estimates of A_1 and A_2 with (25) we arrive at

(26)

$$r^{-1}\int_{J_r} q\,\mathrm{dvar}\,\varrho_\eta \leqq (2R + 4R^{-2})\int_J q(t)\,\mathrm{dvar}_t\,\vartheta_z(t) + 4R^{-2}\int_J q(t)\,\mathrm{dvar}_t\,\vartheta_\eta(t; J)\,.$$

Since ϱ_η vanishes on

$$(27)\qquad Z = \langle a, b\rangle \setminus \cup\mathcal{S}(\eta)\,,$$

we have var $\varrho_\eta(Z) = 0$ by lemma 5 in [2]. Using the notation (11) and writing $\sum_J$ for the sum extended over all $J \in \mathscr{S}(\eta)$ we have thus by (12)

$$u^q_{\psi r}(\eta) = \int_{G_r(\eta)} q \, \mathrm{dvar}\, \varrho_\eta = \sum_J \int_{J_r} q \, \mathrm{dvar}\, \varrho_\eta ,$$

so that, by virtue of (26),

$$r^{-1} u^q_{\psi r}(\eta) \leqq (2R^{-1} + 4R^{-2}) \int_a^b q \, \mathrm{dvar}\, \vartheta_z + 4R^{-2} \sum_J \int_J q(t) \, \mathrm{dvar}_t \, \vartheta_\eta(t; J) =$$
$$= (\text{see (19) and (13)}) = (2R^{-1} + 4R^{-2}) \, v^q_\psi(z) + 4R^{-2} \, v^q_\psi(\eta) .$$

Let $s = \sup \{v^q_\psi(z); z \in S\}$, so that $s \geqq v^q_\psi(\eta)$ by the lower-semicontinuity of $v^q_\psi(.)$ (cf. lemma 2). We have thus derived the inequality

$$\sup \{r^{-1} u^q_{\psi r}(\eta); \; 0 < r < R\} \leqq (2R^{-1} + 8R^{-2}) \, s .$$

On the other hand, we have for $r \geqq R$ by (12)

$$r^{-1} u^q_{\psi r}(\eta) \leqq R^{-1} \int_a^b q \, \mathrm{dvar}\, \varrho_\eta \leqq R^{-1} \int_a^b q \, \mathrm{dvar}\, \psi .$$

Consequently,

$$\sup_{r>0} r^{-1} u^q_{\psi r}(\eta) \leqq \max \left\{R^{-1} \int_a^b q \, \mathrm{dvar}\, \psi, \; (2R^{-1} + 8R^{-2}) \, s\right\}$$

and the proof is complete.

6. Remark. We shall also need the converse of proposition 5 which will be proved below. The equivalence of (20) and

(28) $$v^q_\psi(\eta) + \sup_{r>0} r^{-1} u^q_{\psi r}(\eta) < \infty$$

was proved in [4] for the special case $q \equiv 1$.

It represents a counterpart to the equivalence of

(29) $$\sup_{z \in S} u^q_\psi(z) < \infty$$

and

(30) $$u^q_\psi(\eta) + \sup_{r>0} r^{-1} v^q_{\psi r}(\eta) < \infty$$

established in [2] (see remark 10 and proposition 12 as well as the definition of $u^q_\psi(.)$ and $v^q_{\psi r}(.)$ presented in lemma 7).

For the proof of the implication (28) $\Rightarrow$ (20) the following slight generalization of lemma 1.6 in [4] will be needed.

7. Lemma. *Let f, g, h be continuous functions of locally finite variation on an interval J satisfying there the inequalities*

$$0 < k \leqq |f| \leqq K < \infty, \quad |g| \leqq |f|.$$

Then there is an $L \in R^1$ (depending only on the constants k, K) such that, for every lower-semicontinuous function $F \geqq 0$ on J,

$$\int_J F \, \mathrm{dvar\,arccotg}\,(g + fh) \leqq L \left(\int_J F \, \mathrm{dvar}\, g + \int_J F \, \mathrm{dvar}\, f + \int_J F \, \mathrm{dvar\,arccotg}\, h \right). \tag{31}$$

Proof. If F is the characteristic function of a compact interval $I \subset J$, then (31) holds with a suitable $L \in (0, \infty)$ (depending on k, K only) by lemma 1.6 in [4]. Now it is sufficient to observe that every lower-semicontinuous function $F \geqq 0$ on J can be expressed as a limit of a non-decreasing sequence of functions $F_n \geqq 0$ each of them is a finite combination, with non-negative coefficients, of characteristic functions of compact subintervals of J.

8. Proposition. *If S satisfies the assumptions described in proposition* 5, *then* (28) *implies* (20).

Proof. The symbols $\mathscr{S}(\eta)$, $\vartheta_\eta(t; J)$, $\omega_z(t; J)$ and $\varphi_z(t)$ will have the same meaning as in the proof of proposition 5. We shall also fix an $R > 0$ fulfilling (22) whenever $t \in J \in \mathscr{S}(\eta)$. Fix now $J \in \mathscr{S}(\eta)$ and consider $t \in J_R = \{t \in J;\ |\psi(t) - \eta| < R\}$. Further let $z \in S$, $|z - \eta| = r \leqq \frac{1}{2}R$. Then we have with the notation from (23) (compare the proof of theorem 2.8 in [4])

$$\frac{\sin [\varphi_z(t - \omega_z(t; J)]}{\sin \varphi_z(t)} = \frac{r}{\varrho_\eta(t)},$$

$$\mathrm{cotg}\, \varphi_z(t) = \mathrm{cotg}\, \omega_z(t; J) - \frac{r}{\varrho_\eta(t)} \sin^{-1} \omega_z(t; J),$$

so that $\varphi_z(t)$ can differ only by an additive constant from

$$\mathrm{arccotg} \left[\mathrm{cotg}\, \omega_z(t; J) - \frac{r}{\varrho_\eta(t)} \sin^{-1} \omega_z(t; J) \right]$$

on every component of J_R. Noting that

$$1 \leqq |\sin^{-1} \omega_z(t; J)| \leqq R^{-1}, \quad |\mathrm{cotg}\, \omega_z(t; J)| \leqq |\sin^{-1} \omega_z(t; J)|$$

we are now in position to apply lemma 7 (where we set $g = \mathrm{cotg}\, \omega_z(.; J)$, $f = \sin^{-1} \omega_z(.; J)$, $h = r \varrho_\eta^{-1}(.)$, $F =$ the characteristic function of J_R multiplied by q) concluding that there is a constant L(depending on R only) such that

$$\int_{J_R} q \operatorname{dvar} \varphi_z \leqq L\left[\int_{J_R} q(t) \operatorname{dvar}_t \operatorname{cotg} \omega_z(t; J) + \right. \tag{32}$$

$$\left. + \int_{J_R} q(t) \operatorname{dvar}_t \sin^{-1} \omega_z(t; J) + \int_{J_R} q(t) \operatorname{dvar}_t \operatorname{arccotg} \frac{r}{\varrho_\eta(t)}\right].$$

It follows easily from lemma 2 in [2] that

$$\int_{J_R} q(t) \operatorname{dvar}_t \operatorname{cotg} \omega_z(t; J) = \tag{33}$$

$$= \int_{J_R} |\sin^{-2} \omega_z(t; J)| \, q(t) \operatorname{dvar}_t \omega_z(t; J) \leqq R^{-2} \int_J q(t) \operatorname{dvar}_t \omega_z(t; J),$$

$$\int_{J_R} q(t) \operatorname{dvar}_t \sin^{-1} \omega_z(t; J) = \tag{34}$$

$$= \int_{J_R} |\sin^{-2} \omega_z(t; J)| \, . \, |\cos \omega_z(t; J)| \, q(t) \operatorname{dvar}_t \omega_z(t; J) \leqq R^{-2} \int_J q(t) \operatorname{dvar}_t \omega_z(t; J),$$

$$\int_{J_R} q(t) \operatorname{dvar}_t \operatorname{arccotg} \frac{r}{\varrho_\eta(t)} = \int_{J_R} q(t) \frac{r}{\varrho_\eta^2(t) + r^2} \operatorname{dvar}_t \varrho_\eta(t) . \tag{35}$$

Combining (33)–(35) with (32) we arrive at

$$\int_{J_R} q \operatorname{dvar} \varphi_z \leqq L\left(2R^{-2} \int_J q(t) \operatorname{dvar}_t \omega_z(t; J) + \int_{J_R} q \frac{r}{\varrho_\eta^2 + r^2} \operatorname{dvar} \varrho_\eta\right). \tag{36}$$

Note that, with the notation from (27) and (11),

$$\bigcup_{J \in \mathscr{S}(\eta)} J_R = G_R(\eta) \setminus Z, \quad \psi(Z) = \{\eta\}.$$

Since every element of $\varphi_z(Z)$ is an argument of $(\eta - z)/(z - \eta) = -1$, we conclude from lemma 5 in [2] that $\operatorname{var} \varphi_z(Z) = 0$. Writing $\sum_J$ for the sum extended over all $J \in \mathscr{S}(\eta)$ we get from (36)

$$\int_{G_R(\eta)} q \operatorname{dvar} \varphi_z \leqq L\left(2R^{-2} \sum_J \int_J q(t) \operatorname{dvar}_t \omega_z(t; J) + \int_{G_R(\eta)} q \frac{r}{r^2 + \varrho_\eta^2} \operatorname{dvar} \varrho_\eta\right).$$

Using (13) and noting that $\omega_z(.; J) - \vartheta_\eta(.; J)$ is constant on $J(\in \mathscr{S}(\eta))$ we may rewrite the last inequality in the form

$$\int_{G_R(\eta)} q \operatorname{dvar} \varphi_z \leqq L\left(2R^{-2} v_\psi^q(\eta) + \int_{G_R(\eta)} q \frac{r}{r^2 + \varrho_\eta^2} \operatorname{dvar} \varrho_\eta\right). \tag{37}$$

The last integral can be transformed by lemma 4 in [2] into

$$\int_{G_R(\eta)} q \frac{r}{r^2 + \varrho_\eta^2} \,\mathrm{dvar}\, \varrho_\eta = \int_0^R u_\psi^q(\varrho, \eta) \frac{r}{r^2 + \varrho^2} \,\mathrm{d}\varrho ,$$

which in turn can be estimated with help of lemma 1.7 in [4] as follows

$$\int_0^R u_\psi^q(\varrho, \eta) \frac{r}{r^2 + \varrho^2} \,\mathrm{d}\varrho \leqq \frac{\pi}{2} \sup_{0<x<R} x^{-1} \int_0^x u_\psi^q(\varrho, \eta) \,\mathrm{d}\varrho = \frac{\pi}{2} \sup_{0<x<R} x^{-1} u_{\psi x}^q(\eta) .$$

This together with (37) yields

$$\int_{G_R(\eta)} q \,\mathrm{dvar}\, \varphi_z \leqq L\left[2R^{-2} v_\psi^q(\eta) + \frac{\pi}{2} \sup_{x>0} x^{-1} u_{\psi x}^q(\eta)\right] .$$

Since

$$\vartheta_z(t) = \mathrm{Im} \int_a^t \frac{\mathrm{d}\psi(t)}{\psi(t) - z} , \quad a \leqq t \leqq b ,$$

is a continuous argument of $\psi(t) - z$, the difference $\varphi_z - \vartheta_z$ is constant on $\langle a, b \rangle$. Recalling that $|z - \eta| \leqq \frac{1}{2}R$ we get for $t \in Y_R = \langle a, b \rangle \setminus G_R(\eta)$ the estimate

$$|\psi(t) - z| \geqq |\psi(t) - \eta| - |z - \eta| \geqq \tfrac{1}{2}R ,$$

so that

$$\int_{Y_R} q \,\mathrm{dvar}\, \vartheta_z \leqq \int_{Y_R} \frac{q(t)}{|\psi(t) - z|} \,\mathrm{dvar}\, \psi(t) \leqq 2R^{-1} \int_a^b q \,\mathrm{dvar}\, \psi .$$

Finally,

$$v_\psi^q(z) = \int_a^b q \,\mathrm{dvar}\, \vartheta_z = \int_{G_R(\eta)} q \,\mathrm{dvar}\, \varphi_z + \int_{Y_R} q \,\mathrm{dvar}\, \vartheta_z \leqq$$

$$\leqq \left[2R^{-2} v_\psi^q(\eta) + \frac{\pi}{2} \sup_{x>0} x^{-1} u_{\psi x}^q(\eta)\right] + 2R^{-1} \int_a^b q \,\mathrm{dvar}\, \psi$$

whenever $z \in S$, $|z - \eta| \leqq \frac{1}{2}R$. Since the set $S_R = \{z \in S; |z - \eta| \geqq \frac{1}{2}R\}$ has a positive distance d from $\psi(\langle a, b \rangle)$, we have for $z \in S_R$

$$v_\psi^q(z) = \int_a^b q \,\mathrm{dvar}\, \vartheta_z \leqq \int_a^b \frac{q(t)}{|\psi(t) - z|} \,\mathrm{dvar}\, \psi(t) \leqq d^{-1} \int_a^b q \,\mathrm{dvar}\, \psi .$$

In any case,

$$z \in S \Rightarrow v_\psi^q(z) \leqq L\left[2R^{-2} v_\psi^q(\eta) + \frac{\pi}{2} \sup_{x>0} x^{-1} u_{\psi x}^q(\eta)\right] + (d^{-1} + 2R^{-1}) \int_a^b q \,\mathrm{dvar}\, \psi$$

and the proof is complete.

9. Notation. Given $z \in R^2 \setminus \psi(\langle a, b \rangle)$ we shall denote by $\Delta \log [\psi - z; \langle a, b \rangle]$ and $\Delta \arg [\psi - z; \langle a, b \rangle]$ the increment of $\log [\psi(t) - z]$ and of the argument of $[\psi(t) - z]$ on $a \leqq t \leqq b$, respectively.

We have so far defined $w_\psi f(z)$ for $z \in R^2 \setminus \psi(\langle a, b \rangle)$ only. If $\eta \in \psi(\langle a, b \rangle)$ we fix a continuous argument $\vartheta_\eta(t; J)$ of $\psi(t) - \eta$ on every J in the system $\mathscr{S}(\eta)$ of all components of $\{t \in \langle a, b \rangle; \psi(t) \neq \eta\}$ and define for a Baire function f on $\langle a, b \rangle$

$$w_\psi f(\eta) = \sum_{J \in \mathscr{S}(\eta)} \int_J f(t) \, \mathrm{d}_t \, \vartheta_\eta (t; J) \tag{38}$$

provided the Lebesgue-Stieltjes integrals on the right-hand side of (38) exist and their sum is meaningful; this definition is clearly independent of the choice of the arguments $\vartheta_\eta(.; J)$.

Now we can prove the following counterpart of theorem 14 in [2].

10. Theorem. *Let $S \subset R^2 \setminus \psi\langle a, b \rangle)$ be a connected set whose closure meets $\psi(\langle a, b \rangle)$ at a single point η such that* (21) *holds and assume* (28). *Let $\Theta(r) \geqq 0$ be a continuous non-decreasing function of the variable $r \geqq 0$, $\sup_{r>0} \Theta(r) > 0$. If $\varkappa \in R^1$ and f is a bounded Baire function on $\langle a, b \rangle$ such that $f_\varkappa = f - \varkappa$ satisfies*

$$f_\varkappa(t) = O(\Theta(|\psi(t) - \eta|) \, q(t)) \quad \textit{as} \quad \psi(t) \to \eta \, , \tag{39}$$

then $w_\psi f_\varkappa(\eta)$ exists and for $z \in S$ the following estimate holds

$$\begin{aligned} w_\psi f(z) - \varkappa \Delta \arg [\psi - z; \langle a, b \rangle] - w_\psi f_\varkappa(\eta) = \\ = O\left(|z - \eta| \int_{|z-\eta|}^{\infty} \Theta(x) \, x^{-2} \, \mathrm{d}x \right) \quad \textit{as} \quad z \to \eta \, . \end{aligned} \tag{40}$$

If (39) *holds with O replaced by o then the the right-hand side in* (40) *may be replaced by*

$$O(|z - \eta|) + o\left(|z - \eta| \int_{|z-\eta|}^{\infty} \Theta(x) \, x^{-2} \, \mathrm{d}x \right).$$

Proof. Fix $\varepsilon > 0$ and $R > 0$ such that, with the notation (23),

$$(t \in \langle a, b \rangle, \ 0 < \varrho_\eta(t) < R) \Rightarrow |f_\varkappa(t)| \leqq \varepsilon \Theta(\varrho_\eta(t)) \, q(t) \, .$$

For $z \in S$ we shall denote by $\vartheta_z(t)$ a continuous argument of $\psi(t) - z$ on $\langle a, b \rangle$ and with every $J \in \mathscr{S}(\eta)$ we shall associate a continuous argument $\vartheta_\eta(t; J)$ of $\psi(t) - \eta$ as well as a continuous argument $\omega_z(t; J)$ of $(\psi(t) - \eta)/(z - \eta)$ on J. We shall suppose that R has been chosen small enough to guarantee (22) whenever $z \in S$ and $t \in J \in \mathscr{S}(\eta)$.

Consider now $z \in S$ with $|z - \eta| = r < R$ and let for $J \in \mathscr{S}(\eta)$

$$J_r = J \cap G_r(\eta),$$

where $G_r(\eta)$ is defined by (11). We have then

$$(41) \qquad \left|\sum_J \int_{J_r} f_\varkappa(t)\, \mathrm{d}\vartheta_\eta(t; J)\right| \leqq \varepsilon \sum_J \int_{J_r} \Theta(\varrho_\eta(t))\, q(t)\, \mathrm{dvar}\, \vartheta_\eta(t; J) \leqq$$
$$\leqq (\text{see } (13)) \leqq \varepsilon\, \Theta(r)\, v_\psi^q(\eta),$$

$$(42) \qquad \left|\int_{G_r(\eta)} f_\varkappa\, \mathrm{d}\vartheta_z\right| \leqq \varepsilon \left|\int_{G_r(\eta)} \Theta(\varrho_\eta(t))\, q(t)\, \mathrm{dvar}\, \vartheta_z(t)\right| \leqq \varepsilon\, \Theta(r)\, v_\psi^q(z).$$

Put, for the sake of brevity, $L^r = G_R(\eta) \setminus G_r(\eta)$, $J^r = J \cap L^r$, and consider next

$$(43) \qquad A(z) = \left|\int_{L^r} f_\varkappa\, \mathrm{d}\vartheta_z - \sum_J \int_{J^r} f_\varkappa(t)\, \mathrm{d}\vartheta_\eta(t; J)\right| =$$
$$= \left|\sum_J \operatorname{Im} \int_{J^r} f_\varkappa(t) \left[\frac{1}{\psi(t) - z} - \frac{1}{\psi(t) - \eta}\right] \mathrm{d}\psi(t)\right| \leqq$$
$$\leqq \sum_J \int_{J^r} |f_\varkappa(t)| \frac{|z - \eta|}{\varrho_z(t)\, \varrho_\eta(t)}\, \mathrm{dvar}\, \psi(t) \leqq$$
$$\leqq \sum_J \varepsilon r \int_{J^r} q(t)\, \Theta(\varrho_\eta(t))\, \varrho_z^{-1}(t)\, \varrho_\eta^{-1}(t)\, \mathrm{dvar}\, \psi(t).$$

Noting that, for $t \in J$,

$$\psi(t) = \eta + \varrho_\eta(t)\, \mathrm{e}^{\mathrm{i}\vartheta_\eta(t; J)}$$

we get by lemma 3 in [2] and lemma 1 above for any $J \in \mathscr{S}(\eta)$

$$\int_{J^r} q(t)\, \Theta(\varrho_\eta(t))\, \varrho_z^{-1}(t)\, \varrho_\eta^{-1}(t)\, \mathrm{dvar}\, \psi(t) \leqq \int_{J^r} q(t)\, \Theta(\varrho_\eta(t))\, \varrho_z^{-1}(t)\, \varrho_\eta^{-1}(t)\, \mathrm{dvar}\, \varrho_\eta(t) +$$
$$+ \int_{J^r} \Theta(\varrho_\eta(t))\, q(t)\, \varrho_z^{-1}(t)\, \mathrm{dvar}\, \vartheta_\eta(t; J).$$

In view of (22) we have for $t \in J_r$

$$\varrho_z(t)\, \varrho_\eta^{-1}(t) \geqq |\sin \omega_z(t; J)| \geqq R,$$

whence

$$(44) \qquad A(z) \leqq \varepsilon r R^{-1}(B + C),$$

where

$$B = \int_{L^r} \Theta(\varrho_\eta(t))\, q(t)\, \varrho_\eta^{-1}(t) \,\mathrm{dvar}\, \varrho_\eta(t)\,,$$

$$C = \sum_J \int_{J^r} \Theta(\varrho_\eta(t))\, q(t)\, \varrho_\eta^{-1}(t) \,\mathrm{dvar}\, \vartheta_\eta(t; J)\,.$$

According to lemma 5 in [2], var $\varrho_\eta(\{t \in \langle a, b\rangle;\ \varrho_\eta(t) = r\}) = 0$. Denoting by χ the characteristic function of $\{t \in \langle a, b\rangle;\ r < \varrho_\eta(t) < R\}$ we thus obtain by lemma 4 in [2]

$$B = \int_a^b \chi(t)\, \Theta(\varrho_\eta(t))\, q(t)\, \varrho_\eta^{-2}(t) \,\mathrm{dvar}\, \varrho_\eta(t) =$$

$$= \int_r^R \Theta(x)\, x^{-2}\, u_\psi^q(x, \eta) \,\mathrm{d}x \leqq \Theta(R)\, R^{-2}\, u_{\psi R}^q(\eta) -$$

$$- \int_r^R u_{\psi x}^q(\eta) \,\mathrm{d}_x[\Theta(x)\, x^{-2}] \leqq \Theta(R)\, R^{-2}\, u_{\psi R}^q(\eta) +$$

$$+ 2 \int_r^R u_{\psi x}^q(\eta)\, \Theta(x)\, x^{-3} \,\mathrm{d}x\,.$$

Writing

$$U = \sup_{x > 0} x^{-1}\, u_{\psi x}^q(\eta)$$

we arrive at

$$B \leqq U \left[\Theta(R)\, R^{-1} + 2 \int_r^R \Theta(x)\, x^{-2} \,\mathrm{d}x \right]. \tag{45}$$

In order to get a suitable estimate for C we first fix a $J \in \mathscr{S}(\eta)$ and define measures μ and μ_1 on Borel sets $M \subset J$ by

$$\mu(M) = \int_M q(t) \,\mathrm{dvar}_t\, \vartheta_\eta(t; J)\,, \quad \mu_1(M) = \int_M \Theta(\varrho_\eta(t))\, q(t) \,\mathrm{dvar}_t\, \vartheta_\eta(t; J)\,,$$

so that the integrals occurring in the definition of C can be written in the form

$$\int_{J^r} \Theta(\varrho_\eta(t))\, q(t)\, \varrho_\eta^{-1}(t) \,\mathrm{dvar}\, \vartheta_\eta(t; J) = \int_{J^r} \varrho_\eta^{-1}(t) \,\mathrm{d}\mu_1(t)\,.$$

Let for $\tau > 0$

$$J^r(\tau) = J^r \cap \{t \in \langle a, b\rangle;\ \varrho_\eta^{-1}(t) > \tau\}$$

and notice that $J^r(\tau) = \emptyset$ for $\tau > r^{-1}$ while

$$0 < \tau \leqq r^{-1} \Rightarrow J^r(\tau) = \{t \in J;\ r \leqq \varrho_\eta(t) < \min(\tau^{-1}, R)\}\,.$$

Hence

$$\mu_1(J^r(\tau)) \leqq \Theta(\tau^{-1})\, \mu(J^r(\tau))$$

and we conclude by lemma 11 in [2] that

$$\int_{J^r} \varrho_\eta^{-1}(t)\,\mathrm{d}\mu_1(t) = \int_0^{r^{-1}} \mu_1(J^r(\tau))\,\mathrm{d}\tau \leqq \int_0^{r^{-1}} \Theta(\tau^{-1})\,\mu(J^r(\tau))\,\mathrm{d}\tau =$$

$$= \int_r^\infty \Theta(x)\,\mu(J^r(x^{-1}))\,x^{-2}\,\mathrm{d}x \leqq \mu(J)\int_r^\infty \Theta(x)\,x^{-2}\,\mathrm{d}x\,.$$

Keeping in mind the definition of the measure μ as well as that of the quantity C we get by (13)

$$(46)\qquad C \leqq \left(\int_r^\infty \Theta(x)\,x^{-2}\,\mathrm{d}x\right)\sum_J \int_J q(t)\,\mathrm{dvar}_t\,\vartheta_\eta(t;J) = v_\psi^q(\eta)\int_r^\infty \Theta(x)\,x^{-2}\,\mathrm{d}x\,.$$

Combining (46) and (45) with (44) and taking into account that

$$\Theta(R)\,R^{-1} \leqq \int_R^\infty \Theta(x)\,x^{-2}\,\mathrm{d}x \leqq \int_r^\infty \Theta(x)\,x^{-2}\,\mathrm{d}x$$

we obtain

$$(47)\qquad A(z) \leqq \varepsilon r R^{-1}(v_\psi^q(\eta) + 3U)\int_R^\infty \Theta(x)\,x^{-2}\,\mathrm{d}x\,.$$

Finally consider the set

$$Y_R = \langle a, b\rangle \setminus G_R(\eta)\,,$$

put

$$m = \sup\{|f_\varkappa(t)|;\ t \in \langle a, b\rangle\}$$

and observe that

$$(48)\qquad \left|\int_{Y_R} f_\varkappa\,\mathrm{d}\vartheta_z - \sum_J \int_{J\cap Y_R} f_\varkappa(t)\,\mathrm{d}\vartheta_\eta(t;J)\right| =$$

$$= \left|\operatorname{Im}\int_{Y_R} f_\varkappa(t)\left[\frac{1}{\psi(t)-z} - \frac{1}{\psi(t)-\eta}\right]\mathrm{d}\psi(t)\right| \leqq$$

$$\leqq \int_{Y_R} |f_\varkappa(t)|\, r\,\varrho_z^{-1}(t)\,\varrho_\eta^{-1}(t)\,\mathrm{dvar}\,\psi(t) \leqq mr(R-r)^{-1}\,R^{-1}\,\mathrm{var}\,\psi(\langle a, b\rangle)\,.$$

Since

$$w_\psi f(z) - \varkappa\Delta \arg[\psi - z] - w_\psi f_\varkappa(\eta) =$$

$$= \left(\int_{G_R(\eta)} + \int_{L^r} + \int_{Y_R}\right) f_\varkappa\,\mathrm{d}\vartheta_z - \sum_J \left(\int_{J_r} + \int_{J^r} + \int_{J\cap Y_R}\right) f_\varkappa(t)\,\mathrm{d}\vartheta_\eta(t;J)$$

we see from (42), (41), (48), (47) and (43) that the left-hand side of (40) is dominated in absolute value by

$$\varepsilon r\int_r^\infty \Theta(x)\,x^{-2}\,\mathrm{d}x[(2 + R^{-1})\,V + 3R^{-1}U] + mrR^{-1}(R-r)^{-1}\,\mathrm{var}\,\psi(\langle a, b\rangle)\,,$$

where $V = \sup_{z \in S} v_\psi^q(z)$ (cf. also proposition 9 and lemma 2 and notice that $\Theta(r)\, r^{-1} \leqq$ $\leqq \int_r^\infty \Theta(x)\, x^{-2}\, \mathrm{d}x$).

11. Notation. If f is a (real- or complex-valued) Baire function on $\langle a, b\rangle$ we define for $z \in R^2$

$$i_\psi f(z) = \int_a^b \frac{f(t)}{\psi(t) - z}\, \mathrm{d}\psi(t)$$

provided the Lebesgue-Stieltjes integral on the right-hand side exists (note that $\operatorname{var} \psi(\{t \in \langle a, b\rangle;\ \psi(t) = z\}) = 0$ by lemma 5 in [2], so that $1/[\psi(t) - z]$ is defined $\operatorname{var} \psi$-almost everywhere in $\langle a, b\rangle$).

In accordance with [2] we put

$$u_\psi^q(z) = \int_0^\infty \varrho^{-1} u_\psi^q(\varrho, z)\, \mathrm{d}\varrho\,.$$

Combining theorems 10 and 4 with the corresponding results in [2] we obtain the following

12. Theorem. *Let $S \subset R^2 \setminus \psi(\langle a, b\rangle)$ be a connected set whose closure meets $\psi(\langle a, b\rangle)$ at a single point η such that (21) holds. If*

$$\limsup_{z \to \eta, z \in S} |i_\psi f(z)| < \infty$$

for every continuous real-valued function f on $\langle a, b\rangle$ satisfying (15), then

$$u_\psi^q(\eta) + v_\psi^q(\eta) < \infty\,. \tag{49}$$

Conversely, assume (49) and suppose that $\Theta(r) \geqq 0$ is a bounded continuous function of the variable $r \geqq 0$ such that $\sup_{r \geqq 0} \Theta(r) > 0$. If $\alpha \in R^2$ and f is a bounded complex-valued Baire function on $\langle a, b\rangle$ such that $f_\alpha = f - \alpha$ satisfies

$$f_\alpha(t) = O(\Theta(|\psi(t) - \eta|)\, q(t)) \quad as \quad \psi(t) \to \eta\,, \tag{50}$$

then $i_\psi f_\alpha(\eta)$ exists and for $z \in S$ the following estimate holds

$$\begin{aligned} i_\psi f(z) - \alpha\Delta \log[\psi - z; \langle a, b\rangle] - i_\psi f_\alpha(\eta) &= \\ = O\left(|z - \eta| \int_{|z-\eta|}^\infty \Theta(x)\, x^{-2}\, \mathrm{d}x\right) \quad &as \quad z \to \eta\,; \end{aligned} \tag{51}$$

if (50) holds with O replaced by o, then the right-hand side in (51) may be replaced by

$$O(|z - \eta|) + o\left(|z - \eta| \int_{|z-\eta|}^\infty \Theta(x)\, x^{-2}\, \mathrm{d}x\right).$$

Proof. Note that, for real-valued f,

$$i_\psi f(.) = p_\psi f(.) + \mathrm{i} w_\psi f(.),$$

where $w_\psi f(.)$ is defined in sections 3 and 9 above while, with the notation (23),

$$p_\psi f(\xi) = \int_a^b f(t)\, \varrho_\xi^{-1}(t)\, \mathrm{d}\varrho_\xi(t)$$

provided the Lebesgue-Stieltjes integral on the right-hand side exists. The first part of the theorem follows immediately from theorem 4 above and theorem 9 in [2]. Next observe that, for any $r > 0$ and $\eta \in R^2$,

$$r^{-1}\, u_{\psi r}^q(\eta) \leqq \int_0^r \varrho^{-1}\, u_\psi^q(\varrho, \eta)\, \mathrm{d}\varrho \leqq u_\psi^q(\eta)$$

and, with the notation from lemma 7 in [2],

$$r^{-1}\, v_{\psi r}^q(\eta) \leqq v_\psi^q(\eta),$$

so that (49) implies both

$$u_\psi^q(\eta) + \sup_{r>0} r^{-1}\, v_{\psi r}^q(\eta) < \infty$$

and

$$v_\psi^q(\eta) + \sup_{r>0} r^{-1}\, u_{\psi r}^q(\eta) < \infty .$$

The rest of the theorem is an obvious consequence of theorem 10 above and theorem 14 in [2].

13. Proposition. *If ϱ_η has the meaning described in* (23), *then the condition*

$$\int_a^b q\varrho_\eta^{-1}\, \mathrm{dvar}\, \psi < \infty \tag{52}$$

is equivalent with (49).

Proof. Fix a continuous argument $\vartheta_\eta(t; J)$ of $\psi(t) - \eta$ on every J in the system $\mathscr{S}(\eta)$ of all components of $\{t \in \langle a, b\rangle;\ \psi(t) \neq \eta\}$ and note that, for $t \in J \in \mathscr{S}(\eta)$,

$$\psi(t) = \eta + \varrho_\eta(t)\, \mathrm{e}^{\mathrm{i}\vartheta_\eta(t;\, J)} . \tag{53}$$

Hence it follows by lemma 3 in [2] and lemma 1 above that

$$\int_J q\varrho_\eta^{-1}\, \mathrm{dvar}\, \psi \leqq \int_J q\varrho_\eta^{-1}\, \mathrm{dvar}\, \varrho_\eta + \int_J q(t)\, \mathrm{dvar}_t\, \vartheta_\eta(t; J) .$$

Since

$$\operatorname{var} \psi(\langle a, b\rangle \setminus \cup \mathscr{S}(\eta)) = 0$$

by lemma 5 in [2], we conclude from (11) in [2] and (13) above that

$$\int_a^b q\varrho_\eta^{-1} \operatorname{dvar} \psi = \sum_J \int_J q\varrho_\eta^{-1} \operatorname{dvar} \psi \leqq \sum_J \int_J q\varrho_\eta^{-1} \operatorname{dvar} \varrho_\eta +$$

$$+ \sum_J \int_J q(t) \operatorname{dvar}_t \vartheta_\eta(t; J) = u_\psi^q(\eta) + v_\psi^q(\eta),$$

so that (49) ⇒ (52). The implication

$$(52) \Rightarrow u_\psi^q(\eta) < \infty$$

follows easily from the inequality $\operatorname{var} \varrho_\eta(.) \leqq \operatorname{var} \psi(.)$. Employing (53) and lemma 1 above we get for any $J \in \mathscr{S}(\eta)$

$$\int_J q(t) \operatorname{dvar}_t \vartheta_\eta(t; J) = \int_J q(t) \operatorname{dvar}_t \mathrm{e}^{\mathrm{i}\vartheta_\eta(t; J)} =$$

$$= \int_J q(t) \operatorname{dvar}_t \frac{\psi(t) - \eta}{\varrho_\eta(t)} \leqq (\text{see lemma 3 in [2]}) \leqq$$

$$\leqq \int_J q\varrho_\eta^{-1} \operatorname{dvar} \psi + \int_J q\varrho_\eta \operatorname{dvar} \varrho_\eta^{-1} = (\text{see lemma 2 in [2]}) =$$

$$= \int_J q\varrho_\eta^{-1} \operatorname{dvar} \psi + \int_J q\varrho_\eta^{-1} \operatorname{dvar} \varrho_\eta \leqq 2 \int_J q\varrho_\eta^{-1} \operatorname{dvar} \psi,$$

whence

$$v_\psi^q(\eta) = \sum_J \int_J q(t) \operatorname{dvar} \vartheta_\eta(t; J) \leqq 2 \int_a^b q\varrho_\eta^{-1} \operatorname{dvar} \psi$$

and the implication (52) ⇒ (49) is verified.

14. Remark. Suppose now that ψ is simple, which means that for $t_1, t_2 \in \langle a, b\rangle$

$$0 < |t_1 - t_2| < b - a \Rightarrow \psi(t_1) \neq \psi(t_2),$$

and denote by λ the Hausdorff linear measure (= length in the sense of [9], chap II, § 8). If $Q \geqq 0$ is a lower-semicontinuous function on $K = \psi(\langle a, b\rangle)$ and $q(t) = Q(\psi(t))$, $t \in \langle a, b\rangle$, then

$$\int_K \frac{Q(\xi)}{|\xi - \eta|} \, \mathrm{d}\lambda(\xi) = \int_a^b q\varrho_\eta^{-1} \operatorname{dvar} \psi. \tag{54}$$

Indeed, it follows from lemma 3.3 in [5] that for any $\tau > 0$

$$\lambda\left(\left\{\xi \in K; \frac{Q(\xi)}{|\xi - \eta|} > \tau\right\}\right) = \operatorname{var} \psi(\{t \in \langle a, b\rangle;\ q(t)\,\varrho^{-1}(t) > \tau\})$$

and integrating this equality $d\tau$ over $(0, \infty)$ we obtain (54) by lemma 11 in [2].

The quantities $U^Q_{Kr}(\eta)$ and $V^Q_K(\eta)$ defined in the introduction are easily seen to coincide with $u^q_{\psi r}(\eta)$ and $v^q_\psi(\eta)$, respectively. Hence it follows that the assertions announced in the introduction are simple corollaries of the results established above.

There is an extensive literature dealing with integrals of the Cauchy type and their applications. We refer the reader to [1], [7], [8], [11] for further references.

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