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A CHARACTERIZATION OF SEMILATTICES OF LEFT OR RIGHT GROUPS

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In this note we give a characterization of right regular or periodic semigroups, which are semilattices of left groups.

Let S be a semigroup. The mapping $\mathbf{U} : \exp S \to \exp S$ is said to be a *$\mathscr{C}$-closure operation* if the mapping $\mathbf{U}$ satisfies the following conditions:

i) $$\mathbf{U}(\emptyset) = \emptyset ;$$

ii) $$A \subset B \subset S \Rightarrow \mathbf{U}(A) \subset \mathbf{U}(B) ;$$

iii) $$A \subset \mathbf{U}(A) \quad \text{for each} \quad A \subset S ;$$

iv) $$\mathbf{U}(\mathbf{U}(A)) = \mathbf{U}(A) \quad \text{for each} \quad A \subset S .$$

For $x \in S$ we write simply $\mathbf{U}(x)$ instead of $\mathbf{U}(\{x\})$. The set of all $\mathscr{C}$-closure operations for a semigroup S will be denoted by $\mathscr{C}(S)$.

Let $\mathbf{U} \in \mathscr{C}(S)$. A subset A of S will be called *$\mathbf{U}$-closed* if $\mathbf{U}(A) = A$. Let $\mathscr{F}(\mathbf{U})$ denote the set of all $\mathbf{U}$-closed subsets of S.

We recall the following notion introduced in [1]. If $\mathbf{U} \in \mathscr{C}(S)$, $\mathbf{V} \in \mathscr{C}(S)$ we define $\mathbf{U}\ \varrho\ \mathbf{V}$ if and only if the following holds: For any $\mathbf{U}$-closed (non-empty) subset $A \subset S$ and any $\mathbf{V}$-closed (non-empty) subset $B \subset S$, we have

(1) $$A \cap B = AB .$$

Let $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$, then we define $\mathbf{U} \leqq \mathbf{V}$ if and only if $\mathbf{U}(A) \subset \mathbf{V}(A)$ for any subset $A \subset S$. The ordered set $\mathscr{C}(S)$ is a lattice $(\wedge, \vee)$. If $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$, then there holds:

(2) $$\mathbf{U} \leqq \mathbf{V} \Leftrightarrow \mathscr{F}(\mathbf{V}) \subset \mathscr{F}(\mathbf{U}) ;$$

(3) $$\mathscr{F}(\mathbf{U} \wedge \mathbf{V}) = \{A \cap B / A \in \mathscr{F}(\mathbf{U}),\ B \in \mathscr{F}(\mathbf{V})\} ;$$

(4) $$\mathscr{F}(\mathbf{U} \vee \mathbf{V}) = \mathscr{F}(\mathbf{U}) \cap \mathscr{F}(\mathbf{V}) .$$

(See [1].)

From (1) and (2) there follows:

Lemma 1. *Let* $\mathbf{U}_1, \mathbf{U}_2, \mathbf{V}_1, \mathbf{V}_2 \in \mathscr{C}(S)$ *and let* $\mathbf{U}_1 \leqq \mathbf{U}_2$, $\mathbf{V}_1 \leqq \mathbf{V}_2$. *If* $\mathbf{U}_1 \varrho \mathbf{V}_1$, *then* $\mathbf{U}_2 \varrho \mathbf{V}_2$.

Let $A \subset S$, $A \neq \emptyset$. Put $\mathbf{L}(A) = SA \cup A$ and $\mathbf{R}(A) = AS \cup A$. Finally $\mathbf{L}(\emptyset) = \emptyset = \mathbf{R}(\emptyset)$. It is clear that $\mathbf{L}, \mathbf{R} \in \mathscr{C}(S)$ and $\mathscr{F}(\mathbf{L})$ is the set of all left ideals of S (including $\emptyset$), $\mathscr{F}(\mathbf{R})$ is the set of all right ideals of S (including $\emptyset$). Put $\mathbf{M} = \mathbf{L} \vee \mathbf{R}$, $\mathbf{H} = \mathbf{L} \wedge \mathbf{R}$. Evidently $\mathbf{M}, \mathbf{H} \in \mathscr{C}(S)$. It follows from (3) and (4) that $\mathscr{F}(\mathbf{M})$ is the set of all two-sided ideals of S (including $\emptyset$) and $\mathscr{F}(\mathbf{H})$ is the set of all quasi-ideals of S (including $\emptyset$).

Lemma 2. *Let* $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$. *Then* $\mathbf{U} \varrho \mathbf{V}$ *if and only if* $\mathbf{R} \leqq \mathbf{U}$, $\mathbf{L} \leqq \mathbf{V}$ *and* $x \in \mathbf{U}(x)\, \mathbf{V}(x)$ *for every* $x \in S$.

Proof. See Theorem 9 [1].

A semigroup S is called *left* (*right*) *regular* if $x \in \mathbf{L}(x^2)$ ($x \in \mathbf{R}(x^2)$) for every x of S (see Lemma 3 [1]). A semigroup S is said to be *left* (*right*) *cancellative* if in S the left (right) cancellation law holds, that is $ax = ay$ ($xa = ya$) implies $x = y$ for all $a, x, y \in S$. A semigroup S is called *left* (*right*) *simple* if S does not contain a left (right) ideal different from S. A semigroup S is called a *left* (*right*) *group* if it is left (right) simple and right (left) cancellative.

Lemma 3. *The following conditions on a semigroup S are equivalent*:

1. *S is a semilattice of left groups*;
2. *S is a union of groups and* $\mathbf{R} \leqq \mathbf{L}$;
3. *S is a right regular and* $\mathbf{R} \leqq \mathbf{L}$.

Proof. $1 \Leftrightarrow 2$. This follows from Theorem 11 [2].

$2 \Rightarrow 3$. Evident.

$3 \Rightarrow 2$. Let S be right regular and $\mathbf{R} \leqq \mathbf{L}$. We show that S is left regular, which implies (see Theorem 8 [2]) that S is a union of groups. Let x be an arbitrary element of S. Then $x \in \mathbf{R}(x^2) \subset \mathbf{L}(x^2)$. Hence, S is left regular.

From Remark 1 [2] we obtain the following:

Lemma 4. *The following conditions on a periodic semigroup S are equivalent*:

1. *S is a union of groups*;
2. *S is right regular*;
3. *S is left regular.*

Theorem 1. *The following conditions on a right regular or periodic semigroup S are equivalent:*

1. $\boldsymbol{H} \varrho \boldsymbol{M}$;
2. $\boldsymbol{H} \varrho \boldsymbol{L}$;
3. $\boldsymbol{L} \varrho \boldsymbol{L}$;
4. $\boldsymbol{L} \varrho \boldsymbol{M}$;
5. *S is a semilattice of left groups.*

Proof. $2 \Rightarrow 3 \Rightarrow 4$. This follows from Lemma 1.

$4 \Rightarrow 5$. Let $\boldsymbol{L} \varrho \boldsymbol{M}$ and let S be a right regular semigroup. By Lemma 2, we have $\boldsymbol{R} \leqq \boldsymbol{L}$. Hence, by Lemma 3, S is a semilattice of left groups.

Let $\boldsymbol{L} \varrho \boldsymbol{M}$ and let S be a periodic semigroup. It follows from Theorem 11 [1] that S is left regular and $\boldsymbol{R} \leqq \boldsymbol{L}$. According to Lemma 4 and Lemma 3, S is a semilattice of left groups.

$5 \Rightarrow 1$. Let S be a semilattice of left groups. By Lemma 3, S is a union of groups and $\boldsymbol{R} \leqq \boldsymbol{L}$. Since S is regular, by Theorem 10 [1], we have $\boldsymbol{R} \varrho \boldsymbol{L}$. This implies that $\boldsymbol{H} \varrho \boldsymbol{M}$.

$1 \Rightarrow 2$. Let $\boldsymbol{H} \varrho \boldsymbol{M}$. By Lemma 2 we have $\boldsymbol{R} \leqq \boldsymbol{H} \leqq \boldsymbol{L}$ and so $\boldsymbol{M} = \boldsymbol{L}$. This implies $\boldsymbol{H} \varrho \boldsymbol{L}$.

Dually we have the following:

Theorem 2. *The following conditions on a left regular or periodic semigroup S are equivalent:*

1. $\boldsymbol{M} \varrho \boldsymbol{H}$;
2. $\boldsymbol{R} \varrho \boldsymbol{H}$;
3. $\boldsymbol{R} \varrho \boldsymbol{R}$;
4. $\boldsymbol{M} \varrho \boldsymbol{R}$;
5. *S is a semilattice of right groups.*

References

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[2] *B. Pondělíček:* A certain equivalence on a semigroup, Czechoslovak Math. J. *21* (*96*), (1971), 109–117.

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