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FREE PRODUCTS IN VARIETIES OF LATTICE-ORDERED GROUPS

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Introduction. A lattice-ordered group G (abbr. l-group) is an algebra with two binary operations $+$ and $\vee$, a unary operation $-(.)$ and a nullary operation 0 satisfying all of the following:

i) $G(+, -(.), 0)$ is a group,

ii) the operation $\vee$ is idempotent, associative and commutative,

iii) $a + (b \vee c) = (a + b) \vee (a + c)$, for all $a, b, c \in G$.

Of course, $\vee$ turns out to be the join operation in the lattice and the meet operation is obtained by setting: $a \wedge b = -(-a \vee -b)$. It is a trivial matter to verify this is indeed equivalent to the customary definition of an l-group.

With the above setting in mind let us denote by $\mathscr{L}$ (resp. $\mathscr{L}R$, resp. $\mathscr{L}A$) the category of all l-groups (resp. representable l-groups, resp. abelian l-groups) together with all lattice preserving homomorphisms (henceforth: l-homomorphisms). An l-group is *representable* if it is l-isomorphic to a subdirect product of totally ordered groups (henceforth: o-groups); the product of o-groups is assumed to have the *cardinal* or coordinate-wise ordering. It is well known this condition is equivalent to saying that $a \wedge b = 0$ implies $a \wedge b^x = 0$, where $b^x = -x + b + x$, and $a, b, x \in \in G$. The three categories in question are therefore all varieties of the type described in the first paragraph, subject to the appropriate laws that define the class. Equivalently, these are all closed under subalgebras (l-subgroups), l-homomorphic images and (cardinal) products; this is G. Birkhoff's great theorem on varieties of universal algebras.

It follows then that all three have a free object $F(X)$ over a given set X. WEINBERG first gave a representation of the free object in $\mathscr{L}A$; Bernau corrected and refined some of Weinberg's results in this connection. CONRAD then gave a representation of the free objects in $\mathscr{L}$ and $\mathscr{L}R$ by generalizing Weinberg's ideas and construction. Very little is known about these free objects beyond the above mentioned representations; for example, the word problem is yet to be considered. WEINBERG has shown that the free object in $\mathscr{L}A$ is a subdirect product of integers (and hence archimedean), and indecomposable into cardinal summands, unless we are dealing with the free object on one generator, which is $Z \boxplus Z$ (cardinally ordered) in all three varieties. (Z will denote the additive group of integers, ordered as usual.)

It is known that varieties of universal algebras are complete as well as co-complete categories. In particular they have co-products or free products; briefly, we recall the definition of the free product: let A_λ ($\lambda \in \Lambda$) be a family of objects in a category $\mathscr{A}$. The object A is the *free product* of the A_λ if there are morphisms $u_\lambda : A_\lambda \to A$ (called *coprojections*) having the property that whenever a family of morphisms $f_\lambda : A_\lambda \to B$ is given, there is a unique morphism $f : A \to B$ such that $u_\lambda f = f_\lambda$ for each $\lambda \in \Lambda$. It is our task then to study the free product in each of the above varieties. We shall carry out the construction in $\mathscr{L}$, the corresponding schemes in $\mathscr{L}R$ and $\mathscr{L}A$ being analogous.

For the basic facts concerning l-groups we suggest the reader consult [5]. Weinberg's crucial results are contained in [8] and [9]; Bernau [2] gives some interesting refinements of Weinberg's results, and a general treatment that applies to vector lattices. For the construction of the free vector lattice see [1]. The reader is referred to [4] for a discussion of the free object in $\mathscr{L}$ and $\mathscr{L}R$. For the appropriate backgrounds in universal algebra and category theory in general the author wishes to suggest [3] and [7] respectively. Special notation and terminology will be discussed in context.

1. Construction and basic properties of the free products. Let G_λ ($\lambda \in \Lambda$) be a family of l-groups. Consider the G_λ as sets and let F be the free l-group on the disjoint union of the G_λ. Let N be the l-ideal generated by all elements of the form

$$(g + h) \;-\; [(g) + \;(h)]\,, \qquad g, h \in G_\lambda\,, \quad \lambda \in \Lambda\,,$$
$$(g \vee_\lambda h) - [(g) \vee_F (h)]\,, \qquad g, h \in G\,, \quad \lambda \in \Lambda\,.$$

Let $G = F/N$ and $u_\lambda : G_\lambda \to G$ be defined by

$$xu_\lambda = (x) + N\,, \quad \text{for each} \quad x \in G_\lambda \quad \text{and} \quad \lambda \in \Lambda\,.$$

1.1. Theorem. *In $\mathscr{L}$ G is the free product of the G_λ with u_λ as co-projections. We write $G = \amalg\{G_\lambda \mid \lambda \in \Lambda\}$.*

Proof. Of course, the effect of factoring out N is precisely making the u_λ ($\lambda \in \Lambda$) l-homomorphisms. So let $\phi_\lambda : G_\lambda \to H$ be a family of l-homomorphisms into the l-group H; define $\theta : \bigcup G_\lambda \to H$ by $(x)\,\theta = x\phi_\lambda$ if $x \in G_\lambda$. There is a unique l-homomorphism $\bar{\theta} : F \to H$ extending θ. For all $g, h \in G_\lambda$ we get

$$[(g + h)]\,\bar{\theta} = (g + h)\,\theta = [g + h]\,\phi_\lambda = g\phi_\lambda + h\phi_\lambda = (g)\,\theta + (h)\theta = \\ = [(g) + (h)]\,\bar{\theta}\,,$$

and also that $[(g \vee_\lambda h)]\,\bar{\theta} = [(g) \vee_F (h)]\,\bar{\theta}$. Consequently $N\bar{\theta} = 0$, that is $\bar{\theta}$ factors uniquely through G; let $\phi : G \to H$ be this unique l-homomorphism. Then for each $x \in G_\lambda$, $xu_\lambda\phi = ((x) + N)\,\phi = (x)\,\bar{\theta} = (x)\,\theta = x\phi_\lambda$, i.e. $u_\lambda\phi = \phi_\lambda$. Since it is clear that the ϕ_λ determine ϕ uniquely, our proof is complete.

Remarks. 1) We should point out that if co-products exist in a given category they are completely determined by the components up to an isomorphism.

2) If G_λ $(\lambda \in \Lambda)$ are representable (resp. abelian) l-groups we denote their free product in $\mathscr{L}R$ (resp. $\mathscr{L}A$) by $\amalg^R\{G_\lambda \mid \lambda \in \Lambda\}$ (resp. $\amalg^A\{G_\lambda \mid \lambda \in \Lambda\}$). We shall refer to the *free R-product* (resp. *free A-product*) of representable (resp. abelian) l-groups.

3) In view of the Weinberg-Conrad representations for free objects we deduce that an arbitrary element x of $\amalg\{G_\lambda \mid \lambda \in \Lambda\}$ may be written

$$x = \vee_\alpha \wedge_\beta \sum_{i=1}^{n} [g(\alpha, \beta, i)]\, u_{\lambda_i},$$

where the indicated joins and meets are taken over finite sets, and each $g(\alpha, \beta, i) \in G_{\lambda_i}$, $1 \leqq i \leqq n$. With suitable interpretation, similar expressions hold in $\mathscr{L}R$ and $\mathscr{L}A$ as well.

The following observations are more or less trivial:

1.2. *Let $G = \amalg\{G_\lambda \mid \lambda \in \Lambda\}$; for each $\lambda \in \Lambda$ there is a unique l-homomorphism $\pi_\lambda : G \to G_\lambda$ such that $u_{\lambda'}\pi_\lambda = 1_{G_\lambda}$ if $\lambda' = \lambda$, and 0 if not. In particular each co-projection is $1-1$. The corresponding results hold in $\mathscr{L}R$ and $\mathscr{L}A$.*

1.3. *Let $\bar{G} = \boxplus\{G_\lambda \mid \lambda \in \Lambda\}$ and $\sigma_\lambda : G_\lambda \to \bar{G}$ be the usual embeddings; (in our notation $\bar{G}$ stands for the direct sum of the G_λ, with coordinatewise ordering.) There is a unique l-homomorphism $\sigma : \amalg\{G_\lambda \mid \lambda \in \Lambda\} \to \bar{G}$ such that $u_\lambda\sigma = \sigma_\lambda$ for each $\lambda \in \Lambda$. Similar formulations are valid in $\mathscr{L}R$ and $\mathscr{L}A$.*

1.4. *Let $G = \amalg\{G_\lambda \mid \lambda \in \Lambda\}$ and $\mu, \nu \in \Lambda$, $\mu \neq \nu$. Then $G_\mu u_\mu \cap G_\nu u_\nu = 0$, and if $0 < x \in G_\mu$, $0 < y \in G_\nu$ then $xu_\mu \parallel yu_\nu$.*

Proof. If $0 \leqq a \in G_\mu u_\mu \cap G_\nu u_\nu$ then $a = a_\mu u_\mu = b_\nu u_\nu$, so $a_\mu = a\pi_\mu = b_\nu u_\nu \pi_\mu = 0$ and therefore $a = 0$. Next, suppose $xu \leqq yu_\nu$; then $0 < x = xu_\mu\pi_\mu \leqq yu_\nu\pi_\mu = 0$, a contradiction; similarly $xu_\mu \ngeqq yu_\nu$.

(Again, the corresponding results hold in $\mathscr{L}R$ and $\mathscr{L}A$.)

1.5. *In $G = \amalg^A\{G_\lambda \mid \lambda \in \Lambda\}$ the subgroup generated by the $G_\lambda u_\lambda$ is precisely $\bar{G} = \boxplus\{G_\lambda \mid \lambda \in \Lambda\}$.*

1.6. *Within the context of 1.5, G is the l-ideal generated by $\bar{G}$.*

Proof. We may express $0 < x \in G$ as

$$x = [\vee_\alpha \wedge_\beta \sum_{i=1}^{n} [g(\alpha, \beta, i)] u_{\lambda_i}] \vee 0 \leqq \vee_\alpha \wedge_\beta \sum_{i=1}^{n} (g(\alpha, \beta, i) \vee_{\lambda_i} 0) u_{\lambda_i} \leqq$$
$$\leqq \sum_\alpha \sum_\beta \sum_{i=1}^{n} (g(\alpha, \beta, i) \vee_{\lambda_i} 0) u_{\lambda_i} .$$

1.7. *Once again, let G and $\bar{G}$ be as in* 1.5; *an l-homomorphism $\phi : \bar{G} \to L$ into the abelian l-group L has a unique extension $\bar{\phi} : G \to L$.*

We shall see later that the containment of the cardinal sum in the free A-product is always proper. Thus the kernel of the mapping described in 1.3 is always non-trivial. We shall now look at this kernel more closely; immediately prior to this we wish to give a categorical characterization of the cardinal sum for each of the varieties under consideration.

1.8. Theorem. *Let G_λ $(\lambda \in \Lambda)$ be a family of l-groups in $\mathscr{L}$ $(\mathscr{L}R, \mathscr{L}A)$; G in $\mathscr{L}$ $(\mathscr{L}R, \mathscr{L}A)$ is the cardinal sum of the G_λ if and only if there exist l-homomorphisms $\sigma_\lambda : G_\lambda \to G$ having the property that*

1) *$0 < x \in G_\lambda$, $0 < y \in G_\mu$ with $\lambda \neq \mu$ imlies $x\sigma_\lambda \wedge y\sigma_\mu = 0$, and*

2) *if $\phi_\lambda : G_\lambda \to L$ is a family of l-homomorphisms into the l-group L in $\mathscr{L}$ $(\mathscr{L}R, \mathscr{L}A)$ such that $0 < x \in G_\lambda$, $0 < y \in G_\mu$ and $\lambda \neq \mu$ imply $x\phi_\lambda \wedge y\phi_\lambda = 0$,*
then there is a unique l-homomorphism $\phi : G \to L$ such that $\sigma_\lambda\phi = \phi_\lambda$, for each λ.

Proof. It suffices to prove necessity, for it is clear that any two l-groups with the above properties are isomorphic. If G is the cardinal sum of the G_λ, and σ_λ is the λ-th coordinate embedding then the family of the σ_λ obviously satisfies the first condition.

Now suppose $\phi_\lambda : G_\lambda \to L$ is a family of l-homomorphisms having the desired disjointness property; let $\phi : G \to L$ be the induced *group* homomorphism. To prove it preserves the lattice structure we need only show it preserves disjointness; our assumption about the family of φ_λ and the coordinatewise ordering of G guarantee exactly that. It is evident moreover, that ϕ is uniquely determined by the ϕ_λ in the category $\mathscr{L}(\mathscr{L}R, \mathscr{L}A)$.

1.8.1. Corollary. *Let G_λ $(\lambda \in \Lambda)$ be a family of l-groups in $\mathscr{L}$ $(\mathscr{L}R, \mathscr{L}A)$. In the free product* ($R$-product, A-product) *G of the G_λ let K be the l-ideal generated by all $a_\lambda u_\lambda \wedge b_\mu u_\mu$ where $0 < a_\lambda \in G_\lambda$, $0 < b_\mu \in G_\mu$ and $\lambda \neq \mu$. Then $\boxplus\{G_\lambda \mid \lambda \in \in \Lambda\} \simeq G/K$, and K is the kernel of the map in* 1.3.

Proof. Let $\varrho : G \to G/K$ be the canonical l-homomorphism, and let $\sigma_\lambda = u_\lambda\varrho$, for each $\lambda \in \Lambda$. Now if $0 < a \in G_\lambda$ and $0 < b \in G_\mu$ $(\lambda \neq \mu)$ then $a\sigma_\lambda \wedge b\sigma_\mu = = au_\lambda\varrho \wedge bu_\mu\varrho = (au_\lambda \wedge bu_\mu)\,\varrho = 0$. Furthermore, suppose $\phi_\lambda : G_\lambda \to L$ is a family of l-homomorphisms into the l-group L in $\mathscr{L}$ $(\mathscr{L}R, \mathscr{L}A)$ satisfying the disjointness condition. There is an induced map $\bar{\phi} : G \to L$ such that $(au_\lambda \wedge bu_\mu)\,\bar{\phi} = a\phi_\lambda \wedge$

$\wedge\ b\phi_\mu = 0$, which means that the kernel of $\bar{\phi}$ contains K and hence $\bar{\phi}$ factors uniquely through K, say by $\phi : G/K \to L$ with $\varrho\phi = \bar{\phi}$. We have $\sigma_\lambda\phi = u_\lambda\varrho\phi = u_\lambda\bar{\phi} = \phi_\lambda$, $(\lambda \in \Lambda)$ and it is clear that the ϕ_λ determine ϕ uniquely. The corollary is hereby proved.

Let us denote by $\mathscr{S}$ the category of sets, and use $F : \mathscr{S} \to \mathscr{L}$ $(F_R : \mathscr{S} \to \mathscr{L}R$, $F_A : \mathscr{S} \to \mathscr{L}A)$ for the „free" functor; that is, if S is a set $F(S)$ $(F_R(S), F_A(S))$ is the free object in $\mathscr{L}$ $(\mathscr{L}R, \mathscr{L}A)$ on S. This functor preserves co-limits ([7], p. 44 & p. 67), consequently if S_λ $(\lambda \in \Lambda)$ is a collection of pairwise disjoint sets and $S = \bigcup S_\lambda$ then

$$F(S) = \amalg\{F(S_\lambda) \mid \lambda \in \Lambda\}\,.$$

(Of course, similar formulas hold in $\mathscr{L}R$ and $\mathscr{L}A$.) For the variety $\mathscr{L}A$ we have the following generalization of the above.

1.9. Proposition. *Let $\mathscr{P}$ denote the category of semiclosed*) partially ordered abelian groups and o-homomorphisms; the free functor $\Phi : \mathscr{P} \to \mathscr{L}A$ which assigns to a semiclosed p. o. group the free abelian l-group over it preserves co-limits. Moreover, the cardinal sum is the co-product in $\mathscr{P}$ so that*

$$\Phi(\boxplus\{G_\lambda \mid \lambda \in \Lambda\}) = \amalg^A\{\Phi(G_\lambda) \mid \lambda \in \Lambda\}\,.$$

Proof. Φ preserves co-limits since it is adjoint to the limit preserving functor $U : \mathscr{L}A \to \mathscr{P}$ which "forgets" the lattice structure but remembers the partial order; ([7], p. 44 & *p.* 67.) That the direct sum with pointwise ordering is indeed the co-product is left to the reader as an exercise. The formula in the statement follows immediately.

1.9.1. Corollary. *Let G_λ $(\lambda \in \Lambda)$ be a family of abelian o-groups; then*

$$\amalg^A\{G_\lambda \mid \lambda \in \Lambda\} = \Phi(\boxplus\{G_\lambda \mid \lambda \in \Lambda\})\,.$$

Proof. Merely observe that if H is an o-group then $\Phi(H) = H$. ([4], 3.10)

(Note: this corollary says right away that in the case of abelian o-groups $\amalg^A\{G_\lambda \mid \lambda \in$ $\in \Lambda\} \supset \boxplus\{G_\lambda \mid \lambda \in \Lambda\}$, and this containment is *never* as an l-subgroup.)

2. Separation of the free products. It is well known in group theory that the free product of two non-trivial groups is non-abelian even if the two factors are themselves abelian. We shall prove the corresponding results for varieties of l-groups in this section. For simplicity of notation we shall deal with free products of two factors; we shall also omit mention of the co-projections, and think of the factors simply as l-subgroups of the free product.

(*) A p. o. group is *semiclosed* if $nx \geqq 0$ with n a positive integer, implies $x \geqq 0$. This is equivalent to the statement that the cone of positive elements is an intersection of cones of total orders. Weinberg [8] showed that these are precisely the abelian p. o. groups over which free l-groups exist.)

We consider the free product in $\mathscr{L}$ first; there are two questions to be asked here: given non-zero l-groups G and H is $G \perp\!\!\!\perp H$ always non-abelian? And more specifically, if $g \in G$ and $h \in H$ with both non-zero, does g always fail to commute with h? (Incidentally the stronger version holds in group theory.) The answer to the first question is yes; the second is also answered in the affirmative provided one of the groups is representable. The proofs of these facts depend upon an interesting unstated lemma and the notion of the wreath product.

Let G be an o-group, H be any l-group; the *wreath product* H Wr G *of* H by G is the set $G \times \prod_{a \in G} H_a$, where each $H_a = H$, together with the operation

$$(g; \ldots, x_a, \ldots) + (g'; \ldots, y_a, \ldots) = (g + g'; \ldots, z_a, \ldots),$$

where $z_a = x_{a-g'} + y_a$, and finally the lattice order whose cone is $\{(g; \ldots, x_a, \ldots) \mid g > 0$, or $g = 0$ and each $x_a \geqq 0\}$. H Wr G is a splitting extension of ΠH_a by G, and is not representable unless $G = 0$ and H is representable.

We now state the first main theorem.

2.1. Theorem. *Let G be a non-trivial representable l-group, H be any non-trivial l-group; then in $G \perp\!\!\!\perp H$ no $0 \neq g \in G$ and $0 \neq h \in H$ commute. Furthermore, $G \perp\!\!\!\perp H$ is not representable.*

2.1.1. Corollary. *The free product $G \perp\!\!\!\perp H$ of two non-trivial l-groups is non-representable, (and hence non-abelian.)*

Proof of 2.1. Let us suppose we've proved the theorem for o-groups G. If $G \neq 0$ now denotes an arbitrary representable l-group, let $0 \neq g \in G$ and $0 \neq h \in H$. Let N be a minimal prime subgroup of G containing g; it is well known however, that a minimal prime of a representable l-group is normal, (the result is due to R. Byrd), so we may consider the canonical map $\varrho : G \to G/N$. The induced map $\bar{\varrho} : G \perp\!\!\!\perp H \to G/N \perp\!\!\!\perp H$ is onto, and since G/N is an o-group $g\bar{\varrho} = g\varrho$ does not commute with h in $G/N \perp\!\!\!\perp H$. It follows that g cannot commute with h in $G \perp\!\!\!\perp H$, and since $G/N \perp\!\!\!\perp H$ is not representable neither is $G \perp\!\!\!\perp H$.

So let us now show that our assertions hold when G is already totally ordered. Let $K = H$ Wr G and $\psi : G \to K$ be defined by $b\psi = (b; \ldots, 0, \ldots)$, $\psi' : H \to K$ be defined by $x\psi' = (o; \ldots, x_a, \ldots)$ where x_a is 0 if $a \neq 0$ and x if $a = 0$. These are both l-homomorphisms, so let $\Psi : G \perp\!\!\!\perp H \to K$ be the unique extension of ψ and ψ' to the free product. Now it suffices to show $0 < g \in G$ and $0 < h \in H$ fail to commute.

$$(h^g)\,\Psi = (h\psi')^{g\psi} = (0; \ldots, z_a, \ldots) \quad \text{where} \quad z_a = \begin{cases} 0 & \text{if} \quad a \neq g\,; \\ h & \text{if} \quad a = g \end{cases}$$

clearly then $(h^g)\,\Psi \neq h\Psi$, and so $h^g \neq h$.

Note in the above that $(h^g \wedge h)\,\Psi = (h^g)\,\Psi \wedge h\Psi = 0$. We can therefore pick $0 < a \leqq h$ and $0 < b \leqq h^g$ in $G \perp\!\!\!\perp H$ such that $a \wedge b = 0$ and $a\Psi = h\Psi$, $b\Psi = (h^g)\,\Psi$. But then

$$(a^g \wedge b)\,\Psi = a^g\Psi \wedge b\Psi = a\Psi^{g\psi} \wedge b\Psi = h^g\Psi \wedge h^g\Psi = h^g\Psi = b\Psi > 0\,.$$

Thus $a^g \wedge b > 0$, and therefore $G \perp\!\!\!\perp H$ is not representable. This completes the proof of theorem 2.1.

We now turn to the free product in $\mathscr{L}R$; once again we are able to show the free R-product of two non-trivial l-groups is non-abelian. In fact:

2.2. Theorem. *Let G and H be two non-trivial representable l-groups; then in $G \perp\!\!\!\perp^R H$ no $0 \neq g \in G$ commutes with any $0 \neq h \in H$.*

Proof. Fix $0 \neq g \in G$ and $0 \neq h \in H$; by factoring out primes we may assume both G and H are o-groups and g and h are positive. Let K be the *restricted* wreath product of H by G; that is, K consists of all $(g; \ldots, x_a, \ldots)$ for which x_a is 0 for all but finitely many $a \in G$. We then totally order K as follows: $(g; \ldots, x_a, \ldots) \geqq 0$ if $g > 0$, or else $g = 0$ and the non-zero component x_a with the largest index is positive. As in the proof of theorem 2.1 we use the injections of G and H into K given by $c \to (c; \ldots, 0, \ldots)$ and $d \to (0; \ldots, x_a, \ldots)$ where $x_a = 0$ if $a \neq 0$ and d otherwise. Then as in that proof we show that under the induced map $\Psi : G \perp\!\!\!\perp^R H \to K$ the images of g and h do not commute; it follows that g and h fail to commute in $G \perp\!\!\!\perp^R H$.

(Warning: it is necessary here to restrict the wreath product; the full wreath product discussed before admits no total orders.)

We now settle the question of how the free A-product compares to the cardinal sum.

2.3. Theorem. *If G and H are two non-trivial abelian l-groups neither G nor H is convex in $G \perp\!\!\!\perp^A H$, so that $G \perp\!\!\!\perp^A H$ is not isomorphic to $G \boxplus H$,*

Proof. We will show that for all $0 < a \in G$ and $0 < b \in H$ $a \wedge b > 0$. As before by factoring out a suitable prime subgroup of G it suffices to take G to be totally ordered. Let $L = G \overrightarrow{\times} H$; this notation signifies the cartesian product with the lexicographic ordering placing G over H. One has an l-homomorphism $\gamma : G \perp\!\!\!\perp^A H \to L$ induced by the coordinatewise injections of G and H in L. If $0 < a \in G$ and $0 < b \in H$ then

$$(a \wedge b)\,\gamma = a\gamma \wedge b\gamma = (a, 0) \wedge (0, b) = (0, b) > 0\,;$$

hence $a \wedge b > 0$ in $G \perp\!\!\!\perp^A H$, proving neither G nor H is convex in the free A-product, thereby completing the proof of this theorem.

2.3.1. Corollary. *Let G and H be non-trivial abelian l-groups;*

i) *for* $0 < g \in G$ *and* $0 < h \in H$, $0 < g \wedge h \in G \amalg^A H$.

ii) *the canonical injection of* $G \boxplus H$ *into* $G \amalg^A H$ *is proper and never lattice preserving.*

At this point a general word about varieties of l-groups is perhaps in order. We have looked at three of them; but most important classes of l-groups are not varieties of l-groups. For example, all of the following fail to be quotient-closed: archimedean l-groups, l-groups with basis, subdirect products of reals (integers). The next group features some of those that are not l-subgroup-closed: finite valued l-groups, archimedean l-groups with strong or weak order unit, complete l-groups, laterally complete l-groups, completely distributive l-groups, etc. The class of finite basis l-groups is l-subgroup- and quotient-closed, but certainly not closed under products; the same is true of l-groups with property (F).

One important class which does turn out to be a variety of l-groups is the class of normal valued l-groups. (An l-group G is said to be *normal valued* if each value M of any non-zero element $g \in G$ is normal in its cover.) One establishes this fact as a corollary of a remarkable result of WOLFENSTEIN [10] which says that G is normal valued if and only if for each $0 \leqq a, b \in G$, $a + b \leqq 2b + 2a$. Consequently, we have a free normal valued l-group over a given set X; in this variety, which we shall denote by $\mathscr{L}N$, we can also produce the co-product, or *free N-product* of an arbitrary family of normal valued l-groups. It is well known that a representable l-group is normal valued; (this is due to R. BYRD.)

Suppose G and H are non-trivial normal valued l-groups; we wish to prove that $G \amalg^N H$ is non-representable. Since both factors are l-subgroups of the free N-product all is well when either of them is non-representable. If both are representable then the proof of theorem 2.1 shows that $G \amalg^N H$ is not, (the wreath product constructed there is normal valued!)

2.4. Theorem. *The free N-product of two non-trivial normal valued l-groups G and H is non-representable.*

In view of theorem 2.1 one would certainly like to know whether the free product in $\mathscr{L}$ of two non-zero l-groups is always outside $\mathscr{L}N$.

2.5. Theorem. *Let G and H be two non-trivial representable l-groups; then if* $0 < g \in G$ *and* $0 < h \in H$, $g + h \nleqq 2h + 2g$ *in* $G \amalg H$. *In particular* $G \amalg H$ *is not normal valued.*

Proof. Since the two factors are representable we may again factor out suitable primes and assume instead that G and H are totally ordered. Let H^* be the restricted lexicographic product of copies of H indexed by G. We shall consider o-permutations on H^*: for each $g \in G$ define $g\pi_G$ to be the permutation given by $(\ldots, x_a, \ldots)\, g\pi_G = (\ldots, z_a, \ldots)$, where $z_a = x_{-g+a}$. It is clear that $g\pi_G$ is an o-permutation on H^*, and that the map $\pi_G : G \to \mathscr{P}(H^*)$, the l-group of all o-permutations on H^*, is an o-isomorphism. (In fact each $g\pi_G$ is an o-automorphism of H^*.) For each $h \in H$ let $h\pi_H$ be the o-permutation of right translation by $(\ldots, h_a, \ldots)$, where $h_a = h$ when $a = 0$ and $h_a = 0$ otherwise. Again $\pi_H : H \to \mathscr{P}(H^*)$ is an o-isomorphism; let $\pi : G \amalg H \to \mathscr{P}(H^*)$ be the unique extension of π_G and π_H to the free product. Consider the permutation $(2g + 2h - g - h)\,\pi = (g\pi_G)^2\,(h\pi_H)^2\,(g\pi_G)^{-1}\,(h\pi_H)^{-1}$. One can easily compute that

$$0(g\pi_G)^2\,(h\pi_H)^2\,(g\pi_G)^{-1}\,(h\pi_H)^{-1} = \begin{cases} -\,h & \text{if } a = 0 \\ +2h & \text{if } a = -g\,; \\ 0 & \text{otherwise} \end{cases}$$

clearly, this is negative in H^*, so that the permutation $(2g + 2h - g - h)\,\pi$ is not positive. It follows then the $2g + 2h \not\geqq h + g$; this proves the theorem.

Summarizing then, the free product of two non-zero l-groups is not in $\mathscr{L}A$ or $\mathscr{L}R$, and fails to be in $\mathscr{L}N$ if one of the factors is outside $\mathscr{L}N$ or both are represetnable. The free R-product is not in $\mathscr{L}A$, and the free A-product is always different from the cardinal sum.

In the following section we discuss a new variety of l-groups and exhibit once more the phenomenon presented here for free products.

3. Weakly abelian l-groups. The following bit of information may serve to motivate the variety of l-groups we are about to introduce. The variety of abelian l-groups is the smallest variety of l-groups. For a non-trivial variety of l-groups contains some non-zero l-group G, and in it a copy of Z ordered as usual. This variety then contains all subdirect products of integers, and hence by Weinberg's theorem all free abelian l-groups. It is clear then that all abelian l-groups must be in the variety.

In view of the above, and taking into consideration the fact that so few varieties of l-groups have been identified, we seek some information concerning say the varieties between $\mathscr{L}R$ and $\mathscr{L}A$. It turns out there is at least one, and we finally get on with the discussion thereof.

An l-group G is *weakly abelian* if for each $0 < x \in G$ and $g \in G$ $2x \geqq x^g$. We proceed to list some of the basic properties of weakly abelian l-groups, proofs being omitted whenever they are trivial or fairly straightforward.

3.1. *The class of weakly abelian l-groups forms a variety of l-groups containing $\mathscr{L}A$ and contained in $\mathscr{L}R$.*

3.2. *In a weakly abelian l-group all convex l-subgroups are normal,* (*but the converse is false as we shall soon see.*)

3.3. *If G is a weakly abelian l-group it can be represented as a subdirect product of weakly abelian o-groups having a minimal convex subgroup.*

3.4. *There are non-abelian weakly abelian o-groups, and not every representable l-group is weakly abelian.*

There are many examples of the type of l-group which is representable but not weakly abelian; we will leave it to the reader's imagination to produce such an example. As for the other case consider the following:

3.5. Let $G = Z \times Z \times Z$, and define addition of triples as

$$(a, b, c) + (x, y, z) = (a + x, b + y, c + z + ay) .$$

G is then a (non-abelian) group, and becomes an o-group when we order $(a, b, c) \geqq 0$ if $a > 0$, or $a = 0$ and $b > 0$, or $a = b = 0$ and $c \geqq 0$. One may easily verify that

$$(a, b, c)^{(x,y,z)} = (a, b, c + ay - bx) ,$$

and from this one derives without much trouble that G is weakly abelian.

After proposition 3.7 we shall have an example of a weakly abelian o-group with a trivial center. We shall use this same o-group in an example in § 6.

We now give our main structure theorem for weakly abelian l-groups.

3.6. Theorem. *Let G be a weakly abelian l-group, $0 < x \in G$, $g \in G$ and N be a value of x; then $x + N = x^g + N$. Conversely if $x + N = x^g + N$ for all $0 < x \in$ $\in G$, $g \in G$ and values N of x, then G is weakly abelian.*

Proof. We know $2x \geqq x^{ng}$ for all $n = 1, 2, \ldots$; hence $2x + N \geqq x^{ng} + N$. Yet it is a consequence of Holder's theorem (see [5]) that conjugation, since it induces an o-automorphism of N^*/N – where N^* is the cover of N in the lattice of convex l-subgroups of G – must in fact represent multiplication by some positive real number. It follows then that $x^g + N = x + N$; (we have assumed tacitly that $x^g +$ $+ N \geqq x + N$; this we can do without loss of generality.)

Conversely if $x^g + N = x + N$ for all $0 < x \in G$, $g \in G$ and values N of x, then all such values are normal in G. It is evident that a value of x is also a value of x^g and of $-2x + x^g$ as well. On the other hand a value of $-2x + x^g$ is contained in a value of x, and therefore itself a value of x. It then becomes clear that $2x \geqq x^g$.

3.6.1. Corollary. *Let G be an l-group; the following are equivalent.*

i) *G is weakly abelian.*
ii) *For each $0 < x \varepsilon G$, $g \in G$ and each value N of x, $x + N = x^g + N$.*
iii) $2x > x^g$ *for all* $0 < x \in G$ *and* $g \in G$.
iv) *For a fixed positive integer n $nx \geqq x^g$ for all $0 < x \in G$ and $g \in G$.*
v) *For a fixed positive integer n $nx > x^g$ for all $0 < x \in G$ and $g \in G$.*
vi) $x \geqq |[x, g]|$, *all* $0 < x \in G$ *and* $g \in G$; *of course* $[x, g] = -x + x^g$.
vii) $x \gg |[x, g]|$, *all* $0 < x \in G$ *and* $g \in G$.

(Recall: $a \gg b$ for positive elements a and b if a exceeds every positive integral multiple of b.)

Proof. The equivalence of the first five items follows directly from theorem 3.6. However in view of ii) we need only show the equivalence of vi) or vii) with say i).

i) $\to$ vii) Suppose $0 < x \in G$ and $g \in G$; since $2x \geqq x^g$ we have that $x \geqq [x, g]$. Also if M is value of $[x, g]$ then either $x \notin M$ or $g \notin M$. If $x \in M$ then so is x^g and thus $[x, g]$ is too; it follows then that $x \notin M$, and hence that M is contained in value of x. In view of ii) once again, this containment must be proper. This suffices to show $x \gg |[x, g]|$. vii) $\to$ i) is trivial.

3.6.2. Corollary. *If a weakly abelian o-group has a minimal (non-zero) convex subgroup it has a non-trivial center. Thus every weakly abelian l-group can be realized as a subdirect product of weakly abelian o-groups with non-trivial center.*

Let G be an l-group, A be an l-deal of G. G is a *lex (icographic) extension* of A (notation: $G = \operatorname{lex}(A)$) if 1) G/A is an o-group and 2) each $0 < g \in G \setminus A$ exceeds every element in A. It is fairly well known that $G = \operatorname{lex}(A)$ if and only if the non-zero cosets of G/A consists entirely of positive or entirely of negative elements.

3.6.3. Corollary. *Suppose* $G = \operatorname{lex}(A)$ *and A is weakly abelian; if A is central in G and G/A is weakly abelian, so is G.*

3.6.4. Corollary. *A weakly abelian l-group has no non-central atoms.*

Although we can improve upon the centrality condition in corollary 3.6.3, it is nevertheless an important condition. Consider the next example: let G be the restricted wreath product of the rationals Q by Z as follows; think of the direct sum of copies of Q as a group of "polynomials"

$$\sum_{n=-\infty}^{\infty} a_n \pi^n ,$$

where at most a finite number of the a_n are nonzero. In effect then we have an extension of a subgroup of R by the integers. We order G by declaring $x = (p; \sum_{n=-\infty}^{\infty} a_n \pi^n) > 0$ if p is a positive integer, or $p = 0$ and $\sum_{n=-\infty}^{\infty} a_n \pi^n$ is a positive real number. If $x = (0; \sum_{n=-\infty}^{\infty} a_n \pi^n)$ and $y = (q; \sum_{n=-\infty}^{\infty} b_n \pi^n)$ then

$$x^y = (0; \sum_{n=-\infty}^{\infty} a_{n-q} \pi^n) = (0; \sum_{n=-\infty}^{\infty} a_n \pi^{n+q}) .$$

In this case the net effect of conjugating x by y is simply to "multiply" x by π^q; obviously then G is not weakly abelian. Notice that G is the lex extension of an abelian o-group by another; (they are both archimedean, in fact.) Observe also that G has the property that every convex sugroup is normal in G.

It is well known that a free abstract group admits a total order (see [5], theorem 8, p. 49). The same proof shows

3.7. Proposition. *A free group admits a weakly abelian total order.*

Remark. The proof referred to in [5] actually shows that a free group admits a weakly abelian total order in which the commutator subgroup is convex. Upon factoring out this subgroup we get of course an abelian o-group. This shows that the centrality in 3.6.3 can indeed be improved.

We next turn to the „separation" problem discussed in § 2.

3.8. Proposition. *The free R-product of two non-trivial representable l-groups is always not weakly abelian.*

Proof. Apply the proof of theorem 2.2. This is possible, for if G and H are non-trivial o-groups the restricted wreath product of G and H, ordered as in the proof of 2.2 is not weakly abelian.

3.9. Theorem. *Let G and H be non-trivial weakly abelian l-groups; the free product of G and H in the variety $\mathscr{L}W$ of weakly abelian l-groups is never abelian.*

Proof. If either of the groups is non-abelian there is nothing to prove, so we can assume both are abelian. As usual for representable l-groups we can also suppose both are o-groups. We embed G and H in their respective divisible hulls $\bar{G}$ and $\bar{H}$. Fix $0 < a \in G$ and $0 < b \in H$ and let M and N be the values of a and b in $\bar{G}$ and $\bar{H}$ respectively; further let M^* and N^* be the respective covers of M and N. Then

$$\bar{G} = \bar{G}/M^* \overrightarrow{\times} M^* \quad \text{and} \quad \bar{H} = \bar{H}/N^* \overrightarrow{\times} N^* .$$

Define an o-group K as follows: $K = G/M^* \times H/N^* \times M^*/M \times N^*/N \times R$ ($R =$ = reals) where addition is defined by

$$(a_1, b_1, a_2, b_2, r) + (x_1, y_1, x_2, y_2, s) =$$
$$= (a_1 + x_1,\ b_1 + y_1,\ a_2 + x_2,\ b_2 + y_2,\ r + s + a_2 y_2) .$$

(We think of M^*/M and N^*/N, as we indeed may, as subgroups of R.) With the lexicographic order from left to right K becomes a non-abelian weakly abelian o-group.

There is a natural embedding of G in K by $g\tau_1 = (g_1, 0, g_2 + M, 0, 0)$ where $g = g_1 + g_2$ is the unique decomposition of g with $g_1 \in \bar{G}/M^*$ and $g_2 \in M^*$. (The reader will no doubt appreciate that τ_1 is not a $1-1$ mapping, nevertheless it is appealing to think of the map in this way. Furthermore no use will be made of it as a global embedding, rather as a local one.) Likewise the natural map τ_2 arises from H to K by $h\tau_2 = (0, h_1, 0, h_2 + N, 0)$, where $h = h_1 + h_2$ and $h_1 \in \bar{H}/N^*$, $h_2 \in N^*$.

We extend τ_1 and τ_2 to $\tau : G \amalg^W H \to K$; now $a\tau = (0, 0, a + M, 0, 0)$ and $b\tau = (0, 0, 0, b + N, 0)$; so

$$(a + b)\tau = a\tau + b\tau = (0, 0, a + M,\ b + N,\ (a + M)(b + N)),$$

while

$$(b + a)\tau = b\tau + a\tau = (0, 0, a + M,\ b + N, 0).$$

Since $(a + M)(b + N) \neq 0$ it follows that $(a + b)\tau \neq (b + a)\tau$, and hence $a + b \neq b + a$. (We've shown a bit more than was stated in the theorem; we have proved in fact that for all $0 < a \in G$, $0 < b \in H$ a and b fail to commute in $G \amalg^W H$.)

The condition defining weakly abelian l-groups may lead the reader to speculate that $\mathscr{L}W$ is somehow minimal over $\mathscr{L}A$. Unfortunately, we must report this is not so; (see example 6.7) Notice that $\mathscr{L}W$ (and $\mathscr{L}R$ for that matter) has the following vague property: the "law" defining this variety does not involve any group-theoretical restrictions; this observation is borne out by the fact that free groups admit such orders (3.7). The study of such varieties is quite intriguing, and one wonders whether $\mathscr{L}W$ is not minimal over $\mathscr{L}A$ in this sense.

4. Projectives in $\mathscr{L}A$. Given a category $\mathscr{C}$ an epimorphism ϕ is said to be *extremal* if whenever $\phi = \phi_1\phi_2$ with ϕ_2 monic, then ϕ_2 is an isomorphism. It is known that in a variety of universal algebras a homomorphism is an extremal epimorphism if and only if it is onto. The proof is quite simple and we shall present it here for sake of completeness. Suppose $\phi : A \to B$ is an extremal epic in some variety of universal algebras. Factor ϕ through its image; $\phi = \bar{\phi}\iota$; the inclusion ι of the image of ϕ in B is monic and therefore an isomorphism. It follows that ϕ is onto. The converse is trivial; the reader is invited to try it if he wishes.

One is very seldom sure of what the epics are in a variety, so one must settle for the extremal epimorphisms in dealing with projectives. Accordingly then an object P in a variety is *projective* if for each extremal epimorphism $\phi : M \to N$ and each homomorphism $\alpha : P \to N$ there is a homomorphism $\bar{\alpha} : P \to M$ such that $\bar{\alpha}\phi = \alpha$. It is immediate that free objects are projective under this definition; the reader knows of course that in the category of abelian groups these are the only projectives. As we shall see presently the above is far from true in $\mathscr{L}A$.

We should point out in passing that in $\mathscr{L}A$ there are epimorphisms which are not onto. Witness for example the canonical embedding of Z into Q; two- l-homomorphisms from Q into an l-group A which agree on Z are identical. Suppose σ_1 and σ_2 are the indicated mappings; we have $1\sigma_1 = 1\sigma_2$. Then it is clear that $n(1/n)\sigma_1 = n(1/n)\sigma_2$, for each non-zero integer n, and consequently since l-groups are torsion-free $(1/n)\sigma_1 = (1/n)\sigma_2$; this suffices to prove $\sigma_1 = \sigma_2$.

Here then is a non-free projective of $\mathscr{L}A$.

4.1. Lemma. *Z (with the usual order) is projective.*

Proof. Let M and N be abelian l-groups and $\phi : M \to N$ be an onto l-homomorphism; further let $\alpha : Z \to N$ be any l-homomorphism. Consider 1α; it has a *positive* pre-image under ϕ, say x, in M. Define $1\bar{\alpha} = x$ and extend to Z in the obvious way; the resulting homomorphism preserves order, proving Z is projective.

(Note: CONRAD pointed out that a free object in $\mathscr{L}$, $\mathscr{L}A$ or $\mathscr{L}R$ is never totally ordered, consequently Z is a projective of $\mathscr{L}A$ – and of $\mathscr{L}$ and $\mathscr{L}R$ for that matter, since the commutativity of M and N was not used in the above – which is not free.)

All our discussion for the remainder of this section takes place in $\mathscr{L}A$; the reader may spot some obvious generalizations to non-abelian cases: he is free to formulate these generalizations if he wishes. Call the l-group A a *retract* of the l-group B if there is an l-homomorphism $\beta : B \to A$ and an l-homomorphism $\alpha : A \to B$ such that $\alpha\beta = 1_A$. The next result is quite straightforward to prove, and we shall leave it to the reader.

4.2. Proposition. *In $\mathscr{L}A$:*

i) *a retract of a projective is projective;*

ii) *an l-group is projective if and only if it is a retract of every l-group of which it is an l-homomorphic image. (This property is usually called retract projectivity.)*

iii) *An l-group is projective if and only if it is a retract of a free l-group.*

iv) *Let $P = \coprod^{\Lambda} P_\lambda$, $(\lambda \in \Lambda)$. In order that P be projective it is necessary and sufficient that each P_λ also be projective.*

4.2.1. Corollary. *If P is projective it is a subdirect product of integers, and every set of pairwise disjoint elements is at worst countable.*

Proof. A free l-group is a subdirect product of integers; a retract of a free l-group then must have the same property. As for the statement on disjoint elements we need only comment on Weinberg's result in [9], namely that free l-groups have that property also.

Our main result in this section is:

4.3. Theorem. *The cardinal sum of two finitely generated projectives is again projective.*

Before proving the theorem let us assess some of the consequences.

4.3.1. Corollary. *The abstract free abelian roup on $n < \infty$ generators, when equipped with the cardinal order is projective in $\mathscr{L}A$.*

Since free l-groups are necessarily indecomposable into cardinal sums, we have then a very large class of non-free projectives. Theorem 4.3 is the more intriguing since the cardinal sum does not appear to be a co-limit in the category $\mathscr{L}A$.

Now for the proof of the theorem.

Proof. The argument depends strongly on a lemma which we shall cite here informally; (the proof is quite easy). Given an l-homomorphism ϕ of an l-group A onto one B, suppose $\{x_1, x_2, \ldots, x_n\}$ are pairwise disjoint elements in B. Then one may choose a set $\{a_1, a_2, \ldots, a_n\}$ or pairwise disjoint elements in A such that $a_i\phi = x_i.(i = 1, \ldots, n)$. Let us proceed from here.

It clearly suffices, in view of proposition 4.2, to prove the theorem for finitely generated free l-groups. Suppose then that F_1 is the free l-group over the free set of generators $\{a_1, \ldots, a_n\}$, F_2 the free l-group over $\{b_1, \ldots, b_m\}$. Let $\alpha : F_1 \boxplus F_2 \to N$ be an l-homomorphism and $\phi : M \to N$ be an onto l-homomorphism. Consider the following four sets: $\{a_i{}^+\alpha = a_i\alpha^+ \mid i = 1, \ldots, n\}$, $\{a_i{}^-\alpha = a_i\alpha^- \mid i = 1, \ldots, n\}$, $\{b_j{}^+\alpha = b_j\alpha^+ \mid j = 1, \ldots, m\}$ and $\{b_j{}^-\alpha = b_j\alpha^- \mid j = 1, \ldots, m\}$. According to our informal lemma in the previous paragraph we may find preimages as follows: for each $i = 1, \ldots, n$, $0 \leqq x_i, y_i$ such that $x_i\phi = a_i{}^+\alpha$, $y_i\phi = a_i{}^-\alpha$ and $x_i \wedge y_i = 0$; likewise for each $j = 1, \ldots, m$ $0 \leqq z_j, w_j$ with the properties that $z_j\phi = b_j{}^+\alpha$, $w_j\phi = b_j{}^-\alpha$ and $z_j \wedge w_j = 0$. *In addition* we may take each x_i and y_i to be disjoint to each z_j and w_j; *this is the crucial step.*

Now define $a_i\bar{\alpha}_1 = x_i - y_i$ $(1 \leqq i \leqq n)$, and $b_j\bar{\alpha}_2 = z_j - w_j$ $(1 \leqq j \leqq m)$. These assignments can be lifted to l-homomorphisms $\bar{\alpha}_1$ and $\bar{\alpha}_2$ of F_1 and F_2 respectively into M such that $\bar{\alpha}_i\phi = \alpha_{F_i}$ $(i = 1, 2)$. Since the images of $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are elementwise disjoint, the induced homomorphism $\bar{\alpha}$ on $F_1 \boxplus F_2$ preserves the lattice operations. Evidently $\bar{\alpha}\phi = \alpha$, proving at last that $F_1 \boxplus F_2$ is projective.

We point out that we have been unable to determine whether the converse of corollary 4.2.1 holds. We can make the observation that a countable projective must be free as an abstract group; this is a corollary of Weinberg's result in [8] that every countable subgroup of a free l-group is group-theoretically free.

Corollaries 4.2.1 and 4.3.1 completely characterize those projective l-groups with finitely many disjoint elements. They are precisely the finite cardinal sums of copies of Z.

One important question still left open is the following: are all l-subgroups of free l-groups projective? Equivalently, is every l-subgroup of a free l-group a retract? Or is it perhaps the case that for each projective l-group P there is a free l-group F and a (projective) l-group Q such that $F = P \perp\!\!\!\perp^A Q$?

5. Odds and ends. We compile here some partial results and some facts about l-ideal of free products.

Let $\mathscr{L}^*$ stand for any of the varieties $\mathscr{L}$, $\mathscr{L}N$, $\mathscr{L}R$, $\mathscr{L}W$ or $\mathscr{L}A$. Recall that an element $a > 0$ in an l-group G is *basic* if $\{x \mid 0 \leqq x \leqq a\}$ is totally ordered. Our main result here is that if A and B are representable l-groups in $\mathscr{L}^*$ and B is not an o-group, then no element $0 < a \in A$ is basic in $A \perp\!\!\!\perp^* B$. We need a lemma, which by itself is rather curious. Recall corollary 2.3.1: it says that for $\mathscr{L}^* = \mathscr{L}A$ we have $0 < a \wedge b \in A \perp\!\!\!\perp^* B$ whenever $0 < a \in A$ and $0 < b \in B$. The same proof goes through for all the other varieties provided A and B are representable.

5.1. Lemma. *Let A be a representable l-group in $\mathscr{L}^*$ and $B \in \mathscr{L}^*$. Suppose $0 < a_i \in A$, $0 < b_i \in B$ $(i = 1, 2)$ with $b_1 \parallel b_2$. Then in $A \perp\!\!\!\perp^* B$ $a_1 \wedge b_1 \parallel a_2 \wedge b_2$.*

Proof. If $a_1 \wedge a_2 = 0$ then $(a_1 \wedge b_1) \wedge (a_2 \wedge b_2) = 0$ and neither is 0 so $a_1 \wedge b_1 \parallel a_2 \wedge b_2$. So assume $0 < a_1 \wedge a_2$ and let N be a minimal prime of A without $a_1 \wedge a_2$; then $a_i + N > 0$ for $i = 1, 2$. So we may suppose that A is an o-group and that $0 < a_i \in A$ $(i = 1, 2)$.

Let $K = A \overset{\rightarrow}{\times} B$ and $\phi : A \perp\!\!\!\perp^* B \to K$ be the canonical map; (we note that $K \in \mathscr{L}^*$). Then $(a_1 \wedge b_1)\varphi = a_1\phi \wedge b_1\phi = b_1\phi = b_1$, while $(a_2 \wedge b_2)\varphi = b_2$; therefore $(a_1 \wedge b_1)\phi \parallel (a_2 \wedge b_2)\phi$, and hence $a_1 \wedge b_1 \parallel a_2 \wedge b_2$.

5.2. Theorem. *Suppose A is a representable l-group in $\mathscr{L}^*$ and $B \in \mathscr{L}^*$ but not an o-group. Then no $0 < a \in A$ is basic in $A \perp\!\!\!\perp^* B$.*

Let $M(N)$ be an l-ideal of $A(B)$ and consider the natural map $\varrho(M, N) : A \perp\!\!\!\perp^* B \to A/M \perp\!\!\!\perp^* B/N$. Let $K(M, N)$ be the kernel of $\varrho(M, N)$.

5.3. Proposition. *With $A, B \in \mathscr{L}^*$ $K(M, N)$ is an l-ideal of $A \perp\!\!\!\perp^* B$ such that $K(M, N) \cap A = M$ and $K(M, N) \cap B = N$. Moreover $K(M, N)$ is the smallest l-ideal of $A \perp\!\!\!\perp^* B$ which meets A in M and B in N.*

1) *$K(M, N)$ is prime in $A \perp\!\!\!\perp^* B$ if and only if both M and N are prime in A and B respectively, and either $M = A$ or $N = B$.*

2) *If M is proper in A (or N is proper in B) then $K(M, N)$ is proper in $A \perp\!\!\!\perp^* B$.*

5.3.1. Corollary. With the same notation of 5.3, $K(M, N)$ is the l-ideal generated by M and N.

6. EXAMPLES

6.1. It is well known that if the abelian group A is a retract (in the category of abelian groups) of the group B then A is a direct summand of B. This is not true for the variety $\mathscr{L}A$. Let $A = Z$ and $B = Z \overset{\rightarrow}{\times} Z$; the projection of B onto the top component A is a retraction in $\mathscr{L}A$. But A cannot be a free factor of B since B is totally ordered.

6.2. (Bernau) Let $G = \{m + n\sqrt{2} \mid m, n \in Z\}$ and assign to G the usual archimedean total order. Now $G \perp\!\!\!\perp^A G$ is the free abelian l-group over $G \boxplus G$ (1.9);

Bernau showed in [2] that $G \amalg^A G$ is not archimedean. Thus the free A-product of archimedean l-groups need not be archimedean.

6.3. In [6] we defined a tensor product as follows: given two abelian l-groups A and B there is a (unique) pair $(A \otimes B, \tau)$, where $A \otimes B$ is an abelian l-group and τ is an l-bilinear map of $A \times B$ into $A \otimes B$ (l-bilinear $\equiv$ bilinear, and for each positive element in one component, the induced map in the other is lattice preserving), such that if $\phi : A \times B \to L$ is any l-bilinear map into an abelian l-group L there is a unique l-homomorphism $\phi^* : A \otimes B \to L$*). We proved in [6] that the functor $G \otimes (.)$ preserves cardinal sums. Here we show it does not preserve free products in general, and hence need not have an adjoint functor. Let $G = Z \boxplus Z$; if $G \otimes (.)$ preserves free A-products, then $G \otimes (Z \amalg^A Z) \simeq G \otimes Z \amalg^A G \otimes Z$. Since $G \otimes Z = G$ we should have $G \otimes (Z \amalg^A Z) \simeq (Z \boxplus Z) \amalg^A (Z \boxplus Z)$ which is the free abelian l-group on two generators. However

$$G \otimes (Z \amalg^A Z) = (Z \boxplus Z) \otimes (Z \amalg^A Z) \simeq (Z \amalg^A Z) \otimes (Z \boxplus Z) \simeq$$
$$\simeq (Z \amalg^A Z) \otimes Z \boxplus (Z \amalg^A Z) \otimes Z \simeq (Z \amalg^A Z) \boxplus (Z \amalg^A Z).$$

Since the free l-group on two generators is indecomposable this is a contradiction. For $G = Z \boxplus Z$ then $G \otimes (.)$ does not have an adjoint functor.

6.4. An infinite chain of varieties of normal valued l-groups, each of which intersects $\mathscr{L}R$ in $\mathscr{L}A$. Let $\mathscr{L}n$ be the variety of l-groups G satisfying $na + nb = bn + na$ for all $a, b \in G$. That each $\mathscr{L}n$ is a variety of normal valued l-groups is an easy consequence of Wolfenstein's theorem 3 in [10]. Evidently $\mathscr{L}1 = \mathscr{L}A$.

Next suppose A is representable l-group in $\mathscr{L}n$; we show A is necessarily abelian. If not we may suppose $a^b < a$, for some $a, b \in A$. But then $n(a^b) = (na)^b < na$, and so $na = (na)^{nb} < (na)^{(n-1)b} < \ldots < (na)^b < na$, which is abusrd. Therefore A is abelian, and it becomes clear that $\mathscr{L}R \cap \mathscr{L}n = \mathscr{L}A$.

We consider the chain $\mathscr{L}1 \subseteqq \mathscr{L}2 \subseteqq \mathscr{L}4 \subseteqq \ldots \subseteqq \mathscr{L}2^n \subseteqq \ldots$; we wish to show the containment is proper at each step. It will then follow that $\mathscr{L}2^n \subset \mathscr{L}N$.

We construct for each $k = 2^n$ and l-group G_k as follows: let $K_k = Z_0 \boxplus Z_1 \boxplus \ldots \ldots \boxplus Z_{k-1}$, with $Z_m = Z$ $(0 \leqq m \leqq k-1)$. Define a homomorphism r_k of Z into the l-automorphism grop of K_k by

$$(t_0, t_1, \ldots, t_{k-1})(jr_k) = (t_{0-j^*}, t_{1-j^*}, \ldots, t_{(k-1)-j^*}),$$

where $j^* \equiv j \bmod k$. Let $G_k = Z \times K_k$ and define the group operation by

$$(i; t_0, t_1, \ldots, t_{k-1}) + (j; u_0, u_1, \ldots, u_{k-1}) =$$
$$= (i + j; (t_0, t_1, \ldots, t_{k-1})(jr_k) + (u_0, u_1, \ldots, u_{k-1}));$$

*) Such that $\tau\varphi^* = \varphi$.

(the last sum of k-tuples is coordinatewise addition.) G_k becomes a group, and if we set $(i; t_0, t_1, \ldots, t_{k-1}) \geqq 0$ when $i > 0$, or $i = 0$ and each $t_m \geqq 0$ $(0 \leqq m \leqq k-1)$, then G_k becomes an l-group.

Claim: $G_k \in \mathscr{L}k \setminus \mathscr{L}(k/2)$. For let $k(i; t_0, t_1, \ldots, t_{k-1}) = (ki; t'_0, t'_1, \ldots, t'_{k-1})$ and $k(j; u_0, u_1, \ldots, u_{k-1}) = (kj; u'_0, u'_1, \ldots, u'_{k-1})$; (we really do not care about the t'_m and u'_m, as we shall see.) Now,

$$\begin{aligned}(ki + kj; t'_{0-(kj)*} + u'_0, \ldots, t'_{(k-1)-(kj)*} + u'_{(k-1)}) &= \\ = (kj + ki; t'_0 + u'_0, \ldots, t'_{(k-1)} + u'_{(k-1)}) &= \\ = (kj + ki; u'_{0-(ki)*} + t'_0, \ldots, u'_{(k-1)-(ki)*} + t'_{(k-1)}) &= \\ = k(j; u'_0, u'_1, \ldots, u'_{k-1}) + k(i; t'_0, t'_1, \ldots, t'_{k-1}) &.\end{aligned}$$

Hence $G_k \in \mathscr{L}k$.

On the other hand let $x = (0; 1, 0, 0, 0, \ldots, 0)$ and $y = (1; 0, 0, \ldots, 0)$. Then $(k/2)\, x = (0; (k/2), 0, 0, \ldots, 0)$ and $(k/2)\, y = (k/2; 0, 0, \ldots, 0)$; now

$$(k/2)\, x + (k/2)\, y = (k/2; 0, 0, \ldots, 0, k/2, 0, \ldots, 0),$$

where the non-zero entry is in the $(k/2)$-th coordinate. Yet

$$(k/2)\, y + (k/2)\, x = (k/2; k/2, 0, 0, \ldots, 0) \neq (k/2)\, x + (k/2)\, y,$$

proving $G_k \notin \mathscr{L}(k/2)$. We have therefore proved all our assertions about this example.

6.5. Using the proof of theorem 2.1 we can show that $G \amalg^N H$ is never in $\mathscr{L}n$ $(n = 1, 2, \ldots)$, when G and H are non-trivial l-groups and G is an o-group. This implies of course that the same is true when G is a (normal valued) l-group having at least one normal prime. If we knew that every l-group in $\mathscr{L}n$ has a normal prime we could conclude that $G \amalg^N H$ is not in $\mathscr{L}n$ whenever G and H are non-trivial (normal valued) l-groups.

On the other side of the spectrum is it true that the free product in $\mathscr{L}m$ always fails to be in $\mathscr{L}n$ when $\mathscr{L}n \subset \mathscr{L}m$? (The latter containment holds if and only if n is a proper divisor of m.) It can be shown, the reader is invited to try it, that the free product in $\mathscr{L}m$ of Z and $H \in \mathscr{L}m$ is outside $\mathscr{L}n$. Thus $G \amalg^m H$ fails to be in $\mathscr{L}n$ whenever $H \in \mathscr{L}m$ and G is an l-group having a factor isomorphic to Z.

6.6. We proved that each $\mathscr{L}n$ was a variety meeting $\mathscr{L}R$ in $\mathscr{L}A$. Consider now the variety of l-groups G satisfying $a + 2b = 2b + a$ for each pair $a, b \in G$. This variety must of course meet $\mathscr{L}R$ in $\mathscr{L}A$; we show it actually coincides with $\mathscr{L}A$. For suppose G is an l-group in the variety under discussion; let M be a convex l-subgroup of G and $0 < b \in M$; then $2(b^a) = (2b)^a = 2b \in M$. By convexity, since $b^a > 0$ we get $b^a \in M$. Every convex l-subgroup is normal in G, and in particular G is representable, thence abelian.

6.7. The variety $\mathscr{L}C$. The l-group G is said to have *commuting conjugates* if $a + a^g = a^g + a$ for all $a, g \in G$. The class of all such l-groups is a variety denoted by $\mathscr{L}C$.

i) **A weakly abelian l-group in $\mathscr{L}C$ which is non-abelian.** The example of 3.5 is such an l-group. For

$$(a, b, c) + (a, b, c)^{(x,y,z)} = (a, b, c) + (a, b, c + ay - bx) = \\ = (2a, 2b, 2c + ay - bx + ab),$$

while

$$(a, b, c)^{(x,y,z)} + (a, b, c) = (a, b, c + ay - bx) + (a, b, c) = \\ = (2a, 2b, 2c + ay - bx + ab).$$

Hence this l-group has commuting conjugates.

ii) **A weakly abelian l-group not in $\mathscr{L}C$.** Put a weakly abelian total order on a free group with two or more free generators; such a group is clearly not in $\mathscr{L}C$.

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