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ON SUBMULTIPLICATIVE NONNEGATIVE FUNCTIONALS ON LINEAR MAPS OF LINEAR FINITEDIMENSIONAL NORMED SPACES

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It is shown that a certain inequality involving submultiplicative functionals and norms of linear maps cannot be strengthened by diminishing a certain constant.

0. For $m = 1, 2, 3, \ldots$ define

(0,1) $$g(m) = \sup \{\det (\alpha_{i,j}) \mid \alpha_{i,j} \in R^1, \ |\alpha_{i,j}| \leqq 1, \ i, j = 1, 2, \ldots, m\} .$$

Throughout this paper $(Y, \varphi), (Y_i, \varphi_i)$, $i = 1, 2, 3, \ldots$ will denote a linear finite-dimensional normed space (i.e. Y is a linear space and $\varphi(y)$ is the norm of $y \in Y$). Let $A : Y_1 \to Y_2$ be linear and assume that $0 < \dim Y_1 = \dim Y_2$. In [1], section 1 there was introduced a map Dt, which assigned to any such triple $((Y_1, \varphi_1), (Y_2, \varphi_2), A)$ a nonnegative real. (See Note 0,1.) This real was in [1] denoted – for sake of brevity – by DtA, in this paper it will be denoted by $Dt(\varphi_1, \varphi_2, A)$. It was shown in [1] that the map Dt has the following properties:

(0,2) Let $\varphi_2(Ay) = \varphi_1(y)$ for $y \in Y_1$. Then $Dt(\varphi_1, \varphi_2, A) = 1$.

(0,3) Let $\dim Y_1 = \dim Y_2 = \dim Y_3$, let $A : Y_1 \to Y_2$, $B : Y_2 \to Y_3$ be linear. Then $Dt(\varphi_1, \varphi_3, B \circ A) = Dt(\varphi_1, \varphi_2, A) \,.\, Dt(\varphi_2, \varphi_3, B)$.

(0,4) Let $V_1 \supset V_2 \supset \ldots \supset V_m$ be linear subspaces of Y_1,

$$\dim V_j = m - j + 1 \,, \quad \varkappa_j \in R^1 \,, \quad j = 1, 2, \ldots, m$$

and assume that $\varphi_2(Ay) \leqq \varkappa_j \, \varphi_1(y)$ for $y \in V_j$. Then

$$Dt(\varphi_1, \varphi_2, A) \leqq g(m) \, \varkappa_1 \,.\, \varkappa_2 \ldots \varkappa_m \,.$$

(If $A|_{V_j}$ is the restriction of A to V_j and

$$\|A|_{V_j}\| = \sup \{\varphi_2(Ay) \mid y \in V_j, \ \varphi_1(y) \leqq 1\} \,,$$

then the last inequality may be rewritten as

$$Dt(\varphi_1, \varphi_2, A) \leqq g(m) \, \|A|_{V_1}\| \,.\, \|A|_{V_2}\| \ldots \|A|_{V_m}\| \,.)$$

(0,5) Let $(Y_1, \varphi_1), (Y_2, \varphi_2)$ be Euclidean spaces (i.e. let there exist bilinear forms ψ_i on Y_i such that $\varphi_i^2(y) = \psi_i(y, y)$ for $y \in Y_i$, $i = 1, 2$). Then

$$Dt(\varphi_1, \varphi_2, A) = |\det A|$$

($\det A = \det(\psi_2(Ae_i, f_j))$, e_i, $i = 1, 2, \ldots, m$ being an orthonormal basis in Y_1 and f_j, $j = 1, 2, \ldots, m$ being an orthonormal basis in Y_2).

Write $Dt_m(\varphi_1, \varphi_2, A)$ instead of $Dt(\varphi_1, \varphi_2, A)$ to emphasize that $\dim Y_1 = m = \dim Y_2$ so that the map Dt_m is the restriction of Dt to such triples $((Y_1, \varphi_1), (Y_2, \varphi_2), A)$ that $\dim Y_1 = m = \dim Y_2$. In [1] the map Dt was used to derive some properties of systems of operator equations, which in [2] were applied to linear functional differential equations. By the same method as in [1] stronger results on systems of operator equations would be obtained, if – for some m – the map Dt_m could be replaced by a map ϑ satisfying (1,1), (1,2), (1,3) with $h < g(m)$. The aim of this paper is to show that no such map ϑ exists (cf. Theorem 1,1).

Note 0,1. Let there be recalled the definition of Dt. Let $\dim Y_1 = m = \dim Y_2$ and let $\hat{Y}_i$, $i = 1, 2$ be the space of m-linear skew symmetric forms (exterior m-forms) on Y_i. Introduce the norm $\hat{\varphi}_i$ on $\hat{Y}_i$ by

$$\hat{\varphi}_i(\eta) = \sup \{\eta(y_1, \ldots, y_m) \mid y_j \in Y_i,\ \varphi_i(y_j) \leqq 1,\ j = 1, 2, \ldots, m\}.$$

Define $\hat{A} : \hat{Y}_2 \to \hat{Y}_1$ by $(\hat{A}\eta)(y_1, \ldots, y_m) = \eta(Ay_1, \ldots, Ay_m)$ and Dt by $Dt(\varphi_1, \varphi_2, A) = \sup \{\hat{\varphi}_1(\hat{A}\eta) \mid \eta \in \hat{Y}_2,\ \hat{\varphi}_2(\eta) \leqq 1\}$.

As $\hat{Y}_i$ is a one-dimensional linear space, it may be seen that

$$Dt(\varphi_1, \varphi_2, A) = \hat{\varphi}_1(\hat{A}\eta), \quad \text{if} \quad \eta \in \hat{Y}_2, \quad \hat{\varphi}_2(\eta) = 1. \tag{0,6}$$

Note 0,2. A map ζ, which has analogous properties as Dt, may be introduced as follows: Let $\dim Y_1 = m = \dim Y_2$. By F. John, [4] there exists to (Y_i, φ_i), $i = 1, 2$ a positive definite quadratic form $\tilde{\psi}_i$ on Y_i such that the ellipsoid $U_i = \{y \in Y_i \mid \tilde{\psi}_i(y) \leqq 1\}$ contains the unit ball $K_i = \{y \in Y_i \mid \varphi_i(y) \leqq 1\}$ and has the least possible volume; moreover, F. John proved in [4] that

$$\tilde{\psi}_i(y) \leqq \varphi_i^2(y) \leqq m^{m/2}\, \psi_i(y) \quad \text{for} \quad y \in Y_i. \tag{0,7}$$

$\tilde{\psi}_i$ is unique (cf. Note 1,3). Let ψ_i be the corresponding bilinear form. Let e_j, $j = 1, 2, \ldots, m$ be an orthonormal basis in Y_1 and let f_j, $j = 1, 2, \ldots, m$ be an orthonormal basis in Y_2. Define $\zeta(\varphi_1, \varphi_2, A) = |\det(\psi_2(Ae_i, f_j))|$. Obviously $\zeta(\varphi_1, \varphi_2, A)$ is independent of the choice of bases $\{e_i\}$, $\{f_j\}$ and it may be verified that (0,2)–(0,5) are fulfilled, if Dt is replaced by ζ and $g(m)$ in (0,4) is replaced by $m^{m/2}$.

By the Hadamard inequality $g(m) \leqq m^{m/2}$. Relations between the maps Dt and ζ are discussed in Notes 1,6 and 1,7.

1. Definition 1,1. For $m = 1, 2, 3, \ldots$ and $h > 0$ let $\Theta(m, h)$ be the set of such maps ϑ, which assign to any triple $((Y_1, \varphi_1), (Y_2, \varphi_2), A)$, $\dim Y_1 = m = \dim Y_2$,

$A : Y_1 \to Y_2$ being linear, a nonnegative real. This real will be denoted – for sake of brevity – by $\vartheta(\varphi_1, \varphi_2, A)$. Moreover, it is assumed that any $\vartheta \in \Theta(m, h)$ fulfils the following conditions:

(1,1) Let $Y_1 = Y_2$, $\varphi_1 = \varphi_2$, $Iy = y$ for $y \in Y_1$. Then $\vartheta(\varphi_1, \varphi_2, I) \geqq 1$.

(1,2) Let $A : Y_1 \to Y_2$, $B : Y_2 \to Y_3$ be linear. Then

$$\vartheta(\varphi_1, \varphi_3, B \circ A) \leqq \vartheta(\varphi_1, \varphi_2, A) \,.\, \vartheta(\varphi_2, \varphi_3, B)\,.$$

(1,3) Let $V_1 \supset V_2 \supset \ldots \supset V_m$ be linear subspaces of Y_1, $\dim V_j = m - j + 1$, $j = 1, 2, \ldots, m$. Then

$$\vartheta(\varphi_1, \varphi_2, A) \leqq h\|A|_{V_1}\| \,.\, \|A|_{V_2}\| \,.\, \ldots \,.\, \|A|_{V_m}\|$$

($\|A|_{V_j}\|$ being defined in (0,4)). If $\dim Y = m$, let $\Theta(m, h, (Y, \varphi))$ be the set of such maps σ that assign to any linear map $A : Y \to Y$ a nonnegative real $\sigma(A)$ so that (1,1), (1,2) and (1,3) are fulfilled for $Y_1 = Y_2 = Y$, $\varphi_1 = \varphi_2 = \varphi$, $\vartheta(\varphi, \varphi, A) = \sigma(A)$.

Theorem 1,1. *If $h < g(m)$, then $\Theta(m, h) = \emptyset$.*

For $x = (x_1, \ldots, x_m) \in R^m$ put $\tilde{\varphi}(x) = \sum_{j=1}^{m} |x_j|$; for $\vartheta \in \Theta(m, h)$ denote by $\pi(\vartheta)$ the restriction of ϑ to the set of triples $((R^m, \tilde{\varphi}), (R^m, \tilde{\varphi}), A)$ with $A : R^m \to R^m$ linear. Obviously $\pi(\vartheta) \in \Theta(m, h, (R^m, \tilde{\varphi}))$ for every $\vartheta \in \Theta(m, h)$. Therefore Theorem 1,1 is a consequence of

Theorem 1,2. *If $h < g(m)$, then $\Theta(m, h, (R^m, \tilde{\varphi})) = \emptyset$.*

Theorem 1,2 follows directly from Theorems 1,3 and 1,4.

Theorem 1,3. *If $\sigma \in \Theta(m, h, (Y, \varphi))$, then $\sigma(A) \geqq |\det A|$ for any linear map $A : Y \to Y$.*

Theorem 1,4. *Let m be a positive integer, $\tilde{\varphi}(x) = \sum_{j=1}^{m} |x_j|$ for $x = (x_1, \ldots, x_m) \in R^m$. If $A : R^m \to R^m$ is linear and V is a linear subspace of R^m, define $\|A|_V\| = \sup\{\tilde{\varphi}(Ax) \mid x \in V, \tilde{\varphi}(x) \leqq 1\}$. Let $h \in R^1$ be such that*

$$|\det A| \leqq h \,.\, \|A|_{V_1}\| \,.\, \|A|_{V_2}\| \ldots \|A|_{V_m}\| \tag{1,4}$$

for any linear map $A : R^m \to R^m$ and for any chain $V_1 \supset V_2 \supset \ldots \supset V_m$ of linear subspaces of R^m, $\dim V_j = m - j + 1, j = 1, 2, \ldots, m$. Then

$$h \geqq g(m)\,. \tag{1,5}$$

Proof of Theorem 1,3: Assume that there exists such a linear map $A : Y \to Y$ that $\sigma(A) < |\det A|$. A is nonsingular and $\sigma(A) = \alpha|\det A|$, $0 < \alpha < 1$ $(\sigma(A) > 0$

for A nonsingular by (1,1) and (1,2)). Find a positive integer k such that $\alpha^k < h^{-1} . 2^{-m-2}$. By (1,1) and (1,2) again,

(1,6) $$\sigma(A^k) < h^{-1} . 2^{-m-2}|\det A^k| .$$

If A^{-k} is represented by a matrix in Jordan canonical form, it is seen readily that there exists such a linear map $B : Y \to Y$ that

(1,7) $$\|A^{-k}B^{-1}\| < 2 ,$$

(1,8) $$|\det A^{-k}B^{-1}| > \tfrac{1}{2} ,$$

(1,9) All characteristic numbers λ_j, $j = 1, 2, \ldots, m$ of B are distinct and can be ordered in such a way that $|\lambda_1| \geqq |\lambda_2| \geqq \ldots \geqq |\lambda_m|$ and if $|\lambda_j| = |\lambda_{j+1}|$, then $Im\lambda_j \neq 0$, $\lambda_j = \bar{\lambda}_{j+1}$ (i.e. if λ_j is real, then $|\lambda_k| > |\lambda_j|$ for $k < j$ and $|\lambda_j| > |\lambda_k|$ for $k > j$; if $Im\lambda_j \neq 0$, then $\lambda_j = \bar{\lambda}_s$ for $s = j - 1$ or for $s = j + 1$ and $|\lambda_k| > |\lambda_j|$ for $k < \min(j, s)$, $|\lambda_k| < |\lambda_j|$ for $k > \max(j, s)$). By (1,3) and (1,7) $\sigma(A^{-k}B^{-1}) < h . 2^m$, hence by (1,2), (1,6) and (1,8) $\sigma(B^{-1}) \leqq \sigma(A^k)\, \sigma(A^{-k}B^{-1}) < 2^{-2}|\det A^k| = 2^{-2}|\det B^{-1}| . |\det BA^k| < 2^{-1}|\det B^{-1}|$ and $\sigma(B^{-k}) < 2^{-k}|\det B^{-k}|$ for $k = 1, 2, 3, \ldots$ By (1,1) and (1,2)

(1,10) $$\sigma(B^k) > 2^k|\det B^k| , \quad k = 1, 2, 3, \ldots$$

Find a basis $u_1, \ldots, u_m$ in Y such that $\varphi(u_j) = 1$, $j = 1, 2, \ldots, m$ and

(1,11) $$\begin{aligned} Bu_j &= \lambda_j u_j , \quad \text{if } \lambda_j \text{ is real} , \\ Bu_j &= |\lambda_j| \left(u_j \cos \mu_j + u_{j+1} \sin \mu_j\right) , \\ Bu_{j+1} &= |\lambda_j| \left(-u_j \sin \mu_j + u_{j+1} \cos \mu_j\right) , \end{aligned}$$

if $\lambda_j = \bar{\lambda}_{j+1}$, $\mu_j = \operatorname{Arg} \lambda_j$ (i.e. the matrix of B is "real-canonical").

It is seen readily that

(1,12) $$|\det B^k| = \prod_{j=1}^{m} |\lambda_j|^k .$$

Obviously there exists a $c > 0$ such that

(1,13) $$\text{if } x \in Y, \quad \varphi(x) \leqq 1 , \quad x = \sum_{j=1}^{m} \xi_j u_j , \quad \text{then } |\xi_j| \leqq c , \quad j = 1, 2, \ldots, m .$$

Let V_j be the space spanned by $u_j, u_{j+1}, \ldots, u_m$. Let $y \in V_j$, $y = \sum_{s=j}^{m} \beta_s u_s$, $\varphi(y) \leqq 1$. Then $B^k y = \sum_{s=j}^{m} \gamma_s u_s$ where

(1,14) $$\begin{aligned} \gamma_s &= \lambda_s^k \beta_s \quad \text{if } \lambda_s \text{ is real} , \\ \gamma_s &= |\lambda_s|^k \left(\beta_s \cos k\mu_s - \beta_{s+1} \sin k\mu_s\right) , \\ \gamma_{s+1} &= |\lambda_s|^k \left(\beta_s \sin k\mu_s + \beta_{s+1} \cos k\mu_s\right) \\ &\text{if } \lambda_s = \bar{\lambda}_{s+1} , \quad k = 1, 2, 3, \ldots \end{aligned}$$

(1,13) and (1,14) imply that

$$(1,15) \qquad |\gamma_s| \leqq |\lambda_s|^k \, 2c \quad \text{for} \quad s = j, j+1, \ldots, m\,.$$

It follows from (1,15), (1,11) and (1,9) that

$$\varphi(B^k y) \leqq \sum_{s=j}^{m} |\gamma_s| \leqq 2c \sum_{s=j}^{m} |\lambda_s|^k \leqq |\lambda_j|^k \, 4cm$$

so that

$$(1,16) \qquad \|B^k|_{V_j}\| \leqq |\lambda_j|^k \, 4cm\,.$$

By (1,3) and (1,16)

$$(1,17) \qquad \sigma(B^k) \leqq h(4c)^m \, m^m \prod_{j=1}^{m} |\lambda_j|^k\,.$$

Hence (1,10) and (1,17) cannot hold simultaneously for sufficiently large k (cf. (1,12)) which makes the proof of Theorem 1,3 complete.

Proof of Theorem 1,4: Choose $\varepsilon > 0$ and find reals $d_1, d_2, \ldots, d_m$ such that $0 < d_m < d_{m-1} < \ldots < d_1$ and

$$(1,18) \qquad d_m + \ldots + d_j \leqq d_j(1 + \varepsilon) \quad \text{for} \quad j = m, m-1, \ldots, 1\,.$$

Define

$$K = \{x \in R^m \mid \tilde{\varphi}(x) \leqq 1\}\,,$$
$$L = \{x \in R^m \mid |x_j| \leqq m^{-1/2},\ j = 1, 2, \ldots, m\}\,,$$
$$M = \{x \in R^m \mid |x_j| \leqq d_j m^{-1/2},\ j = 1, 2, \ldots, m\}\,.$$

It is easy to see that there exists such a matrix $(b_{i,j})$ that $b_{i,j} \in R^1$, $|b_{i,j}| = 1$ for $i, j = 1, 2, \ldots, m$ and $\det(b_{i,j}) = g(m)$ (cf. (0,1)). Let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on the j-th place, $j = 1, 2, \ldots, m$. Let $B : R^m \to R^m$ be linear, B being represented by the matrix $(b_{i,j}/m^{1/2})$ with respect to the basis $\{e_j\}$ and let $D : R^m \to R^m$ be linear and let D be represented by $\operatorname{diag}(d_j)$ (diagonal matrix). It is easy to see that

$$(1,19) \qquad \det D \circ B = \det D \,.\, \det B = d_1 \,.\, d_2 \,.\, \ldots \,.\, d_m \,.\, g(m) \,.\, m^{-m/2}\,,$$
$$B(K) \subset L, \quad D \circ B(K) \subset D(L) = M\,.$$

Define $W_1 = R^m$, $W_j = \{x = (x_1, \ldots, x_m) \in R^m \mid x_1 = \ldots = x_{j-1} = 0\}$, $j = 2, 3, \ldots$ $\ldots, m$, $V_j = B^{-1}(W_j)$, $\varrho_j = \inf\{\lambda \in R^1 \mid \lambda \geqq 0,\ \lambda K \supset M \cap W_j\}$, $j = 1, 2, \ldots, m$. Obviously $D \circ B(K \cap V_j) = D(W_j \cap B(K)) \subset D(W_j \cap L) = W_j \cap M$, $\|D \circ B|_{V_j}\| = \inf\{\lambda \in R^1 \mid \lambda \geqq 0,\ \lambda K \supset D \circ B(K \cap V_j)\} \leqq \xi_j = (d_j + \ldots + d_m)\, m^{-1/2}$. Hence by (1,19), (1,4) and (1,18)

$$d_1 \,.\, d_2 \,.\, \ldots \,.\, d_m \,.\, g(m) \,.\, m^{-m/2} = \det D \circ B \leqq h \,.\, \prod_{j=1}^{m} \|D \circ B|_{V_j}\| \leqq$$
$$\leqq h \prod_{j=1}^{m} [(d_j + \ldots + d_m)\, m^{-1/2}] \leqq h \,.\, d_1 \,.\, d_2 \,.\, \ldots \,.\, d_m (1 + \varepsilon)^m \, m^{m/2}$$

and (1,5) holds, as $\varepsilon > 0$ is arbitrary.

Note 1,1. Let $\sigma \in \Theta(m, h, (Y, \varphi))$ fulfil

$$\sigma(B \circ A) = \sigma(B)\,\sigma(A) \tag{1,20}$$

for any linear maps $A : Y \to Y$, $B : Y \to Y$. Then $\sigma(A) = |\det A|$.

To show it observe that (1,1), (1,2) and (1,20) imply that $\sigma(I) = 1$ if $Iy = y$ for $y \in Y$. If $A : Y \to Y$ is linear and singular, then $|\det A| = 0 = \sigma(A)$ by (1,3). If $A : Y \to Y$ is linear and nonsingular, then by Theorem 1,3 $\sigma(A) \geqq |\det A| > 0$ and $\sigma(A^{-1}) \geqq |\det A^{-1}| > 0$. $\sigma(A) > |\det A|$ would imply that $1 = \sigma(I) = \sigma(A)\,\sigma(A^{-1}) > |\det A| \,.\, |\det A^{-1}| = 1$.

Note 1,2. Let $\tilde{\psi}_i$, $i = 1, 2$ be positive definite quadratic forms on R^m, $\tilde{\psi}_1 \neq \tilde{\psi}_2$ and for $\lambda \in \langle 0, 1\rangle$ define

$$U_\lambda = \{x \in R^m \mid \lambda\,\tilde{\psi}_1(x) + (1 - \lambda)\,\tilde{\psi}_2(x) \leqq 1\}$$

and assume that $\operatorname{vol} U_0 = \operatorname{vol} U_1 (\operatorname{vol} U_\lambda = \int_{U_\lambda} \mathrm{d}x_1 \ldots \mathrm{d}x_m)$. Then $\operatorname{vol} U_\lambda < \operatorname{vol} U_0$ for $\lambda \in (0, 1)$.

Let us show it. In a suitable coordinate system both $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are represented by diagonal matrices so that without loss of generality it may be assumed that

$$\tilde{\psi}_1(x) = \sum_{i=1}^m \alpha_i x_i^2\,, \quad \tilde{\psi}_2(x) = \sum_{i=1}^m \beta_i x_i^2\,, \quad \alpha_i > 0\,, \quad \beta_i > 0\,, \quad i = 1, 2, \ldots, m\,,$$
$$(\alpha_1, \ldots, \alpha_m) \neq (\beta_1, \ldots, \beta_m)\,.$$

For $\lambda \in \langle 0, 1\rangle$ $\operatorname{vol} U_\lambda = \omega \prod_{i=1}^m (\alpha_i + \lambda(\beta_i - \alpha_i))^{-1/2}$, ω being a suitable positive constant and

$$\prod_{i=1}^m \alpha_i = \prod_{i=1}^m \beta_i\,.$$

Put $p(\lambda) = \prod_{i=1}^m (\alpha_i + \lambda(\beta_i - \alpha_i))$. The degree r of the polynomial p is equal to the number of i-s such that $\alpha_i \neq \beta_i$ and p has r real roots. Hence $\mathrm{d}p/\mathrm{d}\lambda$ has $r - 1$ real roots and there is just one root of $\mathrm{d}p/\mathrm{d}\lambda$ between the smallest positive root of p (which is greater than 1) and the largest negative root of p and therefore $p(\lambda) > p(0)$ for $\lambda \in (0, 1)$.

Note 1,3. Let $S \subset R^m$ be bounded. Then there exists an ellipsoid $U \subset R^m$ such that $U \supset S$ and U has the least possible volume (the proof is based on the compactness of a suitable set of ellipsoids, cf. [4]). U is unique by Note 1,2.

Note 1,4. Let $K = \{x = (x_1, \ldots, x_m) \in R^m \mid |x_j| \leqq m^{-1/2},\ j = 1, 2, \ldots, m\}$, $U = \{x = (x_1, \ldots, x_m) \in R^m \mid \sum_{i=1}^m x_j^2 \leqq 1\}$. Then U is the ellipsoid of the least volume containing K.

To show it, assume that for some symmetric positive definite matrix $(\gamma_{i,j})$ $U_1 = \{x \in R^m \mid \sum_{i,j} \gamma_{i,j} x_i x_j \leqq 1\}$ is the ellipsoid of the least volume containing K. If $\gamma_{k,l} \neq 0$ for some $k, l, k \neq l$, then define $\beta_{i,j} = \gamma_{i,j}$ if $i \neq k \neq j$ and if $i = k = j$, $\beta_{i,j} = -\gamma_{i,j}$ otherwise. It is easy to see that $U_2 = \{x \in R^m \mid \sum_{i,j} \beta_{i,j} x_i x_j \leqq 1\}$ contains K, $\operatorname{vol} U_2 = \operatorname{vol} U_1$ and U_1 cannot be the ellipsoid of the least volume containing K by Note 1,3. Similarly U_1 cannot be the ellipsoid of the least volume containing K unless $\gamma_{11} = \gamma_{22} = \ldots = \gamma_{m,m}$.

Note 1,5. Let $\vartheta \in \Theta(m, h)$ fulfil $\vartheta(\varphi_1, \varphi_3, B \circ A) = \vartheta(\varphi_1, \varphi_2, A)\, \vartheta(\varphi_2, \varphi_3, B)$ for any linear maps $A : Y_1 \to Y_2$, $B = Y_2 \to Y_3$, $\dim Y_i = m$, $i = 1, 2, 3$. Let Y be a linear space, $\dim Y = m$, let φ_4, φ_5 be norms on Y and let $I : Y \to Y$ be the identity map. Then

$$(1,21) \qquad \vartheta(\varphi_4, \varphi_5, A) = \vartheta(\varphi_4, \varphi_5, I)\, |\det A|\,, \quad A : Y \to Y \quad \text{linear}\,.$$

This follows by Note 1,1.

Especially let $Y = R^m$, $K_4 = \{x \in R^m \mid \varphi_4(x) \leqq 1\}$, $\varphi_5(x) = (\sum_{i=1}^m x_i^2)^{1/2}$. Then

$$(1,22) \quad Dt(\varphi_4, \varphi_5, A) = Dt(\varphi_4, \varphi_5, I)\, |\det A|\,, \quad A : Y \to Y \quad \text{linear},$$
$$(1,23) \quad Dt(\varphi_4, \varphi_5, I) = \sup \{\det (y_j^{(i)}) \mid y^{(i)} = (y_j^{(i)}, \ldots, y_m^{(i)}) \in K_4, \quad i = 1, 2, \ldots, m\}\,.$$

To obtain (1,23), define $\eta \in \hat{Y}$ by

$$\eta(y^{(1)}, \ldots, y^{(m)}) = \det (y_j^{(i)})\,,$$
$$\hat{\varphi}_5(\eta) = \sup \{\eta(y^{(1)}, \ldots, y^{(m)}) \mid y^{(i)} \in Y,\ \varphi_5(y^{(i)}) \leqq 1,\ i = 1, 2, \ldots, m\}\,.$$

By Hadamard inequality $|\det (y_j^{(i)})| \leqq \prod_{i=1}^m (\sum_{j=1}^m (y_j^{(i)})^2)^{1/2} = \prod_{i=1}^m \varphi_5(y^{(i)})$ so that $\hat{\varphi}_5(\eta) = 1$; by (0,6) $Dt(\varphi_4, \varphi_5, I) = \hat{\varphi}_4(\eta)$ and (1,23) holds.

Note 1,6. Let $\varphi_4(x) = m^{-1/2} \max |x_i|$, $\varphi_5(x) = (\sum_{i=1}^m x_i^2)^{1/2}$ for $x = (x_1, \ldots, x_m) \in R^m$. Then

$$(1,24) \qquad \zeta(\varphi_4, \varphi_5, A) = |\det A|\,, \quad A : R^m \to R^m \quad \text{linear}\,,$$
$$(1,25) \qquad Dt(\varphi_4, \varphi_5, A) = g(m)\, m^{-m/2} |\det A|\,, \quad A : R^m \to R^m \quad \text{linear}\,.$$

(1,24) follows directly from the definition of ζ (see Note 0,2) and Note 1,4. (1,25) is a consequence of (1,22) (1,23) and (0,1). For the properties of g see [3], Chapter 14 or [1], Note 1,2.

Note 1,7. Let $y^{(r)} = (y_1^{(r)}, y_2^{(r)})$, $r = 1, 2, \ldots, 6$ be the vertices of the regular sixangle K_6 in R^2, $y^{(1)} = (0, 1)$, $y^{(2)} = (\frac{1}{2}, \frac{1}{2}\sqrt{3})$, $y^{(3)} = (-\frac{1}{2}, \frac{1}{2}\sqrt{3})$, $y^{(4)} = -y^{(1)}$, $y^{(5)} = -y^{(2)}$, $y^{(6)} = -y^{(3)}$. Let φ_6 be such a norm on R^2 that $\varphi_6(x) \leqq 1$ iff $x \in K_6$.

Then

$$(1,26)\qquad \zeta(\varphi_6, \varphi_5, A) = |\det A| , \qquad A : R^2 \to R^2 \quad \text{linear} ,$$

$$(1,27)\qquad Dt(\varphi_6, \varphi_5, A) = \tfrac{1}{2} \sqrt{(3)}\, |\det A| , \quad A : R^2 \to R^2 \quad \text{linear} .$$

(1,26) follows from the definition of ζ and from the fact that $U = \{x \in R^2 | x_1^2 + x_2^2 \leqq 1\}$ is the two-dimensional ellipsoid of the least area containing points $y^{(i)}$, $i = 1, 2, \ldots, 6$. This can be shown in an elementary way or from conditions (19a)–(19d) in [4]. (Conditions (19a)–(19d) of [4] are satisfied for $y_i^r = y_i^{(r)}$, $\lambda_0 = 3$, $\lambda_r = 1$, $r = 1, 2, \ldots, 6$, $s = 6$. It can be concluded in quite the same manner as in [4] that the area of any two-dimensional ellipsoid containing the points $y^{(r)}$, $r = 1, 2, \ldots, 6$ is at least equal to π; it does not matter that $6 = s > \frac{1}{2}m(m + 3) = 5$.)

(1,27) is a consequence of (1,22) and (1,23).

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