Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 22 (1972), No. 3, 393–422

Persistent URL: http://dml.cz/dmlcz/101110

Czechoslovak Mathematical Journal, 22 (97) 1972, Praha

CONDITIONS FOR MAXIMAL LOCAL DIFFUSIONS IN MULTI-DIMENSIONAL CASE

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(Received February 10, 1971)

Let an Îto equation and a region be given. A class of all Îto equations which have "smaller" diffusion coefficients than the given equation is considered. If moreover an initial value is given, then the probabilities that the solutions of the individual equations of the class leave the region can be compared. Conditions on coefficients of the given equation are established in order that the probability should be the greatest for the solution of the given equation. These conditions do not depend on the initial values.

The precise definitions connected with the problem (i.e. of maximal and strongly maximal matrix functions) were formulated for the first time in [1] (Def. 5, 6) and [2] (Def. 1). For the convenience of the reader they are reformulated here as Definitions 1, 2 (in the sense of article [2]). In paper [1], necessary and sufficient conditions for maximality and strong maximality of a matrix function are formulated. (These Theorems are reformulated here as Theorems 2 and 3.)

Since the conditions include a requirement that a solution of a parabolic equation is (for example) convex in spatial variables, more explicit conditions were derived in the one-dimensional case. Only the coefficients of the given Îto equation (or the corresponding parabolic equation) occur in these more explicit conditions. The purpose of this paper is to derive explicit conditions in the multi-dimensional case.

Theorems 2 and 3 imply that the investigation of maximality and strong maximality of a matrix function is equivalent to the investigation of the corresponding partial equation (1,2). Therefore the theorems of this paper will be mostly formulated for parabolic equations of several variables and only the remarks will refer to the significance of the theorems for our problem.

The method of the article is quite different from that used in [2]. With respect to Theorem 2 it is necessary to show that $\partial^2 u/\partial x_i^2 \geqq 0$ where $u(t, x)$ is the bounded solution of (1,2) fulfilling (1,3), (1,4). First it is shown that $\partial^2 u/\partial x_i^2 \geqq 0$ in the whole region for the parabolic equations of the type (2,1) if this is fulfilled on the boundary. Immediately a new problem arises: under what conditions the solution $u(t, x)$ is convex near the boundary surface? This new problem is solved for sufficiently small t (see Theorem 5) and for sufficiently small right-hand sides of the given Îto (or corresponding parabolic) equations (see Theorem 6). Finally equations for $\partial u/\partial \nu$ and $\partial^2 u/\partial \nu\, \partial l$ are derived. The values $\partial u/\partial \nu$ and $\partial^2 u/\partial \nu\, \partial l$ on the boundary may be used for the solution of the problem.

1. Definitions and notations

Let R_n denote the n-dimensional Euclidean space with a norm $|.|$. Let D be a given region in R_n and $Q = (0, L) \times D$ a region in R_{n+1} (L is a positive number). $\bar{D}$ denotes the closure of D and $\dot{D}$ the boundary of D. Let $a(t, x)$ be a vector function (which has values in R_n) and $B(t, x)$ an $n \times n$ matrix function defined in $\bar{Q}$. Wiener process is denoted by $w(t)$ (i.e. $w_i(t)$, $i = 1, \ldots, n$ are continuous and stochastically independent processes with stochastically independent increments fulfilling $Ew_i(t) = 0$, $E|w(t)|^2 = t$ where E is the mathematical expectation. The concept of *a solution* $x(t)$ *of*

$$(1,1) \qquad \mathrm{d}x = a(t, x)\,\mathrm{d}t + B(t, x)\,\mathrm{d}w(t)$$

with an adhesive barrier $\langle 0, L\rangle \times \dot{D}$ is the following: first the domain of definition of $a(t, x)$ and $B(t, x)$ is extended onto the whole $\langle 0, L\rangle \times R_n$ so that, there exists the solution $x^*(t)$ of the extended equation fulfilling $x^*(0) = x(0)$. $x(0)$ is usually assumed to be nonstochastic in this article. Otherwise it is assumed that $x(0)$ is independent of the increments of $w(t)$. Put $x(t) = x^*(t)$ for $t \leqq \tau$ and $x(t) = x^*(\tau)$ for $t > \tau$ where τ is the Markovian moment of the first exit of $x^*(t)$ from D. In [1] it was proved that under certain conditions (which are fulfilled in the present paper) this definition determines $x(t)$ uniquely for the equivalence. Let a nonnegative function $f(x)$ be defined on D and $\int_D f(x)\,\mathrm{d}x = 1$. If $P(x(0) \in A) = \int_A f(x)\,\mathrm{d}x$ for every Borel subset A of D, then *the solution* $x(t)$ *has the initial density* $f(x)$. Such solution will be denoted by $x_f(t)$. If $x(0) = x_0$ is nonstochastic, then instead of f we shall write $\delta(x_0)$. Let

$$P(B, a, f, Q) = P\{\exists\, \{\xi : x_f(\xi) \notin D,\ \xi \in \langle 0, L\rangle\}\}$$

be the probability that the solution $x_f(t)$ of (1,1) leaves the region D during the time interval $\langle 0, L\rangle$. *A region D is regular* if it is bounded and if it has the outside strong sphere property [3]. We say that *a region D fulfils condition* (B) if it is bounded and if to every point $\bar{x} \in \dot{D}$ there exists a ball K with the centre at $\bar{x}$ and a system of orthogonal coordinates $x_1, \ldots, x_n$ where x_n has the direction of the inward normal to $\dot{D}$ with respect to D at the point $\bar{x}$ such that the boundary $\dot{D}$ can be expressed in the ball as a function $x_n = h(x_1, \ldots, x_{n-1})$ for $[x_1, \ldots, x_{n-1}] \in K^+ \subset K^*$ with Hölder continuous second derivatives. The set K^* is defined by $K^* = \{[x_1, \ldots, x_{n-1}] : : [x_1, \ldots, x_{n-1}, 0] \in K\}$ and K^+ is an open subset of K^* containing the origin of $x_1, \ldots, x_{n-1}$ – coordinate system.

This definition slightly differs from that in [1] (Definition 2) since there the axis x_n has the direction of the outward normal. This new definition is advantageous since convex regions are treated.

Now, a theorem proved in [2] will be reformulated:

Theorem 1. *Let a vector function $a(t, x)$ and an $n \times n$ matrix function $B(t, x)$ be defined on $\bar{Q}$ ($Q = (0, L) \times D$), Lipschitz continuous in x and Hölder continuous in t. Let the matrix function $\Lambda(t, x) = B(t, x)\, B^T(t, x)$ ($B^T(t, x)$ is the transposed matrix) be positively definite in $\bar{Q}$. If the region D is regular, then:*

i) *Îto equation (1,1) has the unique solution with the adhesive barrier $\langle 0, L\rangle \times \dot{D}$ for every initial density in D and for every nonstochastic initial value from D.*

ii) *The parabolic equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j} \Lambda_{ij}(L - t, x) \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_i a_i(L - t, x) \frac{\partial u}{\partial x_i} \tag{1,2}$$

has the unique bounded solution fulfilling

$$u(0, x) = 0 \quad \text{for} \quad x \in D\,, \tag{1,3}$$

$$u(t, x) = 1 \quad \text{for} \quad t > 0\,, \quad x \in \dot{D}\,. \tag{1,4}$$

iii) *The bounded solution $u(t, x)$ fulfils*

$$P(B, a, f, Q) = \int_D f(x)\, u(L, x)\, \mathrm{d}x$$

for every density f in D and also for $f = \delta(x)$, $x \in D$:

$$P(B, a, \delta(x), Q) = u(L, x)\,.$$

Now the precise meaning to the notions of the maximal and strongly maximal matrix functions will be given.

Definition 1. *A matrix function $B(t, x)$ is maximal with respect to a vector function $a(t, x)$ and to a region Q ($Q = (0, L) \times D$)* if i) a, B, D fulfil conditions of Theorem 1, ii) $\Lambda(t, x) = B(t, x)\, B^T(t, x)$ is a diagonal matrix in Q, iii)

$$P(B, a, f, Q) = \max P(B', a, f, Q)$$

for all densities f in D where the maximum is taken over the set of matrix functions $B'(t, x)$ fulfilling conditions i) ii) and $\Lambda'_{ii}(t, x) \leqq \Lambda_{ii}(t, x)$ where $\Lambda'_{ii}(t, x)$, $\Lambda_{ii}(t, x)$ are the diagonal elements of the matrix $\Lambda'(t, x) = B'(t, x)\, B'^T(t, x)$ and $\Lambda(t, x)$, respectively.

Definition 2. *A matrix function $B(t, x)$ is strongly maximal with respect to a vector function $a(t, x)$ and to a region Q* if condition i) from Definition 1 is fulfilled and if

$$P(B, a, f, Q) = \max P(B', a, f, Q)$$

for all densities f in D where the maximum is taken over the set of matrix functions $B'(t, x)$ fulfilling conditions of Theorem 1 and $\Lambda(t, x) - \Lambda'(t, x)$ is a positively semi-definite matrix for every $[t, x] \in Q$ ($\Lambda'(t, x)$, $\Lambda(t, x)$ are defined in the same manner as in Definition 1).

The following terms are introduced for the sake of brevity of further formulations.

Definition 3. Let a function $f(x_1, \ldots, x_n)$ be defined in D such that the partial derivatives of the second order exist (on the boundary they are defined as limits). *The function is convex or sharply convex at a point* $[\bar{x}_1, \ldots, \bar{x}_n] \in \bar{D}$ if the matrix $\{\partial^2 f / \partial x_i \, \partial x_j(\bar{x}_1, \ldots, \bar{x}_n)\}_{i,j}$ is positively semi-definite or positively definite, respectively. The convexity of f in $\bar{D}$ is defined as usual. *The function f is convex or sharply convex along the axes* $x_1, \ldots, x_n$ at a point $[\bar{x}_1, \ldots, \bar{x}_n] \in \bar{D}$ if $\partial^2 f / \partial x_i^2 \geqq 0$ or if $\partial^2 f / \partial x_i^2 > 0$, respectively. *The function f is convex along the axes in* $\bar{D}$ if it is convex along the axes at any point from $\bar{D}$. Let D fulfil condition (B) and $\bar{x} \in \dot{D}$. *Region D is convex or sharply convex at* $\bar{x}$ if the function $h(x_1, \ldots, x_{n-1})$ expressing the boundary $\dot{D}$ in K is convex or sharply convex at 0, respectively.

If the function f has continuous partial derivatives of the second order, then the convexity in D implies the convexity at every point of D. The converse statement holds for convex regions D and similar statement is valid for the convexity along axes.

Now, the main results from [1] and [2] can be reformulated.

Theorem 2. *Let $a(t, x)$, $B(t, x)$ and D fulfil conditions of Theorem* 1. *The matrix function $B(t, x)$ fulfilling* ii) *from Definition* 1 *is maximal with respect to the vector function $a(t, x)$ and to the region $Q = (0, L) \times D$ if and only if the bounded solution of* (1,2) *fulfilling* (1,3) *and* (1,4) *is convex along the spatial axes* (*i.e.* $x_1, \ldots, x_n$) *in Q.*

Theorem 3. *Let $a(t, x)$, $B(t, x)$ and D fulfil conditions of Theorem* 1. *The matrix function $B(t, x)$ is strongly maximal with respect to the vector function $a(t, x)$ and to the region $Q = (0, L) \times D$ if and only if the bounded solution of* (1,2) *fulfilling* (1,3) *and* (1,4) *is convex with respect to the spatial variables x in Q.*

2.

The method developed in the paper can be applied only to a certain class of Îto stochastic equations (or corresponding parabolic equations). In the following theorem the class of parabolic equations is shown whose solutions (given by (1,3) and (1,4)) are convex along the spatial axes if they are sharply convex at every point of the boundary $(0, L\rangle \times \dot{D}$. The problem under what conditions such solutions are sharply convex at every point of the boundary $(0, L\rangle \times \dot{D}$ is studied in further sectiones.

Theorem 4. *Let $\alpha_i(t)$, $\beta_i(t)$ be Hölder continuous functions on $\langle 0, L\rangle$. Let $A_{ij}(t, x)$ be defined on $\bar{Q}$ and form a symmetric positively definite matrix at every point of $\bar{Q}$. Assume that A_{ij} depends only on t, x_i, x_j and it is a linear function of x_i and x_j for $i \neq j$, A_{ii} depends only on t and x_i such that $\partial^2 A_{ii}/\partial x_i^2$ is Hölder continuous and let the region D be regular. Denote by $u(t, x)$ the bounded solution of*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{i,j} A_{ij}(t, x_i, x_j)\frac{\partial^2 u}{\partial x_i\,\partial x_j} + \sum_i(\alpha_i(t) + x_i\,\beta_i(t))\frac{\partial u}{\partial x_i} \tag{2,1}$$

fulfilling

$$u(0, x) = 0 \quad for \quad x \in D\,, \tag{2,2}$$

$$u(t, x) = 1 \quad for \quad t > 0\,, \quad x \in \dot{D}\,. \tag{2,3}$$

If $u(t, x)$ is sharply convex along the spatial axes at every point of $(0, L\rangle \times \dot{D}$, then $u(t, x)$ is convex along the spatial axes in Q.

Remark 1. Theorem 4 is applicable to a system of Îto equations

$$\mathrm{d}x_i = (\alpha_i(t) + x_i\,\beta_i(t))\,\mathrm{d}t + \sum_j B_{ij}(t, x)\,\mathrm{d}w_j(t) \tag{2,4}$$

where the matrix function $A(t, x) = B(t, x)\,B^T(t, x)$ fulfils the conditions of the Theorem. (For example $B_{ij}(t, x) = \gamma_{ij}(t) + x_i\,\eta_{ij}(t)$ and $B(t, x)$ is a regular matrix at every point in $\bar{Q}$.) With respect to Theorem 1 the corresponding parabolic equation is (2,1) if only the variable t is substituted by $L - t$. Theorem 2 then implies that under the conditions of Theorem 4 the matrix function $B(t, x)$ is maximal with respect to $a_i(t, x) = \alpha_i(t) + x_i\,\beta_i(t)$ and to the region $Q = (0, L) \times D$.

A necessary condition for the region D is that the intersections of the straight lines parallel to the spatial axes $x_1, \ldots, x_n$ with D are intervals only.

Remark 2. Let the solution $u(t, x)$ of (2,1) fulfilling (2,2) and (2,3) be sharply convex at every point of $(0, L\rangle \times \dot{D}$. Let $x_1^*, \ldots, x_n^*$ be any orthogonal system of coordinates. Denote by (2,1)* ((2,4)*) the parabolic (or Îto) equation transformed to the coordinate system $x_1^*, \ldots, x_n^*$. If (2,1)* (or (2,4)*) is of the type (2,1) (or (2,4)) again, then the matrix function $B(t, x)$ is strongly maximal with respect to $a_i(t, x) = \alpha_i(t) + x_i\,\beta_i(t)$ and with respect to the region Q.

Proof of Theorem 4. Put $v_k = \partial^2 u/\partial x_k^2$. With respect to (2,1) and to the assumptions of Theorem 4

$$\frac{\partial v_k}{\partial t} = \left(2\beta_k(t) + \frac{1}{2}\frac{\partial^2 A_{kk}}{\partial x_k^2}\right)v_k + \sum_{i \neq k}\left(\alpha_i(t) + x_i\,\beta_i(t) + 2\frac{\partial A_{ik}}{\partial x_k}\right)\frac{\partial v_k}{\partial x_i} + \\ + \left(\alpha_k(t) + x_k\,\beta_k(t) + \frac{\partial A_{kk}}{\partial x_k}\right)\frac{\partial v_k}{\partial x_k} + \frac{1}{2}\sum_{i,j} A_{ij}\frac{\partial^2 v_k}{\partial x_i\,\partial x_j}\,. \tag{2,5}$$

The initial condition (2,2) implies $v_k(0, x) = 0$ for $x \in D$ and the condition about the behaviour of u near to the boundary surface implies $\liminf v_k(t, x) > 0$ for $t > 0$ and $[x_1, \ldots, x_n] \to [\bar{x}_1, \ldots, \bar{x}_n] \in \dot{D}$. Denote $\zeta = \max(2\beta_k + \frac{1}{2}(\partial^2 A_{kk}/\partial x_k^2))$ for $[t, x_1, \ldots, x_n] \in \bar{Q}$. By means of the transformation $v_k = w \exp(\zeta t)$, (2,5) changes to

$$\frac{\partial w}{\partial t} = \left(2\beta_k + \frac{1}{2}\frac{\partial^2 A_{kk}}{\partial x_k^2} - \zeta\right) w + \sum_{i \neq k}\left(\alpha_i + x_i\beta_i + 2\frac{\partial A_{ik}}{\partial x_k}\right)\frac{\partial w}{\partial x_i} +$$

$$+ \left(\alpha_k + x_k\beta_k + \frac{\partial A_{kk}}{\partial x_k}\right)\frac{\partial w}{\partial x_k} + \tfrac{1}{2}\sum A_{ij}\frac{\partial^2 w}{\partial x_i\,\partial x_j},$$

the initial and boundary value conditions remaining unchanged. The principle of maximum yields $w(t, x) \geqq 0$ and thus $\partial^2 u/\partial x_k^2 = v_k(t, x) \geqq 0$.

3.

This section will be devoted to the investigation of the behaviour of the solution $u(t, x)$ near to the boundary surface $(0, L\rangle \times \dot{D}$. Let a region D fulfil condition (B), let P be a point on the boundary $\dot{D}$ and $x_1, \ldots, x_n$ a local orthogonal system at the point P with respect to D. Suppose that $u(t, x)$ is the bounded solution of

$$(3,1) \qquad \frac{\partial u}{\partial t} = \tfrac{1}{2}\sum_{i,j} A_{ij}(t, x)\frac{\partial^2 u}{\partial x_i\,\partial x_j} + \sum_i a_i(t, x)\frac{\partial u}{\partial x_i}$$

fulfilling

$$(3,2) \qquad u(0, x) = 0 \quad \text{for} \quad x \in D,$$

$$(3,3) \qquad u(t, x) = 1 \quad \text{for} \quad t > 0, \quad x \in \dot{D}.$$

With respect to condition (B) the boundary $\dot{D}$ can be locally expressed by means of a function $x_n = h(x_1, \ldots, x_{n-1})$. Due to (3,3) the equality $u(t, x_1, \ldots, x_{n-1}, h(x_1, \ldots \ldots, x_{n-1})) = 1$ holds. With respect to Theorem 10.1 Chap. 4 [4] the second derivatives of $u(t, x)$ are continuous including the boundary $\dot{D}$ for $t > 0$. The last equality yields

$$(3,4) \qquad \frac{\partial u}{\partial x_i}(t, P) = 0 \quad \text{for} \quad t > 0, \quad i = 1, \ldots, n-1,$$

$$\frac{\partial^2 u}{\partial x_i\,\partial x_j}(t, P) + \frac{\partial u}{\partial x_n}(t, P)\frac{\partial^2 h}{\partial x_i\,\partial x_j}(P) = 0 \quad \text{for} \quad t > 0, \quad i, j = 1, \ldots, n-1.$$

With regard to Theorem 14, Chap. II [3] the derivative $\partial u/\partial x_n(t, P)$ is negative. Let H be an $(n-1) \times (n-1)$ matrix with elements $\partial^2 h/\partial x_i\,\partial x_j(P)$, $i, j = 1, \ldots \ldots, n-1$, U an $n \times n$ matrix with elements $\partial^2 u/\partial x_i\,\partial x_j(t,P)$ for $i, j = 1, \ldots, n$

and V an $n \times n$ matrix with elements $V_{ij} = \partial^2 h/\partial x_i \, \partial x_j(P)$ for $i, j = 1, \ldots, n-1$, $V_{in} = V_{ni} = \partial^2 u/\partial x_i \, \partial x_n(t, P)$ for $i = 1, \ldots, n-1$ and $V_{nn} = (\partial^2 u/\partial x_n^2(t, P))\,.\,(-\partial u/\partial x_n(t, P))$. Equations (3,4) imply $\det U = (-\partial u/\partial x_n(t, P))^{n-2} \det V$ and similar relations for all main subdeterminants. It means that the matrix U is positively semi-definite (positively definite) if and only if the matrix V is positively semi-definite (positively definite). Considering Silvester Theorem we can conclude that the solution $u(t, x)$ is sharply convex at the point $[t, P]$ if and only if the matrix H is positively definite (i.e. the boundary $\dot{D}$ is sharply convex at P) and the determinant $\det V$ is positive. Since the solution $u(t, x)$ has the first and the second derivatives on the boundary $\dot{D}$ for $t > 0$, equation (3,1) gives with respect to (3,4)

$$a_n(t, P) \frac{\partial u}{\partial x_n}(t, P) + \frac{1}{2} \sum_{i,j} A_{ij}(t, P) \frac{\partial^2 u}{\partial x_i \, \partial x_j}(t, P) = 0 .$$

Denote by $|H|$ the determinant of the matrix H and by $H^{i,j}$ its subdeterminants. The condition $\det V \geqq 0$ is equivalent to the inequality

$$\text{(3,5)} \qquad |H| \left[2a_n(t, P) - \sum_{i,j=1}^{n-1} A_{ij}(t, P) \frac{\partial^2 h}{\partial x_i \, \partial x_j}(P) \right] \left(\frac{\partial u}{\partial x_n}(t, P) \right)^2 + \\ + 2|H| \frac{\partial u}{\partial x_n}(t, P) \sum_{i=1}^{n-1} A_{in}(t, P) \frac{\partial^2 u}{\partial x_i \, \partial x_n}(t, P) - \\ - A_{nn}(t, P) \sum_{i,j=1}^{n-1} (-1)^{i+j} H^{i,j} \frac{\partial^2 u}{\partial x_i \, \partial x_n}(t, P) \frac{\partial^2 u}{\partial x_j \, \partial x_n}(t, P) \geqq 0 .$$

Remark 3. If the solution $u(t, x)$ is convex at the point $[t, P]$, then the region D is convex at the point P and (3,5) holds. These conditions are also sufficient.

If the solution $u(t, x)$ is sharply convex at the point $[t, P]$, then the region D is sharply convex at the point P and (3,5) holds as a sharp inequality. These conditions are also sufficient conditions.

From (3,5) it follows for any real number Θ

$$\text{(3,6)} \qquad |H| \left[2a_n(t, P) - \sum_{i,j=1}^{n-1} A_{ij}(t, P) \frac{\partial^2 h}{\partial x_i \, \partial x_j}(P) + \Theta^{-2} \right] \left(\frac{\partial u}{\partial x_n} \right)^2 \geqq \\ \geqq A_{nn}(t, P) \sum_{i,j=1}^{n-1} (-1)^{i+j} H^{i,j} \frac{\partial^2 u}{\partial x_i \, \partial x_n} \frac{\partial^2 u}{\partial x_j \, \partial x_n} - \\ - \Theta^2 |H| \left(\sum_{i=1}^{n-1} A_{in}(t, P) \frac{\partial^2 u}{\partial x_i \, \partial x_n} \right)^2 .$$

Since H is a positively definite matrix also the matrix of elements $(-1)^{i+j} H^{i,j}$ is positively definite. Choosing Θ as the least positive solution of

$$\text{(3,7)} \qquad \det \left[A_{nn}(t, P) (-1)^{i+j} H^{i,j} - \Theta^2 |H| A_{in}(t, P) A_{jn}(t, P) \right] = 0 ,$$

then

$$(3{,}8)\qquad 2a_n(t, P) - \sum_{i,j=1}^{n-1} A_{ij}(t, P)\, \frac{\partial^2 h}{\partial x_i\, \partial x_j}(P) + \Theta^{-2} \geqq 0$$

holds since the matrix in (3,7) is symmetric. The result is worth formulate as a lemma.

Lemma 1. *If $u(t, x)$ is convex at the point $[t, P]$, then (3,8) holds where Θ is the least positive solution of (3,7). If $A_{in}(t, P) = 0$ for $i = 1, \ldots, n - 1$, then necessary condition (3,8) becomes*

$$2a_n(t, P) - \sum_{i,j=1}^{n-1} A_{ij}(t, P)\, \frac{\partial^2 h}{\partial x_i\, \partial x_j}(P) \geqq 0\,.$$

Remark 4. With regard to Lemma 3 [1] it is $\Sigma A_{ij}(\partial^2 h/\partial x_i\, \partial x_j) \geqq 0$ so that the necessary condition (3,8) is a refinement of the necessary condition following from Remark 7 [1].

The following lemma will be useful in further considerations.

Lemma 2. *Let the region D fulfil condition* (B), *$P \in \dot{D}$ and $x_1, \ldots, x_n$ be the local coordinate system at P with respect to D. For every $t_0 \geqq 0$ there exists a linear regular transformation $z = Lx$, $u(t, x) = w(t, z)$ such that $L_{ni} = 0$ for $i = 1, \ldots$ $\ldots, n - 1$, $L_{nn} > 0$ and such that (3,1) is transformed onto*

$$(3{,}9)\qquad \frac{\partial w}{\partial t} = \frac{1}{2} \sum_{i,j} A^*_{ij}(t, z)\, \frac{\partial^2 w}{\partial z_i\, \partial z_j} + \sum_i a^*(t, z)\, \frac{\partial w}{\partial z_i}$$

where $a^*_i(t, z) = La(t, L^{-1}z)$ and $A^*_{ij}(t, z)$ are elements of the matrix $LA(t, L^{-1}z)\, L^T$ and $A^*(t_0, P)$ forms the unit matrix. If the region D can be locally expressed by means of $x_n = h(x_1, \ldots, x_{n-1})$, then D^* can be locally expressed by means of $z_n =$ $= h^*(z_1, \ldots, z_{n-1})$ where $\partial h^*/\partial z_i(P) = 0$ and $\partial^2 h^*/\partial z_i\, \partial z_j(P)$ for $i, j = 1, \ldots, n - 1$ are elements of a matrix $L_{nn}(L^{-1})^T\, \hat{H} L^{-1}$. The matrix $\hat{H}$ is defined as follows: $\hat{H}_{ij} = \partial^2 h/(\partial x_i\, \partial x_j)(P)$ for $i, j = 1, \ldots, n - 1$ while all other $\hat{H}_{ij}$ equal zero.

If a point $P \in \dot{D}$ and a positive number δ_1 are given, there exist a number $\delta_2(\delta_1) > 0$ (which converges to zero for $\delta_1 \to 0$) and a function $h^\circ(x_1, \ldots, x_{n-1})$ defined on the whole R_{n-1} such that the function h° and its partial derivatives of the first and the second order are Hölder continuous, $h^\circ(x_1, \ldots, x_{n-1}) = h(x_1, \ldots$ $\ldots, x_{n-1})$ for $\{x : |x| < \delta_1\}$ and *$h^\circ = 0$ for $\{x : |x| > \delta_2\}$ and the transformation $x_i = y_i$, $i = 1, \ldots, n - 1$, $x_n = y_n + h^\circ(y_1, \ldots, y_{n-1})$, $u(t, x) = v(t, y)$ maps (3,1) onto*

$$(3{,}10)\quad \frac{\partial v}{\partial t} = \sum_{i=1}^{n} a^\circ_i(t, y)\, \frac{\partial v}{\partial y_i} - \left[\sum_{i=1}^{n-1} a^\circ_i(t, y)\, \frac{\partial h^\circ}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^{n-1} A^\circ_{ij}(t, y)\, \frac{\partial^2 h^\circ}{\partial y_i\, \partial y_j}\right] \frac{\partial v}{\partial y_n} +$$
$$+ \frac{1}{2} \sum_{i,j=1}^{n} A^\circ_{ij}(t, y)\, \frac{\partial^2 v}{\partial y_i\, \partial y_j} - \frac{1}{2} \sum_{i,j=1}^{n-1} A^\circ_{ij}(t, y) \left[\frac{\partial^2 v}{\partial y_i\, \partial y_n}\, \frac{\partial h^\circ}{\partial y_j} + \frac{\partial^2 v}{\partial y_j\, \partial y_n}\, \frac{\partial h^\circ}{\partial y_i}\right] +$$
$$+ \frac{1}{2}\left[\sum_{i,j=1}^{n-1} A^\circ_{ij}(t, y)\, \frac{\partial h^\circ}{\partial y_i}\, \frac{\partial h^\circ}{\partial y_j} - 2 \sum_{i=1}^{n-1} A^\circ_{in}(t, y)\, \frac{\partial h^\circ}{\partial y_i}\right] \frac{\partial^2 v}{\partial y_n^2}\,,$$

where $a_i^\circ(t, y) = a_i(t, x)$, $A_{ij}^\circ(t, y) = A_{ij}(t, x)$ *and*

$$\limsup_{\delta_1 \to 0} \max_y \max_i |\partial h^\circ / \partial y_i| = 0 \,, \quad \limsup_{\delta_1 \to 0} \max_y \max_{i,j} |\partial^2 h^\circ / \partial y_i \, \partial y_j| < \infty \,.$$

The region D is transformed onto a region D° *so that the boundary of* D° *in the* δ_1*-neighbourhood of P is given by* $y_n = 0$.

Proof. It can be assumed that $L_{ni} = 0$ for $i \neq n$ and $L_{nn} > 0$ which means that the half-space $x_n > 0$ is transformed onto the half-space $z_n > 0$. Let C be the matrix with elements $C_{ij} = A_{ij}(t_0, P) - A_{in}(t_0, P)\, A_{jn}(t_0, P) / A_{nn}(t_0, P)$, $i, j = 1, \ldots, n-1$. With respect to a well-known theorem there is a regular $(n-1) \times (n-1)$ matrix $L^{(1)}$ such that $L^{(1)} C L^{(1)T}$ is a diagonal matrix and the diagonal elements are plus one, minus one or zero. Let L_{ij} be the elements of $L^{(1)}$ for $i, j = 1, \ldots, n-1$, $L_{in} = - \sum_{j=1}^{n-1} L_{ij} A_{jn}(t_0, P) / A_{nn}(t_0, P)$ for $i < n$ and $L_{nn} = A_{nn}^{-1/2}(t_0, P)$. L is a regular matrix of the type $n \times n$ since $L^{(1)}$ is regular ($A_{nn} > 0$ holds since (3,1) is parabolic). It can be easily verified that $z = Lx$ maps (3,1) onto (3,9). As L is regular and A is positively definite, the matrix LAL^T has to be also positively definite which implies that the diagonal elements of $L^{(1)} C L^{(1)T}$ equal to one. Since the elements of this matrix are simultaneously elements of LAL^T this last matrix is the unit matrix.

The second statement of Lemma can be obtained if h is multiplied by a function $\varphi(x)$ where

$$\varphi(x) = 1 \quad \text{for} \quad |x| \leqq \delta_1 \,,$$

$$\varphi(x) = \frac{1}{2} \left\{ 1 + \cos \frac{|x| - \delta_1}{\delta_1} \pi - \frac{\pi^2}{\delta_1^5} \left[\left(|x| - \frac{3}{2} \delta_1 \right)^2 - \frac{\delta_1^2}{4} \right]^2 \left(|x| - \frac{3}{2} \delta_1 \right) \right\}$$

$$\text{for} \quad \delta_1 < |x| < 2\delta_1 \,,$$

$$\varphi(x) = 0 \quad \text{for} \quad |x| \geqq 2\delta_1 \,.$$

Remark 5. The second derivatives $\partial^2 h^* / \partial z_i \, \partial z_j(P)$ are the elements of the matrix $L_{nn}((L^{(1)})^T)^{-1} H (L^{(1)})^{-1}$ where the matrix $L^{(1)}$ was introduced in the proof of Lemma 2.

The previous lemmas give immediately an interesting result.

Lemma 3. *Let the assumptions of Lemma* 1 *and* 2 *be fulfilled. A necessary condition for the convexity of the bounded solution* $u(t, x)$ *fulfilling* (3,1) *and* (3,2), (3,3) *at a point* $[t, P]$, $t \geqq 0$, $P \in \dot{D}$ *is* $2a_n(t, P) \geqq \operatorname{trace}((L^{(1)})^{-1})^T H (L^{(1)})^{-1}$.

Provided the region D is a sphere in the n-dimensional Euclidean space with a radius r and A the unit matrix, Lemma 3 gives the neccesary condition $2a_n(t, P) \geqq (n-1)/r$ (a_n is the coefficient if the equation is expressed in the local coordinate system in P). On the other hand the investigation of Example 3 in [1] shows that in the case $n = 2$, D is a circle with radius $r = 1$, the matrix A is the unit matrix and $a_1 = -x/2$, $a_2 = -y/2$ (see (15,16) in [1]) that the solution $u(t, x_1, x_2)$ is convex in x_1, x_2. In this case the last necessary condition is valid even for this equation.

4.

In this section the behaviour of the solution $u(t, x)$ on the boundary surface $(0, L\rangle \times \dot{D}$ will be investigated for small t. The coefficients of (3,1) can satisfy weaker conditions than those of Theorem 1, on the other hand, the region D has to fulfil stronger conditions.

Theorem 5. *Let the region D fulfil condition* (B), $P \in \dot{D}$, $x_1, \ldots, x_n$ *be the local coordinate system at the point P with respect to D. Assume that the coefficients of* (3,1) *are Hölder continuous and*

$$A_{ij}(t, x) = A_{ij}(t, P) + \sum_k \frac{\partial A_{ij}}{\partial x_k}(t, P)(x_k - P_k) + \sum_k (x_k - P_k)\, \varphi_k(t, x)$$

where $A_{ij}(t, P)$ are Hölder continuous with an exponent greater than $\frac{1}{2}$, $\partial A_{ij}/\partial x_k(t, P)$ and $\varphi_k(t, x)$ are Hölder continuous, $\varphi_k(0, P) = 0$. Denote by $u(t, x)$ the bounded solution of (3,1) *fulfilling* (3,2) *and* (3,3). *Let Γ be an $n \times n$ matrix defined as follows: Γ_{ij} are the elements of the matrix $((L^{(1)})^{-1})^T H (L^{(1)})^{-1}$ for $i, j = 1, \ldots$ $\ldots, n - 1$, $\Gamma_{in} = \Gamma_{ni} = \frac{1}{2} \sum_{s=1}^{n-1} \partial A_{nn}/\partial x_s(0, P)\, (L^{(1)})^{-1}_{si}\, (A_{nn}(0, P))^{-\frac{1}{2}}$ for $i < n$ and $\Gamma_{nn} =$ $= 2a_n(0, P) - \sum_{i=j}^{n-1} \Gamma_{ii}$. If H is a positively definite matrix and* $\det \Gamma > 0$, *then $u(t, x)$ is sharply convex at the points $[t, P]$ for small t. If $u(t, x)$ is convex at all $[t, P]$ for sufficiently small t, then the region D is convex at P and* $\det \Gamma \geqq 0$ *for small t.*

Remark 6. The matrix $L^{(1)}$ is defined in the proof of Lemma 2. If the matrix $A(0, P)$ is diagonal, then also $L^{(1)}$ is diagonal.

Remark 7. If $\partial A_{nn}/\partial x_i(0, P) = 0$ for $i = 1, \ldots, n - 1$ and the matrix H is positively definite, then the inequality from Lemma 3 is a sufficient condition for the sharp convexity of $u(t, x)$ at the points $[t, P]$ for small t if only the inequality is valid as a sharp inequality.

Proof of Theorem 5. Assume from the beginning that equation (3,1) is defined in a region $Q = (0, L) \times D$ where D is an unbounded region in R_n defined by $x_n >$ $> h(x_1, \ldots, x_{n-1})$. The function h and the partial derivatives $\partial h/\partial x_i$, $\partial^2 h/\partial x_i\, \partial x_j$ are assumed to be uniformly Hölder continuous, the second partial derivatives $\partial^2 h/\partial x_i\, \partial x_j$ bounded and the first partial derivatives so small that equation (3,10) with $h^\circ = h$ is uniformly parabolic. The coefficients of (3,1) are assumed to be uniformly Hölder continuous, bounded and such that (3,1) is uniformly parabolic. Without any loss of generality put $P = 0$. When the first transformation from Lemma 2 for $t_0 = 0$ and $P = 0$ is applied the transformed equation of (3,1) will satisfy

$$(4,1) \qquad A_{ii}(0, 0) = 1\,, \quad A_{ij}(0, 0) = 0\,, \quad i \neq j,\ i, j = 1, \ldots, n\,.$$

Since the convexity of $u(t, x)$ is preserved by linear transformation it will be assumed that (4,1) is valid from the beginning. With respect to the assumption about the convexity of D at 0 it is sufficient to prove that (3,5) is valid as a sharp inequality for small t. Denote $l_i = \lim \partial^2 u/\partial x_i\, \partial x_n(t, 0)/(-\partial u/\partial x_n(t, 0))$ for $t \to 0$. (The existence of the limit will be proved later.) With regard to the assumptions about coefficients of (3,1) it is sufficient to show

$$|H| \left[2a_n(0, 0) - \sum_{i=1}^{n-1} \frac{\partial^2 h}{\partial x_i^2}(0) \right] > \sum_{i,j=1}^{n-1} (-1)^{i+j} H^{i,j} l_i l_j . \tag{4,2}$$

The second transformation of Lemma 2 maps equation (3,1) onto (3,10) and this equation will be written in the form

$$\frac{\partial v}{\partial t} = \frac{1}{2} \sum_{i,j} A_{ij}^{\circ}(t, x) \frac{\partial^2 v}{\partial x_i\, \partial x_j} + \sum_i a_i^{\circ}(t, x) \frac{\partial v}{\partial x_i}, \tag{4,3}$$

since the transformation does not affect the limits l_i. The region D is transformed onto the half-space $x_n > 0$ so that the boundary value condition for $v(t, x)$ will be now

$$v(0, x) = 0 \quad \text{for} \quad x_n > 0, \tag{4,4}$$

$$v(t, x) = 1 \quad \text{for} \quad t > 0, \quad x_n = 0. \tag{4,5}$$

Let λ be a positive number. Consider a transformation $t = \lambda\tau$, $x = \xi \sqrt{\lambda}$, $v(t,x) = v_\lambda(\tau, \xi)$. Equation (4,3) passes to

$$\frac{\partial v_\lambda}{\partial \tau} = \frac{1}{2} \sum_{i,j} A_{ij}^{\circ}(\lambda\tau, \xi \sqrt{\lambda}) \frac{\partial^2 v_\lambda}{\partial \xi_i\, \partial \xi_j} + \sum_i a_i^{\circ}(\lambda\tau, \xi \sqrt{\lambda}) \sqrt{\lambda} \frac{\partial v_\lambda}{\partial \xi_i} \tag{4,6}$$

with the same initial and boundary value conditions. Evidently

$$\frac{\partial v}{\partial x_i}(\lambda, 0) = \frac{1}{\sqrt{(\lambda)}} \frac{\partial v_\lambda}{\partial \xi_i}(1, 0), \quad \frac{\partial^2 v}{\partial x_i\, \partial x_j}(\lambda, 0) = \frac{1}{\lambda} \frac{\partial^2 v_\lambda}{\partial \xi_i\, \partial \xi_j}(1, 0). \tag{4,7}$$

For $\lambda = 0$ equation (4,6) assumes the form

$$\frac{\partial v_0}{\partial \tau} = \tfrac{1}{2} \sum_i \frac{\partial^2 v_0}{\partial \xi_i^2} \tag{4,8}$$

(see (4,1)). Denote $v_0(\tau, \xi)$ the bounded solution of (4,8) fulfilling (4,4) and (4,5). The solution v_0 depends only on τ and ξ_n. This implies that v_0 is a solution of a one-dimensional heat equation for which the explicit formulas give

$$\frac{\partial v_0}{\partial \xi_n} = -\sqrt{\left(\frac{2}{\pi\tau}\right)} \exp\left\{-\frac{\xi_n^2}{2\tau}\right\}, \quad \frac{\partial^2 v_0}{\partial \xi_n^2} = \frac{\xi_n}{\tau} \sqrt{\left(\frac{2}{\pi\tau}\right)} \exp\left\{-\frac{\xi_n^2}{2\tau}\right\}. \tag{4,9}$$

Put $\Delta_\lambda(\tau, \xi) = (v_\lambda(\tau, \xi) - v_0(\tau, \xi))/\sqrt{\lambda}$ for every $\lambda > 0$. The function Δ_λ is then the bounded solution of

$$(4,10) \qquad \frac{\partial \Delta}{\partial \tau} = \frac{1}{2}\sum_{i,j=1}^{n} A^\circ_{ij}(\lambda\tau, \xi\sqrt{\lambda}) \frac{\partial^2 \Delta}{\partial \xi_i \, \partial \xi_j} + \sum_{i=1}^{n} a^\circ_i(\lambda\tau, \xi\sqrt{\lambda})\sqrt{\lambda}\frac{\partial \Delta}{\partial \xi_i} + \\ + a^\circ_n(\lambda\tau, \xi\sqrt{\lambda})\frac{\partial v_0}{\partial \xi_n} + \frac{A^\circ_{nn}(\lambda\tau, \xi\sqrt{\lambda}) - A^\circ_{nn}(0, 0)}{2\sqrt{\lambda}} \frac{\partial^2 v_0}{\partial \xi_n^2}$$

fulfilling the initial condition $\Delta_\lambda(0, \xi) = 0$ for $\xi_n > 0$ and the boundary value condition $\Delta_\lambda(t, \xi) = 0$ for $t > 0$, $\xi_n = 0$. First some estimates of Δ_λ are necessary. Let $G_\lambda(\tau, \xi; \sigma, \eta)$ be the Green function of (4,10) in the half-space $\xi_n > 0$. Since the coefficients of (4,3) are uniformly Hölder continuous, bounded and since the equation is uniformly parabolic, Theorem 16.3 [4] may be applied yielding

$$(4,11) \qquad |G_\lambda| \leqq \frac{c_1}{(\tau - \sigma)^{n/2}} \exp\left\{-c_2 \frac{|\xi - \eta|^2}{\tau - \sigma}\right\}$$

in the whole region $\langle 0, L\rangle \times R_n^+ \times \langle 0, L\rangle \times R_n^+$ (where R_n^+ is the half-space $\xi_n > 0$) and c_1, c_2 are independent of λ for $0 < \lambda \leqq 1$,

$$(4,12) \qquad \left|\frac{\partial G_\lambda}{\partial \xi_i}\right| \leqq \frac{c_3}{(\tau - \sigma)^{(n+1)/2}} \exp\left\{-c_4 \frac{|\xi - \eta|^2}{\tau - \sigma}\right\}, \\ \left|\frac{\partial^2 G_\lambda}{\partial \xi_i \, \partial \xi_j}\right| \leqq \frac{c_3}{(\tau - \sigma)^{(n+2)/2}} \exp\left\{-c_4 \frac{|\xi - \eta|^2}{\tau - \sigma}\right\}$$

in every compact subset of $\langle 0, L\rangle \times R_n^+ \times \langle 0, L\rangle \times R_n^+$ and c_3, c_4 are independent of λ, $0 < \lambda \leqq 1$. Put

$$(4,13) \qquad \Delta^*_\lambda = \int_0^\tau \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} G_\lambda(\tau, \xi; \sigma, \eta) \left[a^\circ_n(\lambda\sigma, \eta\sqrt{\lambda}) \frac{\partial v_0}{\partial \xi_n}(\sigma, \eta) + \right. \\ \left. + \frac{A^\circ_{nn}(\lambda\sigma, \eta\sqrt{\lambda}) - A^\circ_{nn}(0, 0)}{2\sqrt{\lambda}} \frac{\partial^2 v_0}{\partial \xi_n^2}\right] d\eta_n \, d\eta_{n-1} \ldots d\eta_1 \, d\sigma .$$

Inequalities (4,9) and (4,11) give an estimate

(4,14)

$$|\Delta^*_\lambda| \leqq c_5 \max |a^\circ_n| \exp\left\{-c_6 \frac{\xi_n^2}{\tau}\right\} \sqrt{(\tau)} + c_5 \left[\max \left|\frac{A^\circ_{nn}(\lambda\sigma, \eta\sqrt{(\lambda)} - A^\circ_{nn}(\lambda\sigma, 0)}{2|\eta|\sqrt{\lambda}}\right| + \right. \\ \left. + \max \left|\frac{A^\circ_{nn}(\lambda\sigma, 0) - A^\circ_{nn}(0, 0)}{2\sqrt{\lambda}}\right|\right] (1 + |\xi|) \exp\left\{-c_6 \frac{\xi_n^2}{\tau}\right\},$$

where the constants c_5, c_6 depend only on c_1, c_2. (4,14) implies

(4,15) $\qquad \lim \Delta^*_\lambda(\tau, \xi) = 0$ *for* $\tau \to 0$ *uniformly with respect to* λ,

$0 < \lambda \leqq 1$, *for a given* ξ, $\xi_n > 0$.

Let a point $[\tau^0, \xi^0]$, $\tau^0 > 0$, $\xi_n^0 \geqq 0$ be given. Due to (4,9) there exists a number $N > 0$ such that

$$(4,16) \qquad \left|\frac{\partial v_0}{\partial \xi_n}(\tau^1, \xi_n^1)\right| \leqq N\,, \quad \left|\frac{\partial^2 v_0}{\partial \xi_n^2}(\tau^1, \xi_n^1)\right| \leqq N\,,$$

$$\left|\frac{\partial v_0}{\partial \xi_n}(\tau^1, \xi_n^1) - \frac{\partial v_0}{\partial \xi_n}(\tau^2, \xi_n^2)\right| \leqq N(|\tau^1 - \tau^2|^{\alpha/2} + |\xi_n^1 - \xi_n^2|^\alpha)\,,$$

$$\left|\frac{\partial^2 v_0}{\partial \xi_n^2}(\tau^1, \xi^1) - \frac{\partial^2 v_0}{\partial \xi_n^2}(\tau^2, \xi^2)\right| \leqq N(|\tau^1 - \tau^2|^{\alpha/2} + |\xi_n^1 - \xi_n^2|^\alpha)$$

if the points $[\tau^1, \xi_n^1]$, $[\tau^2, \xi_n^2]$ are in the two-dimensional region

$$D_1 = \{[\tau, \xi_n] : \tau > \tfrac{1}{4}\tau^0(1 - \xi_n),\ \tau > 0,\ \xi_n > 0\}\,.$$

Let v_0° be an extension of v_0 from D_1 such that (4,16) is fulfilled for v_0° on the whole $\langle 0, L\rangle \times R_n^+$. Put $\hat{v}_0 = v_0 - v_0^\circ$. Define $\Delta_\lambda^{\circ *}$, $\hat{\Delta}_\lambda^*$ by means of (4,13) where v_0 is substituted by v_0° or $\hat{v}_0$, respectively. The function $\Delta_\lambda^{\circ *}$ is evidently a solution of (4,10) where v_0 is replaced by v_0° and by Theorem 4, Chap IV. [3] it can be concluded that

$$(4,17) \qquad |\Delta_\lambda^{\circ *}| \leqq M\,, \quad \left|\frac{\partial \Delta_\lambda^{\circ *}}{\partial \xi_i}\right| \leqq M\,, \quad \left|\frac{\partial^2 \Delta_\lambda^{\circ *}}{\partial \xi_i\, \partial \xi_j}\right| \leqq M$$

in the region

$$S = \{[\tau, \xi] : |\tau - \tau^0| < \tfrac{1}{4}\tau^0,\ |\xi - \xi^0| < \tfrac{1}{4}\tau^0,\ \xi_n > 0\}\,,$$

where the constant M does not depend on λ but may depend on τ^0. Since G_λ is the Green function it holds

$$(4,18) \qquad \Delta_\lambda^{\circ *}(\tau, \xi) = 0 \quad \text{for} \quad \xi_n = 0\,.$$

Let D_2 be a two-dimensional region defined by

$$D_2 = \{[\tau, \xi_n] : \tau < \tfrac{1}{4}\tau^0(1 - \xi_n),\ \tau > 0,\ \xi_n > 0\}\,.$$

If $[\sigma, \eta_n] \in D_2$ and $[\tau, \xi] \in S$, then $\tau^0/2 \leqq \tau - \sigma \leqq 5\tau^0/4$ and

$$(4,19) \qquad (\tau - \sigma)^{-\gamma} \exp\left\{-\frac{|\xi - \eta|^2}{\tau - \sigma}\right\} \leqq c_7 \exp\{-c_8|\xi - \eta|^2\}$$

where c_7, c_8 are independent of λ. With respect to (4,11), (4,12), (4,19), the fact that $\hat{v}_0 = 0$ on D_1 and that the integrals

$$\int_{D_2} \left|\frac{\partial \hat{v}_0}{\partial \xi_n}\right| \mathrm{d}\xi_n\, \mathrm{d}\tau\,, \quad \int_{D_2} \left|\frac{\partial^2 \hat{v}_0}{\partial \xi_n^2}\right| \mathrm{d}\xi_n\, \mathrm{d}\tau$$

converge absolutely, the function $\hat{\Delta}_\lambda^*$ is also a solution of (4,10) if only v_0 is substituted by $\hat{v}_0$. With respect to (4,12) inequality (4,17) holds also for $\hat{\Delta}_\lambda^*$ in the region S and since G_λ is the Green function, (4,18) holds as well. Hence it follows that $v_0 + \Delta_\lambda^* \sqrt{\lambda}$ is the solution of (4,6) fulfilling (4,4) and (4,5). This solution fulfils the estimate (4,14). According to [5] the equation $v = v_0 + \Delta_\lambda^* \sqrt{\lambda}$ is proved which means $\Delta_\lambda = \Delta_\lambda^*$.

Let λ_n be a sequence of positive numbers which converge to zero. With respect to (4,14) and Theorem 15, Chap. III [3] it is possible to choose a subsequence λ_n' such that $\Delta_{\lambda'_n}$ converge to a certain solution Δ_0 of

$$(4,20) \qquad \frac{\partial \Delta}{\partial \tau} = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 \Delta}{\partial \xi_i^2} + a_n^\circ(0,0) \frac{\partial v_0}{\partial \xi_n}(\tau, \xi) + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial A_{nn}^\circ(0,0)}{\partial x_i} \xi_i \frac{\partial^2 v_0}{\partial \xi_n^2}(\tau, \xi)$$

fulfilling (4,14). With regard to (4,15) it is $\Delta_0(0, \xi) = 0$ for $\xi_n > 0$ and (4,17), (4,18) imply $\Delta_0(t, \xi) = 0$ for $t > 0$, $\xi_n = 0$. Due to Theorem 2 [5] there exists only one such solution fulfilling (4,14). This yields that also the original sequence Δ_{λ_n} converges to the same Δ_0 such that $\lim_{\lambda \to 0} \Delta_\lambda = \Delta_0$ at every point of the half-space $\xi_n > 0$ and the limit is uniform in the S-region of every point $[\tau^0, \xi^0]$, $\tau^0 > 0$, $\xi_n^0 \geqq 0$. For $\nabla_\lambda = \Delta_\lambda - \Delta_0$ it is

$$\frac{\partial \nabla}{\partial \tau} = \frac{1}{2} \sum_{i,j} A_{ij}^\circ(\tau\lambda, \xi \sqrt{\lambda}) \frac{\partial^2 \nabla}{\partial \xi_i \partial \xi_j} + \sum_i a_i^\circ(\lambda\tau, \xi \sqrt{\lambda}) \sqrt{\lambda} \frac{\partial \nabla}{\partial \xi_i} +$$

$$+ \frac{1}{2} \sum_{i,j} (A_{ij}^\circ(\lambda\tau, \xi \sqrt{\lambda}) - A_{ij}^\circ(0,0)) \frac{\partial^2 \Delta_0}{\partial \xi_i \partial \xi_j} + \sum_i a_i^\circ(\lambda\tau, \xi \sqrt{\lambda}) \sqrt{\lambda} \frac{\partial \Delta_0}{\partial \xi_i} +$$

$$+ \frac{A_{nn}^\circ(\lambda\tau, \xi \sqrt{\lambda}) - A_{nn}^\circ(0,0) - \sum_i \frac{\partial A_{nn}^\circ}{\partial x_i}(0,0) \xi_i \sqrt{\lambda}}{2 \sqrt{\lambda}} \frac{\partial^2 v_0}{\partial \xi_n^2} +$$

$$+ (a_n^\circ(\tau\lambda, \xi \sqrt{\lambda}) - a_n^\circ(0,0)) \frac{\partial v_0}{\partial \xi_n}$$

with the initial and boundary value conditions: $\nabla_\lambda(0, \xi) = 0$ for $\xi_n > 0$, $\nabla_\lambda(t, \xi) = 0$ for $t > 0$, $\xi_n = 0$.

The functions $\partial \Delta_0 / \partial \xi_i$, $\partial^2 \Delta_0 / \partial \xi_i \partial \xi_j$, $\partial v_0 / \partial \xi_n$, $\partial^2 v_0 / \partial \xi_n^2$ are Hölder continuous in the region S of the point [1, 0]. If Theorem 4, Chap. IV [3] is applied to the region $0 < \xi_n < 1$, $-1 < \xi_i < 1$, $\tau > 1/2$ and to the region S of the point [1, 0], it implies

$$\lim_{\lambda \to 0} \frac{\partial \Delta_\lambda}{\partial \xi_i}(1,0) = \frac{\partial \Delta_0}{\partial \xi_i}(1,0), \quad \lim_{\lambda \to 0} \frac{\partial^2 \Delta_\lambda}{\partial \xi_i \partial \xi_j}(1,0) = \frac{\partial^2 \Delta_0}{\partial \xi_i \partial \xi_j}(1,0)$$

with respect to the assumptions about A_{nn}. Considering the definition of Δ_λ and (4,7), (3,4) we can easily calculate

$$(4,20)\quad \lim_{t\to 0}\frac{\partial u}{\partial x_n}(t,0)\sqrt{t} = \lim_{t\to 0}\frac{\partial v}{\partial x_n}(t,0)\sqrt{t} = \lim_{\lambda\to 0}\frac{\partial v_\lambda}{\partial \xi_n}(1,0) = \frac{\partial v_0}{\partial \xi_n}(1,0),$$

$$\lim_{t\to 0}\frac{\partial^2 u}{\partial x_i\,\partial x_n}(t,0)\Big/\left(-\frac{\partial u}{\partial x_n}(t,0)\right) = \lim_{\lambda\to 0}\frac{1}{\sqrt{\lambda}}\frac{\partial^2 v_\lambda}{\partial \xi_i\,\partial \xi_n}(1,0)\Big/\left(-\frac{\partial v_\lambda}{\partial \xi_n}(1,0)\right) =$$

$$= \lim_{\lambda\to 0}\frac{\partial^2 \Delta_\lambda}{\partial \xi_i\,\partial \xi_n}(1,0)\Big/\left(-\frac{\partial v_\lambda}{\partial \xi_n}(1,0)\right) = \frac{\partial^2 \Delta_0}{\partial \xi_i\,\partial \xi_n}(1,0)\Big/\left(-\frac{\partial v_0}{\partial \xi_n}(1,0)\right)$$

with the exception $i = n$. This implies

$$(4,22)\quad \lim_{t\to 0}\frac{\partial^2 u}{\partial x_i\,\partial x_n}(t,0)\Big/\left(-\frac{\partial u}{\partial x_n}(t,0)\right) = \frac{1}{2}\frac{\partial A^\circ_{nn}}{\partial x_i}(0,0) = \frac{1}{2}\frac{\partial A_{nn}}{\partial x_i}(0,0),$$

since

$$\frac{\partial^2 \Delta_0}{\partial \xi_i\,\partial \xi_n}(t,0) = \frac{1}{\sqrt{(2\pi t)}}\frac{\partial A^\circ_{nn}}{\partial x_i}(0,0),\quad \frac{\partial v_0}{\partial \xi_n}(t,0) = -\sqrt{\frac{2}{\pi t}}$$

(see (4,9)) and

$$\frac{\partial A^\circ_{nn}}{\partial x_i}(0,0) = \frac{\partial A_{nn}}{\partial x_i}(0,0).$$

Let the region D be bounded. Let $P = 0$ and let it be possible to express the boundary locally by $x_n = h(x_1, \ldots, x_{n-1})$. First the coefficients of (3,1) can be extended onto the whole half-space $x_n \geqq 0$ so that the assumptions from the beginning of this proof are fulfilled. Since $\partial h/\partial x_i(0) = 0$, Lemma 2 implies that there exists a function $h^\circ(x_1, \ldots, x_{n-1})$ defined on the whole R_{n-1} and a positive number δ_1 such that $h = h^\circ$ in the δ_1-neighbourhood of 0, the first and second partial derivatives of h° being bounded and uniformly Hölder continuous and equation (3,10) being uniformly parabolic in a region $D^+ = \{x;\, x_n > h^\circ(x_1, \ldots, x_{n-1})\}$, $D \subset D^+$. These assumptions imply that D and D^+ have the same boundary in a δ_1-neighbourhood of 0. Let $u_1(t, x)$ be the bounded solution of (3,1) fulfilling (3,2) and (3,3). Denote by $u_2(t, x)$ the solution fulfilling the same conditions with D^+ instead of D. The application of Theorem 4, Chap. IV [3] to $z = u_2 - u_1$ yields

$$\lim_{t\to 0}\left[\frac{\partial u_2}{\partial x_n}(t,0) - \frac{\partial u_1}{\partial x_n}(t,0)\right] = \lim_{t\to 0}\frac{\partial z}{\partial x_n}(t,0) = 0$$

and

$$\lim_{t\to 0}\left[\frac{\partial^2 u_2}{\partial x_i\,\partial x_j}(t,0) - \frac{\partial^2 u_1}{\partial x_i\,\partial x_j}(t,0)\right] = \lim_{t\to 0}\frac{\partial^2 z}{\partial x_i\,\partial x_j}(t,0) = 0.$$

(4,20) implies

$$\lim_{t\to 0}\frac{\partial u_2}{\partial x_n}(t,0) = -\infty$$

so that

$$\lim_{t\to 0}\frac{\partial u_2}{\partial x_n}(t,0)\Big/\left(\frac{\partial u_1}{\partial x_n}(t,0)\right) = 1 \quad\text{and}\quad \lim_{t\to 0}\frac{\partial u_1}{\partial x_n}(t,0) = -\infty.$$

This means

$$\lim_{t\to 0} \frac{\partial^2 u_1}{\partial x_i\, \partial x_j}(t, 0) \Big/ \left(-\frac{\partial u_1}{\partial x_n}(t, 0)\right) = \lim_{t\to 0} \frac{\partial^2 u_2}{\partial x_i\, \partial x_j}(t, 0) \Big/ \left(-\frac{\partial u_2}{\partial x_n}(t, 0)\right)$$

so that (4,21) is valid also for bounded regions.

If equation (3,1) does not fulfil (4,1), then by means of Lemma 2 this relations can be achieved by a linear transformation. Substituting (4,22) into (4,2) for l_i and recalling the considerations which proved Remark 3 we obtain easily the statement of Theorem.

Remark 8. For the derivative of the solution $u(t, x)$ with respect to the inward normal the formula

$$(4,22) \qquad \lim_{t\to 0} \sqrt{(t)}\, \frac{\partial u}{\partial x_n}(t, 0) = - \sqrt{\left(\frac{2}{\pi A_{nn}(0, 0)}\right)}$$

holds.

The remark immediately follows from (4,21), (4,9) and from the fact that $L_{nn} = (A_{nn}(0, 0))^{-1/2}$ where L is the transformation matrix given in Lemma 2.

5.

This section will be devoted to the case when the time t is fixed but the right-hand side of the differential equation is assumed to be small. Let a differential equation

$$(5,1) \qquad \frac{\partial u}{\partial t} = \frac{1}{2}\, \varepsilon^2 \sum_{i,j} A_{ij}(t, x, \varepsilon)\, \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \varepsilon \sum_i a_i(t, x, \varepsilon)\, \frac{\partial u}{\partial x_i}$$

be given where coefficients $a_i(t, x, \varepsilon)$, $A_{ij}(t, x, \varepsilon)$ can depend on ε in a region $Q_\varepsilon = (0, L) \times D_\varepsilon$ where the region D_ε may also depend on ε. The expression $u(t, x, \varepsilon)$ denotes the bounded solution of (5,1) fulfilling conditions

$$(5,2) \qquad u(0, x, \varepsilon) = 0 \quad \text{for} \quad x \in D_\varepsilon\,,$$

$$(5,3) \qquad u(t, x, \varepsilon) = 1 \quad \text{for} \quad t > 0\,, \quad x \in \dot{D}_\varepsilon\,.$$

Theorem 6. *Let points P_ε, $P_\varepsilon \in \dot{D}_\varepsilon$ and a number $t_0 > 0$ be given. Assume that there exist numbers $\delta, \gamma, \delta > 0$ such that the boundaries $\dot{D}_\varepsilon$ in the $\delta\varepsilon^\gamma$-neighbourhood of P_ε can be expressed by means of $x_n = h_\varepsilon(x_1, \ldots, x_{n-1})$ where $x_1, \ldots, x_n$ is the local coordinate system in P_ε (the dependence on ε need not be indicated). Further assume that $|\varepsilon^\gamma\, \partial^2 h_\varepsilon/(\partial x_i\, \partial x_j)| \leqq M$ and*

$$\left|\frac{\partial h_\varepsilon}{\partial x_i}(x_1, \ldots, x_{n-1}) - \frac{\partial h_\varepsilon}{\partial x_i}(\xi_1, \ldots, \xi_{n-1})\right| \leqq K \sum_{i=1}^{n-1} |x_i - \xi_i|^\alpha,$$

$$\left|\frac{\partial^2 h_\varepsilon}{\partial x_i\, \partial x_j}(x_1, \ldots, x_{n-1}) - \frac{\partial^2 h_\varepsilon}{\partial x_i\, \partial x_j}(\xi_1, \ldots, \xi_{n-1})\right| \leqq \frac{K}{\varepsilon} \sum_{i=1}^{n-1} |x_i - \xi_i|^\alpha$$

in the $\delta\varepsilon^\gamma$*-neighbourhood of* P_ε, $\lim_{\varepsilon\to 0} \varepsilon^\gamma \partial^2 h_\varepsilon/(\partial x_i\, \partial x_j)\,(P_\varepsilon) = h_{ij}$ *where* h_{ij} *form a positively definite matrix and* M *is independent of* ε. *The coefficients* $a_i(t, x, \varepsilon)$, $A_{ij}(t, x, \varepsilon)$ *are uniformly Hölder continuous with respect to* ε *with an exponent* α, $\alpha > 0$,

$$|a_i(t, P_\varepsilon, \varepsilon) - a_i(t, P_\varepsilon, 0)| \leqq K\varepsilon^\alpha\,, \quad |A_{ij}(t, P_\varepsilon, \varepsilon) - A_{ij}(t, P_\varepsilon, 0)| \leqq K\varepsilon^\alpha$$

and A_{ij} *are uniformly positively definite with respect to* t, x, ε *on* $\bar{Q}_\varepsilon$. *Let there exist a number* β, $\beta \geqq 0$ *so that* $a_n(t_0, P_\varepsilon, \varepsilon)\, \varepsilon^{-\beta} \to A > 0$. *Assume that the following inequalities are valid*

$$(5,4) \qquad 2\alpha + \gamma - \beta > 1\,, \quad \alpha + 2\gamma \leqq 2\,, \quad \beta \leqq 1 - \gamma \quad (0 < \alpha \leqq 1,\ \beta \geqq 0)\,.$$

If $\beta < 1 - \gamma$ *in the last inequality, then there exists a number* $\varepsilon_0 > 0$ *so that* $u(t, x, \varepsilon)$ *are convex at the points* $[t_0, P_\varepsilon]$ *for* $0 < \varepsilon \leqq \varepsilon_0$. *If* $\beta = 1 - \gamma$ *in the last inequality of* (5,4) *and* $2A - \operatorname{trace}(L^T)^{-1} HL^{-1} > 0$ (*where the matrix* H *consists of elements* $H_{ij} = h_{ij}$ *for* $i, j = 1, \dots, n-1$, $H_{ni} = H_{in} = 0$), *then there exists a number* $\varepsilon_0 > 0$ *so that* $u(t, x, \varepsilon)$ *is again convex at the point* $[t_0, P_\varepsilon]$ *for* $0 < \varepsilon \leqq \varepsilon_0$.

If the solution $u(t, x, \varepsilon)$ *is convex at* $[t_0, P_\varepsilon]$ *for all sufficently small* ε *and* $\beta = 1 - \gamma$, *then* $2A - \operatorname{trace}(L^T)^{-1} HL^{-1} \geqq 0$.

Proof. Without any loss of generality it can be assumed that $P_\varepsilon = 0$ and, with respect to Lemma 2

$$(5,5) \qquad A_{ii}(t_0, 0, \varepsilon) = 1\,, \quad A_{ij}(t_0, 0, \varepsilon) = 0\,, \quad i \neq j\,.$$

By means of the transformation $x = \varepsilon y$, $u(t, x) = \bar{u}(t, y)$ equation (5,1) becomes

$$(5,6) \qquad \frac{\partial \bar{u}}{\partial t} = \tfrac{1}{2} \sum_{i,j} A_{ij}(t, \varepsilon y, \varepsilon) \frac{\partial^2 \bar{u}}{\partial y_i\, \partial y_j} + \sum_i a_i(t, \varepsilon y, \varepsilon) \frac{\partial \bar{u}}{\partial y_i}$$

the regions being transformed onto $\bar{D}_\varepsilon$. The $\delta\varepsilon^{\gamma-1}$-neighbourhood of 0 is the image of the $\delta\varepsilon^\gamma$-neighbourhood of 0. The boundary of $\bar{D}_\varepsilon$ can be then described by the function $y_n = \bar{h}_\varepsilon(y_1, \dots, y_{n-1}) = (1/\varepsilon)\, h_\varepsilon(x_1, \dots, x_{n-1})$. As it was shown in the proof of Remark 3 it is necessary and sufficient to prove that the determinant of the matrix V is positive (or nonnegative) where the matrix is defined as follows: $V_{ij} = \partial^2 \bar{h}_\varepsilon(\partial y_i\, \partial y_j)\,(0) = \varepsilon\, \partial^2 h_\varepsilon(\partial x_i\, \partial x_j)\,(0)$, $V_{ni} = V_{in} = \partial^2 \bar{u}/(\partial y_i\, \partial y_n)\,(t_0, 0)$, $i = 1, \dots, n-1$ and $V_{nn} = \partial^2 \bar{u}/\partial y_n^2(t_0, 0)\,(-\partial \bar{u}/\partial y_n(t_0, 0))$. Define a matrix W_ε so that

$$W_{\varepsilon ij} = \varepsilon\, \partial^2 h_\varepsilon/(\partial x_i\, \partial x_j)\,(0) \quad \text{for} \quad i, j = 1, \dots, n-1\,,$$

$$W_{\varepsilon in} = W_{\varepsilon ni} = \partial^2 \bar{u}/(\partial y_i\, \partial y_n)\,(t_0, 0)/(-\partial \bar{u}/\partial y_n(t_0, 0)) \quad \text{for} \quad i < n$$

$$W_{\varepsilon nn} = 2a_n(t_0, 0, \varepsilon) - \varepsilon \sum_{i=1}^{n-1} \partial^2 h/\partial x_i^2(0)\,.$$

Immediately it follows that $\det V = (-\partial \bar{u}/\partial y_n)^2 \det W$. Hence the necessary and sufficient condition for the convexity of $\bar{u}(t, x, \varepsilon)$ at $[t_0, 0]$ is the positivity of the determinant of W_ε. Put $W_{0ij} = h_{ij}$, $W_{0in} = W_{0ni} = \lim_{\varepsilon\to 0} \varepsilon^{(\gamma-\beta-1)/2}\, \partial^2\bar{u}/\partial y_i\, \partial y_n(t_0, 0)$: : $(-\partial\bar{u}/\partial y_n(t_0, 0))$ for $i \neq n$ and $W_{0nn} = 2A - \lim_{\varepsilon\to 0} \varepsilon^{1-\gamma-\beta} \sum_{i=1}^{n-1} h_{ii}$. Obviously

$$\lim_{\varepsilon\to 0} \varepsilon^{-\beta+(n-1)(1-\gamma)} \det W_\varepsilon = \det W_0 .$$

This yields that the positivity of det W_0 is a sufficient condition for the convexity of $\bar{u}(t_0, x, \varepsilon)$ at $[t_0, 0]$ for small ε and the nonnegativity of det W_0 is a necessary condition. The next part of the proof will be devoted to the fact that W_{0in} equals zero. Under such condition the statement of the Theorem is obvious.

First the coefficients $a_i(t, x, \varepsilon)$, $A_{ij}(t, x, \varepsilon)$ can be extended onto the whole $\langle 0, L\rangle \times R_n$ so that they are Hölder continuous, bounded and the extended equation is uniformly parabolic (with respect to all variables t, x, ε). According to Lemma 2 there exists a function $h^\circ_\varepsilon(y_1, \ldots, y_{n-1})$ defined on the whole R_{n-1} such that $\bar{h}_\varepsilon = h^\circ_\varepsilon$ in the $\delta\varepsilon^{\gamma-1}$-neighbourhood of 0, h°_ε and the first and second partial derivatives being uniformly bounded and uniformly Hölder continuous and such that equation (3,10) is uniformly positively definite. With respect to the assumptions about h°_ε it is $|\partial^2 h^\circ_\varepsilon/(\partial y_i\, \partial y_j)| \leqq M\varepsilon^{1-\gamma}$ in the $\delta\varepsilon^{\gamma-1}$-neighbourhood of 0 and $\partial h^\circ_\varepsilon/\partial y_i(0) = 0$. Hence $|\partial h^\circ_\varepsilon/\partial y_i| \leqq M\delta$ in the $\delta\varepsilon^{\gamma-1}$-neighbourhood of 0. Using the second transformation defined in Lemma 2, i.e. $y_i = z_i$, $i = 1, \ldots, n-1$, $y_n = z_n + h^\circ_\varepsilon(z_1, \ldots, z_{n-1})$, $\bar{u}(t, y) = v(t, z)$, equation (5,6) is transformed onto

$$(5,7) \qquad \frac{\partial v}{\partial t} = \tfrac{1}{2} \sum_{i,j} A^\circ_{ij}(t, z, \varepsilon) \frac{\partial^2 v}{\partial z_i\, \partial z_j} + \sum_i a^\circ_i(t, z, \varepsilon) \frac{\partial v}{\partial z_i},$$

regions $\bar{D}_\varepsilon$ are transformed onto D°_ε such that conditions (5,2) and (5,3) are transformed onto

$$(5,8) \qquad v(0, z) = 0 \quad \text{for} \quad z \in D^\circ_\varepsilon ,$$

$$(5,9) \qquad v(t, z) = 1 \quad \text{for} \quad t > 0 , \quad z \in \dot{D}^\circ_\varepsilon .$$

Certainly there exists a number δ_1, $\delta_1 > 0$ such that the $\delta_1\varepsilon^{\gamma-1}$-neighbourhood of 0 is a subset of the image of the $\delta\varepsilon^{\gamma-1}$-neighbourhood of 0.

Let $B(t, x, \varepsilon))$ be a Hölder continuous matrix function such that $BB^T = A^\circ$. According to Theorem 1, the Îto equation

$$(5,10) \qquad \mathrm{d}z = a^\circ(L - t, z, \varepsilon)\, \mathrm{d}t + B(L - t, z, \varepsilon)\, \mathrm{d}q$$

corresponds to (5,7) where $q(t)$ is an n-dimensional Wiener process. With respect to the Hölder continuity it holds

$$(5,11) \quad |a^\circ(t, z, \varepsilon)| \leqq K_1(1 + |z|), \quad |B(t, z, \varepsilon)| \leqq K_1(1 + |z|), \quad t \in \langle 0, L\rangle$$

where K_1 is independent of ε. It is possible to approximate a, B by Lipschitz continuous (in z) vector and matrix functions $a_m(t, z, \varepsilon)$, $B_m(t, z, \varepsilon)$. Denote by $z_m(t)$ the solution of the corresponding Îto equations

$$dz = a_m(L - t, z, \varepsilon)\, dt + B_m(L - t, z, \varepsilon)\, dq$$

which have the initial value z_0. Due to (5,11), (4,7) and further inequalities from [6] it holds

$$\sqrt{E \sup_{\langle 0,t\rangle} |z_m(\tau)|^2} \leqq \sqrt{E\, |z_m(0)|^2} + K_1 t +$$
$$+ K_1 \int_0^t \sqrt{E \sup_{\langle 0,\tau\rangle} |z_m(\xi)|^2}\, d\tau + 2K_1 \sqrt{(nt)} +$$
$$+ 2K_1 \sqrt{n} \sqrt{\left(\int_0^t E \sup_{\langle 0,\tau\rangle} |z_m(\xi)|^2\, d\tau\right)}\,.$$

By Lemma 2 [6] $\sqrt{E \sup_{\langle 0,L\rangle} |z_m(\tau)|^2} \leqq N(1 + |z_0|)$ where N depends only on K_1 and on the dimension n. This yields

$$(5,12) \qquad P\{\exists\{\tau : \tau \in \langle 0, L\rangle,\ |z_m(\tau)| > \delta_1 \varepsilon^{\gamma-1}\}\} \leqq N^2(1 + |z_0|)^2\, \delta_1^{-2} \varepsilon^{2-2\gamma} \leqq$$
$$\leqq N^2(1 + |z_0|)^2\, \delta_1^{-2} \varepsilon^{\alpha}$$

(the second inequality from (5,4) is used and $\varepsilon < 1$ is assumed). By Theorem 1 iii) the probability $P\{\exists\{\tau \in \langle 0, t\rangle,\ z_m(\tau) \notin D_\varepsilon^\circ\}\}$ equals $v_m(t, z)$ where $v_m(t, z)$ is the bounded solution of

$$(5,13) \qquad \frac{\partial v}{\partial t} = \tfrac{1}{2} \sum_{i,j} A_{ijm}(t, z, \varepsilon) \frac{\partial^2 v}{\partial z_i\, \partial z_j} + \sum_i a_{im}(t, z, \varepsilon) \frac{\partial v}{\partial z_i}$$

$(A_m = B_m B_m^T)$ fulfilling (5,8) and (5,9). Let $w_m(t, z)$ be the bounded solution of (5,13) fulfilling $w_m(0, z) = 0$ for $z_n > 0$ and $w_m(t, z) = 1$ for $t > 0$, $z_n = 0$. By Theorem 1 iii) again

$$w_m(t, z_0) = P\{\exists\{\tau : \tau \in \langle 0, t\rangle,\ (z_m(\tau))_n = 0\}\}$$

where $z_m(t)$ is the same solution of the Îto equation as in (5,12). $(z_m(t))_n$ is the n-th coordinate of the solution $z_m(t)$. Obviously

$$v_m(t, z) - w_m(t, z) \leqq 2P\{\exists\{\tau : \tau \in \langle 0, t\rangle,\ |z_m(\tau)| > \delta_1 \varepsilon^{\gamma-1}\}\} \leqq$$
$$\leqq 2N^2(1 + |z_0|)^2\, \delta_1^{-2} \varepsilon^{\alpha}\,.$$

The last inequality holds due to (5,12). Since the coefficients on the right-hand side do not depend on the Lipschitz coefficients of a_m, B_m, it follows that

$$(5,14) \qquad |v(t, z) - w(t, z)| \leqq 2N^2(1 + |z_0|)^2\, \delta_1^{-2} \varepsilon^{\alpha}\,.$$

The function $w(t, z)$ is the bounded solution of (5,7) fulfilling $w(0, z) = 0$ for $z_n > 0$, $w(t, z) = 1$ for $t > 0$, $z_n = 0$. Since w depends actually on ε it will be written as $w(t, z, \varepsilon)$.

Let $w(t, z, 0)$ be the bounded solution of

$$\frac{\partial w}{\partial t} = \tfrac{1}{2}\sum_{i,j} A^{\circ}_{ij}(t, 0, 0) \frac{\partial^2 w}{\partial z_i \, \partial z_j} + \sum_i a^{\circ}_i(t, 0, 0) \frac{\partial w}{\partial z_i} \tag{5,15}$$

fulfilling $w(0, z, 0) = 0$ for $z_n > 0$, $w(t, z, 0) = 1$ for $t > 0$, $z_n = 0$. The difference $\Delta(t, z, \varepsilon) = w(t, z, \varepsilon) - w(t, z, 0)$ is then the bounded solution of

$$\begin{aligned} \frac{\partial \Delta}{\partial t} &= \frac{1}{2}\sum_{i,j} A^{\circ}_{ij}(t, z, \varepsilon) \frac{\partial^2 \Delta}{\partial z_i \, \partial z_j} + \sum_i a^{\circ}_i(t, z, \varepsilon) \frac{\partial \Delta}{\partial z_i} + \\ &+ \tfrac{1}{2}(A^{\circ}_{nn}(t, z, \varepsilon) - A^{\circ}_{nn}(t, 0, 0)) \frac{\partial^2 w(t, z, 0)}{\partial z_n^2} + (a^{\circ}_n(t, z, \varepsilon) - a^{\circ}_n(t, 0, 0)) \frac{\partial w(t, z, 0)}{\partial z_n} \end{aligned} \tag{5,16}$$

fulfilling $\Delta(0, z, \varepsilon) = 0$ for $z_n > 0$, $\Delta(t, z, \varepsilon) = 0$ for $t \geqq 0$, $z_n = 0$. The corresponding Îto equation to (5,15) is now a scalar equation

$$d\zeta = a^{\circ}_n(L - t, 0, 0)\, dt + (\sqrt{A^{\circ}_{nn}}\,(L - t, 0, 0))\, dq$$

which has the solution

$$\zeta(t) = \zeta_0 + \int_0^t a^{\circ}_n(L - \tau, 0, 0)\, d\tau + q\left(\int_0^t A^{\circ}_{nn}(L - \tau, 0, 0)\, d\tau\right).$$

Since $w(t, \zeta_0, 0) = P\{\exists\{\tau : \tau \in \langle 0, t\rangle, \zeta(\tau) = 0\}\}$ there exist constants K_2, c such that

$$|w(t, z, 0)| \leqq K_2 \exp\left\{-c \frac{z_n^2}{t}\right\}. \tag{5,17}$$

According to Theorem 6 Chap. III [3] and to the fact that the coefficients of (5,15) do not depend on z there exists a constant K_3 such that

(5,18)

$$\left|\frac{\partial w}{\partial z_n}(t, z, 0)\right| \leqq K_3 t^{-1/2} \exp\left\{-c \frac{z_n^2}{t}\right\}, \quad \left|\frac{\partial^2 w}{\partial z_n^2}(t, z, 0)\right| \leqq K_3 t^{-1} \exp\left\{-c \frac{z_n^2}{t}\right\}.$$

Let $G_\varepsilon(t, z; \tau, \eta)$ be the Green function of (5,16) in the half-space $z_n \geqq 0$. Since the coefficients are uniformly Hölder continuous and the equation is uniformly parabolic, Theorem 16.3 [4] can be used to show that G_ε fulfils inequalites (4,11) and (4,12). By the same method as in the proof of Theorem 5 it can be proved

$$\begin{aligned} \Delta(t, z, \varepsilon) &= \int_0^t \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} G_\varepsilon(t, z; \tau, \eta) \Big\{[(a^{\circ}_n(\tau, \eta, \varepsilon) - \\ &- a^{\circ}_n(\tau, 0, 0)] \frac{\partial w}{\partial z_n}(\tau, \eta, 0) + \tfrac{1}{2}[A^{\circ}_{nn}(\tau, \eta, \varepsilon) - A^{\circ}_{nn}(\tau, 0, 0)] \times \\ &\times \frac{\partial^2 w}{\partial z_n^2}(\tau, \eta, 0)\Big\}\, d\eta_n\, d\eta_{n-1} \ldots d\eta_1\, d\tau. \end{aligned} \tag{5,19}$$

By means of (4,11), (5,17) and

$$(5,20) \qquad |a_n^\circ(t, z, \varepsilon) - a_n^\circ(t, 0, 0)| \leqq K_4\varepsilon^\alpha(1 + |z|), \quad t \in \langle 0, L \rangle,$$
$$|A_{ij}^\circ(t, z, \varepsilon) - A_{ij}^\circ(t, 0, 0)| \leqq K_4\varepsilon^\alpha(1 + |z|), \quad t \in \langle 0, L \rangle$$

the following inequality is easily proved

$$(5,21) \qquad |\Delta(t, z, \varepsilon)| \leqq K_5\varepsilon^\alpha \exp\left\{-c\frac{|z|^2}{t}\right\}(1 + |z|),$$

where the constants K_5, c do not depend on ε. (5,14) and (5,21) give the estimate

$$(5,22) \qquad |v(t, z) - w(t, z, 0)| \leqq K_6\varepsilon^\alpha \quad \text{for} \quad |z| < \delta_1/2, \quad t \in \langle 0, L \rangle$$

(inequalities (5,4) imply $\gamma \leqq 1$). The difference $\nabla(t, z, \varepsilon) = v(t, z, \varepsilon) - w(t, z, 0)$ is again the bounded solution of (5,16) fulfilling $\nabla(0, z, \varepsilon) = 0$ for $z_n > 0$, $\nabla(t, z, \varepsilon) = 0$ for $t \geqq 0$, $z_n = 0$, $|z| < \delta_1\varepsilon^{\gamma-1}$ and $|\nabla(t, z, \varepsilon)| \leqq 1$ everywhere. An application of Theorem 4, Chap. IV, [3] to equation (5,16) yields

$$(5,23) \qquad \left|\frac{\partial \nabla}{\partial z_n}(t_0, 0, \varepsilon)\right| \leqq K_7\varepsilon^\alpha, \quad \left|\frac{\partial^2 \nabla}{\partial z_i\, \partial z_n}(t_0, 0, \varepsilon)\right| \leqq K_7\varepsilon^\alpha$$

with respect to (5,20) and (5,22). The constant K_7 is independent of ε. (5,23) implies that $\partial v/\partial z_n(t, 0, \varepsilon)$ converges to $\partial w/\partial z_n(t, 0, 0)$ for $\varepsilon \to 0$ and this value is nonzero according to Theorem 14 [3]. The second inequality from (5,23) gives

$$|\partial^2 v/(\partial z_i\, \partial z_n)(t, 0, \varepsilon)| \leqq K_7\varepsilon^\alpha \quad \text{for} \quad i \neq n$$

and since $\partial^2 \bar{u}/(\partial y_i\, \partial y_n)(t_0, 0) = \partial^2 v/(\partial z_i\, \partial z_n)(t_0, 0)$, it holds $|\partial^2 \bar{u}/(\partial y_i\, \partial y_n)(t_0, 0)| \leqq$ $\leqq K_7\varepsilon^\alpha$. This implies $|W_{0in}| \leqq \lim \varepsilon^{(2\alpha+\gamma-\beta-1)/2}K_7 = 0$, the last equality being a consequence of (5,4).

If (5,5) is not valid, then the first transformation described in Lemma 2 is used and the statement of the Theorem is easily achieved.

6.

In this section a modification of Theorem 4 for symmetric regions will be given. In this sense the variety of regions will be more restrictive. On the other hand, the conditions on $a_i(t, x_i)$ are weaker since they need not be linear in x_i.

Definition 4. *A region D is symmetric with respect to the axes $x_1, \ldots, x_n$, if $[x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n] \in D$ implies $[x_1, x_2, \ldots, x_{k-1}, -x_k, x_{k+1}, \ldots, x_n] \in$ $\in D$ for every index $k = 1, \ldots, n$.*

Let a parabolic differential equation

$$(6,1)\qquad \frac{\partial u}{\partial t} = \frac{1}{2}\sum_{i,j} \Lambda_{ij}(t, x_i, x_j)\, \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_i a_i(t, x_i)\, \frac{\partial u}{\partial x_i}$$

be given. We shall consider bounded solutions fulfilling

$$(6,2)\qquad u(0, x) = 0 \quad \text{for} \quad x \in D\,,$$

$$(6,3)\qquad u(t, x) = 1 \quad \text{for} \quad t > 0\,, \quad x \in \dot{D}\,.$$

Theorem 7. *Let a region D be symmetric with respect to the axes $x_1, \ldots, x_n$ and fulfil condition (B). Let $a_i(t, x_i)$ depend only on t and x_i in the way that they are odd functions in x_i and $\partial^2 a_i/\partial x_i^2(t, x) \geqq 0$. Let $\Lambda_{ij}(t, x_i, x_j)$ be linear odd functions in x_i, x_j for $i \neq j$ and let $\Lambda_{ii}(t, x_i)$ depend only on t and x_i in the way that they are even functions in x_i. All coefficients of (6,1) are assumed to be Hölder continuous. If the bounded solution of (6,1) fulfilling (6,2) and (6,3) is sharply convex along the axes $x_1, \ldots, x_n$ at all points $[t, x_1, \ldots, x_n] \in (0, L\rangle \times \dot{D}$, then the solution $u(t, x)$ is convex along the axes $x_1, \ldots, x_n$ in the region Q.*

Proof. The symmetry of the region and of the coefficients implies $u(t, x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) = u(t, x_1, \ldots, x_{k-1}, -x_k, x_{k+1}, \ldots, x_n)$ for every k which means that it is sufficient to investigate $u(t, x)$ in a domain $\langle 0, L\rangle \times D_k$ where $D_k = \{x; x \in D, x_k \geqq 0\}$. First we shall prove

$$(6,4)\qquad \frac{\partial u}{\partial x_k}(t, x) \geqq 0 \quad \text{in} \quad (0, L) \times D_k\,.$$

The solution $u(t, x)$ can be approximated by $u^{(n)}(t, x)$ where $u^{(n)}(t, x)$ is the bounded solution of (6,1) fulfilling (6,2) and

$$(6,5)\qquad u^{(n)}(t, x) = \varphi^{(n)}(t) \quad \text{for} \quad x \in \dot{D}$$

where $\varphi^{(n)}(t) = 1$ for $t \geqq 1/n$, $\varphi^{(n)}(t) = 0$ for $t \leqq 1/n^2$, $0 \leqq \varphi^{(n)}(t) \leqq 1$ for all t and $\varphi^{(n)}(t)$ has the derivatives of all orders. Put $v_k^{(n)}(t, x) = \partial u^{(n)}/\partial x_k(t, x)$. The function $v_k^{(n)}(t, x)$ fulfils

$$(6,6)\qquad \frac{\partial v_k}{\partial t} = \frac{\partial a_k}{\partial x_k} v_k + \sum_i \left(a_i + \frac{\partial \Lambda_{ik}}{\partial x_k} - \tfrac{1}{2}\delta_{ik} \frac{\partial \Lambda_{kk}}{\partial x_k} \right) \frac{\partial v_k}{\partial x_i} + \frac{1}{2}\sum_{i,j} \Lambda_{ij} \frac{\partial^2 v_k}{\partial x_i\, \partial x_j}\,.$$

By Theorem 14, Chap. II [3] it follows $\partial u/\partial \nu(t, x) < 0$ where ν is the inward normal at the point $x \in \dot{D}$. This yields

$$(6,7)\qquad v_k^{(n)}(t, x) = \frac{\partial u^{(n)}}{\partial x_k}(t, x) > 0 \quad \text{for} \quad x \in \dot{D}\,, \quad x_k > 0\,.$$

The symmetry of the region and of the coefficients gives

(6,8) $\quad v_k^{(n)}(t, x^*) = 0$ *where* x^* *are all points for which* $x_k^* = 0$.

Let $\eta = \max \{\partial a_k/\partial x_k(t, x); \ [t, x] \in \langle 0, L \rangle \times \bar{D}_k\}$ and put $v_k^{(n)}(t, x) = w_k^{(n)}(t, x)$ . . exp $\{\eta t\}$. The function $w_k^{(n)}(t, x)$ fulfils the equations

$$\frac{\partial w_k}{\partial t} = \left(\frac{\partial a_k}{\partial x_k} - \eta\right) w_k + \sum_i \left(a_i + \frac{\partial A_{ik}}{\partial x_k} - \tfrac{1}{2}\delta_{ik} \frac{\partial A_{kk}}{\partial x_k}\right) \frac{\partial w_k}{\partial x_i} + \frac{1}{2} \sum_{i,j} A_{ij} \frac{\partial^2 w_k}{\partial x_i \, \partial x_j}$$

and the conditions (6,2), (6,7) and (6,8). With regard to the maximum principle this implies $w_k^{(n)}(t, x) \geqq 0$ as well as $v_k^{(n)}(t, x) \geqq 0$ in $\langle 0, L \rangle \times D_k$. The meaning of this inequality is that $u^{(n)}(t, x)$ are nondecreasing in x_k in $\langle 0, L \rangle \times D_k$. Since $u^{(n)}(t, x)$ approximate the solution $u(t, x)$, inequality (6,4) is proved.

Put $z_k(t, x) = \partial^2 u/\partial x_k^2(t, x)$. The function $z_k(t, x)$ fulfils the equation

$$\frac{\partial z_k}{\partial t} = \frac{\partial^2 a_k}{\partial x_k^2} \frac{\partial u}{\partial x_k} + \left(2 \frac{\partial a_k}{\partial x_k} + \frac{1}{2} \frac{\partial^2 A_{kk}}{\partial x_k^2}\right) z_k +$$

$$+ \sum_i \left(a_i + 2 \frac{\partial A_{ik}}{\partial x_k} - \delta_{ik} \frac{\partial A_{kk}}{\partial x_k}\right) \frac{\partial z_k}{\partial x_i} + \tfrac{1}{2} \sum_{i,j} A_{ij} \frac{\partial^2 z_k}{\partial x_i \, \partial x_j} .$$

With respect to (6,4) and to the assumption of Theorem about $\partial^2 a_k/\partial x_k^2$,

$$\frac{\partial z_k}{\partial t} \geqq \left(2 \frac{\partial a_k}{\partial x_k} + \frac{1}{2} \frac{\partial^2 A_{kk}}{\partial x_k^2}\right) z_k +$$

$$+ \sum_i \left(a_i + 2 \frac{\partial A_{ik}}{\partial x_k} - \delta_{ik} \frac{\partial A_{kk}}{\partial x_k}\right) \frac{\partial z_k}{\partial x_i} + \tfrac{1}{2} \sum_{i,j} A_{ij} \frac{\partial^2 z_k}{\partial x_i \, \partial x_j} .$$

Now the same method as in the proof of Theorem 4 can be used to complete the proof of Theorem 7.

7.

Theorems 4 and 7 show that it is sufficient to know the behaviour of $u(t, x)$ near to the boundary if the question of maximality is solved. In the previous sections the behaviour of $u(t, x)$ near to the boundary was discussed for small t or for parabolic differential equations with small right-hand sides. Now the problem will be discussed under the assumption that the fundamental solution is known. According to Remark 3 it is sufficient to know the behaviour of $\partial u/\partial x_n$ and $\partial^2 u/\partial x_n \, \partial x_i$ for $i = 1, \ldots, n - 1$ near to the boundary $(0, L\rangle \times \dot{D}$. Assume now for the sake of simplicity that a parabolic differential equation

(7,1) $$\frac{\partial u}{\partial t} = \sum_{i,j} A_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_i a_i(t, x) \frac{\partial u}{\partial x_i}$$

is given in the whole $\langle 0, L \rangle \times R_n$. As usually, $u(t, x)$ denotes the bounded solution of (7,1) fulfilling

$$(7,2) \qquad u(0, x) = 0 \quad \text{for} \quad x \in D\,,$$

$$(7,3) \qquad u(t, x) = 1 \quad \text{for} \quad t > 0\,, \quad x \in \dot{D}$$

where D is a given region in R_n.

Theorem 8. *Let the coefficients $a_i(t, x)$, $A_{ij}(t, x)$ and their derivatives $\partial a_i/\partial x_i$, $\partial A_{ij}/\partial x_j$, $\partial^2 A_{ij}/\partial x_i\, \partial x_j$ be Hölder continuous in $\bar{Q}$, the matrix $A(t, x)$ which consists of elements $A_{ij}(t, x)$ let be positively definite on $\bar{Q}$ and let the region D fulfil condition (B). If $u(t, x)$ is the bounded solution of (7,1) fulfilling (7,2), (7,3) and if $Z(t, x; \tau, \xi)$ is the fundamental solution of (7,1), then*

$$(7,4) \qquad \frac{\partial u}{\partial v^\circ}(t, x) = -2 \int_D \frac{\partial Z}{\partial v^\circ}(t, x; 0, \xi)\, d\xi\, +$$

$$+ 2 \int_0^t d\tau \int_{\dot{D}} \frac{\partial Z}{\partial v^\circ}(t, x; \tau, \xi) \sum_{i,j} A_{ij}(\tau, \xi) \cos(v, \xi_i) \cos(v, \xi_j) \frac{\partial u}{\partial v}(\tau, \xi)\, d\sigma_\xi$$

for $x \in \dot{D}$ where $\partial u/\partial v^\circ$ is the derivative with respect to the outer normal and with respect to the variables $x_1, \ldots, x_n$; $\partial u/\partial v$ is the derivative with respect to the outer normal but with respect to the variables $\xi_1, \ldots, \xi_n$ and $d\sigma_\xi$ denotes the elements of the surface $\dot{D}$, (v, x_i) being the angle between x_i and the normal v at $[\xi_1, \ldots, \xi_n]$.

Proof. With respect to the well-know Green formula and to (7,3) it holds ($\varepsilon > 0$)

$$(7,5) \qquad u(t, x) - 1 = \int_D Z(t, x; \varepsilon, \xi)\,(u(\varepsilon, \xi) - 1)\, d\xi\, +$$

$$+ \int_\varepsilon^t d\tau \int_{\dot{D}} Z(t, x; \tau, \xi) \sum_{i,j} A_{ij}(\tau, \xi) \cos(v, \xi_i) \cos(v, \xi_j) \frac{\partial u}{\partial v}(\tau, \xi)\, d\sigma_\xi\,.$$

Put $\varphi(t, \xi) = \sum_{i,j} A_{ij}(t, \xi) \cos(v, \xi_i) \cos(v, \xi_j)$. Since the conditions for (15,9) Chap. IV [4] are fulfilled where $n_i(x_0) = \cos(v, x_i)$ and

$$V(t, x) = \int_\varepsilon^t d\tau \int_{\dot{D}} Z(t, x; \tau, \xi)\, \varphi(\tau, \xi) \frac{\partial u}{\partial v}(\tau, \xi)\, d\sigma_\xi\,,$$

it holds

$$\varphi(t, x) \frac{\partial u}{\partial v^\circ}(t, x) = \varphi(t, x) \int_D \frac{\partial Z}{\partial v^\circ}(t, x; \varepsilon, \xi)\,(u(\varepsilon, \xi) - 1)\, d\xi\, +$$

$$+ \int_\varepsilon^t d\tau \int_{\dot{D}} \varphi(t, x) \frac{\partial Z}{\partial v^\circ}(t, x; \tau, \xi)\, \varphi(\tau, \xi) \frac{\partial u}{\partial v}(\tau, \xi)\, d\sigma_\xi + \tfrac{1}{2}\varphi(t, x) \frac{\partial u}{\partial v^\circ}(t, x)\,.$$

As $A(t, x)$ is positively definite the function $\varphi(t, x)$ is positive so that

$$\frac{\partial u}{\partial v^\circ}(t, x) = 2\int_D \frac{\partial Z}{\partial v^\circ}(t, x; \varepsilon, \xi)\,(u(\varepsilon, \xi) - 1)\,\mathrm{d}\xi + \\ + 2\int_\varepsilon^t \mathrm{d}\tau \int_{\dot D} \frac{\partial Z}{\partial v^\circ}(t, x; \tau, \xi)\,\varphi(\tau, \xi)\,\frac{\partial u}{\partial v}(\tau, \xi)\,\mathrm{d}\sigma_\xi\,.$$

According to Remark 8 (4,22) it is possible to pass to the limit for $\varepsilon \to 0$. Equation (7,4) is proved.

8.

In this section a formula for the second derivatives $\partial^2 u/(\partial x_i\,\partial x_n)$ on $\dot D$ will be derived. Since the formula is rather complicated it will be derived and introduced only under following simplifying assumptions: the origin 0 of the coordinate system belongs to $\dot D$, the region D is situated in the half-space $x_n > 0$ and there exists $\eta > 0$ such that

(8,1) $$\dot D \cap S(0, \eta) = \{x : x_n = 0,\ \|x\| < \eta\}$$

where $S(y, \eta)$ is the η-neighbourhood of y. It is also assumed that $A(t, 0)$ is the unit matrix for the given $t > 0$. These assumptions can be satisfied by means of Lemma 2. The existence of the principal value of the integral on the right-hand side of (8,2) will be proved during the proof of the Theorem.

Theorem 9. *Let the assumptions formulated in the beginning of this section and the assumptions of Theorem* 8 *be fulfilled. If all first partial derivatives of A_{ij} exist and $\partial A_{ij}/\partial x_k$ are Hölder continuous, then for $i \neq n$*

(8,2) $$\frac{\partial^2 u}{\partial x_i\,\partial x_n}(t, 0) = -\frac{2}{3}\int_D \frac{\partial^2 Z}{\partial x_i\,\partial x_n}(t, x; 0, \xi)\,\mathrm{d}\xi + \\ + \frac{2}{3}\int_0^t \mathrm{d}\tau(\text{v.p.})\int_{\dot D} \frac{\partial^2 Z}{\partial x_i\,\partial x_n}(t, 0; \tau, \xi)\,\varphi(\tau, \xi)\,\frac{\partial u}{\partial v}(\tau, \xi)\,\mathrm{d}\sigma_\xi + \\ + \frac{1}{6}\left[\frac{\partial \det A}{\partial x_i}(t, 0) + \sum_{\alpha=1}^{n-1}\frac{\partial A^{\alpha\alpha}}{\partial x_i}(t, 0) - \frac{\partial A^{nn}}{\partial x_i}(t, 0) - 2\frac{\partial \varphi}{\partial x_i}(0)\right]\frac{\partial u}{\partial v^\circ}(t, 0)\,.$$

Since the precise proof of the theorem would need some complicated calculations of integrals it will be only outlined.

With respect to (11,13), (11,23) [4] it holds

$Z(t, x; \tau, \xi) = \sum_{m=0}^{\infty}(-1)^m Z_m(t, x; \tau, \xi)$ where Z_0 is defined (see (11,2) [4])

$$Z_0(t, x - \xi; \tau, \xi) = (4\pi)^{-n/2}(t - \tau)^{-n/2}(\det A(\tau, \xi))^{-1/2} \times \\ \times \exp\left\{-\frac{\sum A^{\alpha\beta}(\tau, \xi)(x_\alpha - \xi_\alpha)(x_\beta - \xi_\beta)}{4(t - \tau)}\right\}.$$

$A(t, x)$ being the matrix which consists from the elements $A_{ij}(t, x)$ given in equation (7,1) and $A^{\alpha\beta}(t, x)$ being elements of the inverse matrix. The function Z_m are given by

$$(8,3) \qquad Z_m(t, x; \tau, \xi) = \int_\tau^t \mathrm{d}\lambda \int_{R_n} Z_0(t, x - y; \lambda, y)\, K_m(\lambda, y; \tau, \xi)\, \mathrm{d}y$$

and by (11,24) [4]

$$(8,4) \qquad K_m(t, x; \tau, \xi) = \int_\tau^t \mathrm{d}\lambda \int_{R_n} K(t, x; \lambda, y)\, K_{m-1}(\lambda, y; \tau, \xi)\, \mathrm{d}y$$

and $K_1 = K$ where K is given by (11,12) [4]. Inequality (11,25) can be rewritten in the form

$$|K_m(t, x; \tau, \xi)| \leqq c_m(t - \tau)^{(m\alpha - n - 2)/2} \exp\left\{-c\,\frac{|x - \xi|^2}{t - \tau}\right\}$$

where c_m are positive numbers, $\sum c_m < \infty$. Hence

$$(8,5) \qquad \left|\frac{\partial Z_m}{\partial x_i}(t, x; \tau, \xi)\right| \leqq K\bar{c}_m(t - \tau)^{(m\alpha - n - 1)/2} \exp\left\{-c\,\frac{|x - \xi|^2}{t - \tau}\right\}.$$

Equality (7,5) implies

$$\frac{\partial^2 u}{\partial x_i\, \partial x_n}(t, x) = \int_D \frac{\partial^2 Z}{\partial x_i\, \partial x_n}(t, x; \varepsilon, \xi)\,(u(\varepsilon, \xi) - 1)\, \mathrm{d}\xi + \\ + \int_\varepsilon^t \mathrm{d}\tau \int_{\dot{D}} \frac{\partial^2 Z}{\partial x_i\, \partial x_n}(t, x; \tau, \xi)\, \varphi(\tau, \xi)\, \frac{\partial u}{\partial \nu}(\tau, \xi)\, \mathrm{d}\sigma_\xi$$

if $x \in D$. Denote

$$V(t, x) = \int_\varepsilon^t \mathrm{d}\tau \int_{\dot{D}} \frac{\partial^2 Z}{\partial x_i\, \partial x_n}(t, x; \tau, \xi)\, \varphi(\tau, \xi)\, \frac{\partial u}{\partial \nu}(\tau, \xi)\, \mathrm{d}\sigma_\xi$$

and

$$V_m(t, x) = \int_\varepsilon^t \mathrm{d}\tau \int_{\dot{D}} \frac{\partial^2 Z_m}{\partial x_i\, \partial x_n}(t, x; \tau, \xi)\, \varphi(\tau, \xi)\, \frac{\partial u}{\partial \nu}(\tau, \xi)\, \mathrm{d}\sigma_\xi\,.$$

First the limit $\lim_{x \to 0} V_0(t, x)$ will be calculated. Denote $D^* = \dot{D} \cap S(0, \zeta)$, $D^+ = \dot{D} - D^*$ (for the sake of brevity, the dependence of D^* and D^+ on ζ is not indicated). Denote

$$V_{0\zeta}^*(t, x) = \int_\varepsilon^t \mathrm{d}\tau \int_{D^*} \frac{\partial^2 Z_0}{\partial x_i\, \partial x_n}(t, x; \tau, \xi)\, \varphi(\tau, \xi)\, \frac{\partial u}{\partial \nu}(\tau, \xi)\, \mathrm{d}\sigma_\xi$$

and

$$V_{0\zeta}^+(t, x) = \int_\varepsilon^t \mathrm{d}\tau \int_{D^+} \frac{\partial^2 Z_0}{\partial x_i\, \partial x_n}(t, x; \tau, \xi)\, \varphi(\tau, \xi)\, \frac{\partial u}{\partial \nu}(\tau, \xi)\, \mathrm{d}\sigma_\xi\,.$$

Obviously

$$\lim_{x\to 0} V_{0\zeta}^{+}(t, x) = \int_{\varepsilon}^{t} \mathrm{d}\tau \int_{D^+} \frac{\partial^2 Z_0}{\partial x_i\, \partial x_n}(t, 0; \tau, \xi)\, \varphi(\tau, \xi) \frac{\partial u}{\partial \nu}(\tau, \xi)\, \mathrm{d}\sigma_{\xi}\,.$$

If $\zeta < \eta$, then $V_{0\zeta}^{*}(t, x)$ can be rewritten

$$V_{0\zeta}^{*}(t, x) = \int_{\varepsilon}^{t} \mathrm{d}\tau \int_{D^*} \dots \int \frac{\partial^2 Z_0}{\partial x_i\, \partial x_n}(t, x; \tau, \xi)\, \varphi(\tau, \xi) \frac{\partial u}{\partial \nu}(\tau, \xi)\, \mathrm{d}\xi_1 \dots \mathrm{d}\xi_{n-1}$$

where the last component of the vector ξ equals to zero. Denote further

$$f(t, x) = \int_{\varepsilon}^{t} \mathrm{d}\tau \int_{D^*} \dots \int \frac{\partial^2 Z_0}{\partial \xi_i\, \partial x_n}(t, x; \tau, \xi)\, \varphi(\tau, \xi) \frac{\partial u}{\partial \nu}(\tau, \xi)\, \mathrm{d}\xi_1 \dots \mathrm{d}\xi_{n-1}\,.$$

The integration by parts gives

(8,6)

$$f(t, x) = \int_{\varepsilon}^{t} \mathrm{d}\tau \int \dots \int \left[\frac{\partial Z_0}{\partial x_n}(t, x; \tau, \xi)\, \varphi(\tau, \xi) \frac{\partial u}{\partial \nu}(\tau, \xi)\right]_{\xi_i = -\sqrt{(\zeta^2 - \sum_{j=1, j\neq i}^{n-1} \xi_j^2)}}^{\xi_i = \sqrt{(\zeta^2 - \sum_{j=1, j\neq i}^{n-1} \xi_j^2)}} \times$$

$$\times\, \mathrm{d}\xi_1 \dots \mathrm{d}\xi_{i-1}\, \mathrm{d}\xi_{i+1} \dots \mathrm{d}\xi_{n-1} -$$

$$- \int_{\varepsilon}^{t} \mathrm{d}\tau \int_{D^*} \dots \int \frac{\partial Z_0}{\partial x_n}(t, x; \tau, \xi) \frac{\partial}{\partial \xi_i}\left(\varphi(\tau, \xi) \frac{\partial u}{\partial \nu}(\tau, \xi)\right) \mathrm{d}\xi_1 \dots \mathrm{d}\xi_{n-1} =$$

$$= \int_{\varepsilon}^{t} \mathrm{d}\tau \int \dots \int [\dots]\, \mathrm{d}\xi_1 \dots \mathrm{d}\xi_{i-1}\, \mathrm{d}\xi_{i+1} \dots \mathrm{d}\xi_{n-1} +$$

$$+ \frac{1}{2} \int_{\varepsilon}^{t} \mathrm{d}\tau \int_{D^*} \dots \int \frac{\sum_j \Lambda^{nj}(\tau, \xi)(x_j - \xi_j)}{(4\pi)^{n/2} (t - \tau)^{(n+2)/2} (\det \Lambda(\tau, \xi))^{1/2}} \times$$

$$\times \exp\left\{-\frac{\sum \Lambda^{\alpha\beta}(\tau, \xi)(x_\alpha - \xi_\alpha)(x_\beta - \xi_\beta)}{4(t - \tau)}\right\} \frac{\partial}{\partial \xi_i}\left(\varphi(\tau, \xi) \frac{\partial u}{\partial \nu}(\tau, \xi)\right) \mathrm{d}\xi_1 \dots \mathrm{d}\xi_{n-1}\,.$$

Simultaneously,

(8,7)

$$f(t, x) + V_{0\zeta}^{*}(t, x) = \int_{\varepsilon}^{t} \mathrm{d}\tau \int_{D^*} \dots \int (4\pi)^{-(n/2)} (t - \tau)^{-(n+2)/2} (\det \Lambda(\tau, \xi))^{-1/2} \times$$

$$\times \left[-\frac{1}{2} \sum_j \frac{\partial \Lambda^{nj}}{\partial \xi_i}(\tau, \xi)(x_j - \xi_j) + \frac{1}{4} \frac{\sum_j \Lambda^{nj}(\tau, \xi)(x_j - \xi_j)}{\det \Lambda(\tau, \xi)} \frac{\partial \det \Lambda}{\partial \xi_i}(\tau, \xi) + \right.$$

$$\left. + \frac{1}{8} \frac{\sum_{\alpha\beta j} \Lambda^{nj} \frac{\partial \Lambda^{\alpha\beta}}{\partial \xi_i}(x_j - \xi_j)(x_\alpha - \xi_\alpha)(x_\beta - \xi_\beta)}{t - \tau}\right] \times$$

$$\times \exp\left\{-\frac{\sum \Lambda^{\alpha\beta}(\tau, \xi)(x_\alpha - \xi_\alpha)(x_\beta - \xi_\beta)}{4(t - \tau)}\right\} \varphi(\tau, \xi) \frac{\partial u}{\partial \nu}(\tau, \xi)\, \mathrm{d}\xi_1 \dots \mathrm{d}\xi_{n-1}\,.$$

(8,6) and (8,7) consist of integrals of the type

$$g(t, x) = \int_{\varepsilon}^{t} \mathrm{d}\tau \int_{D^*} \ldots \int \frac{(x_i - \xi_i)^{\mu_1} (x_j - \xi_j)^{\mu_2} (x_k - \xi_k)^{\mu_3}}{(t - \tau)^{\Theta}} \times$$
$$\times \exp\left\{- \frac{\sum A^{\alpha\beta}(\tau, \xi)(x_\alpha - \xi_\alpha)(x_\beta - \xi_\beta)}{4(t - \tau)}\right\} \sigma(\tau, \xi)\, \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_{n-1}$$

where numbers μ_i are integers from zero to three, the sum $\mu_1 + \mu_2 + \mu_3$ equals either to one or to three and Θ is a real number fulfilling $\Theta - \frac{1}{2}(\mu_1 + \mu_2 + \mu_3) = \frac{1}{2}(n + 1)$. It can be shown that the integral defining $g(t, 0)$ exists, since if $\sigma(\tau, \xi)$ is written as $\sigma(t, 0) + (\sigma(\tau, \xi) - \sigma(t, 0))$, then the first integral obviously converges and the second integral containing $\sigma(\tau, \xi) - \sigma(t, 0)$ converges owing to the Hölder continuity of $\sigma(\tau, \xi)$ if $A^{\alpha\beta}(\tau, \xi)$ are replaced by $A^{\alpha\beta}(t, 0)$. Similarly writing $A^{\alpha\beta}(\tau, \xi) = A^{\alpha\beta}(t, 0) + (A^{\alpha\beta}(\tau, \xi) - A^{\alpha\beta}(t, 0))$, then the second integral converges owing to the inequality

$$\left|\exp\left\{- \frac{\sum A^{\alpha\beta}(\tau, \xi)(x_\alpha - \xi_\alpha)(x_\beta - \xi_\beta)}{4(t - \tau)}\right\} - \right.$$
$$\left. - \exp\left\{\frac{\sum A^{\alpha\beta}(t, 0)(x_\alpha - \xi_\alpha)(x_\beta - \xi_\beta)}{4(t - \tau)}\right\}\right| \leqq c_1 \sum |A^{\alpha\beta}(\tau, \xi) -$$
$$- A^{\alpha\beta}(t, 0)| \exp\left\{-c_2 \frac{|x - \xi|^2}{t - \tau}\right\}$$

and to the Hölder continuity of $A^{\alpha\beta}(t, x)$.

Similarly as in § 15, Chap. IV, [4] it can be proved

$$\lim_{x \to 0} [g(t, x) - g(t, 0)] = \sigma(t, 0) \lim_{x \to 0} \int_{\varepsilon}^{t} \mathrm{d}\tau \int_{D^*} \ldots \int \left[\frac{(x_i - \xi_i)^{\mu_1} (x_j - \xi_j)^{\mu_2} (x_k - \xi_k)^{\mu_3}}{(t - \tau)^{\Theta}} \times\right.$$
$$\times \exp\left\{- \frac{\sum A^{\alpha\beta}(t, x)(x_\alpha - \xi_\alpha)(x_\beta - \xi_\beta)}{4(t - \tau)}\right\} - \frac{(-\xi_i)^{\mu_1} (-\xi_j)^{\mu_2} (-\xi_k)^{\mu_3}}{(t - \tau)^{\Theta}} \times$$
$$\left.\times \exp\left\{- \frac{\sum A^{\alpha\beta}(t, 0)\, \xi_\alpha \xi_\beta}{4(t - \tau)}\right\}\right] \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_{n-1}\,.$$

Denote

$$h(t, x) = \int_{\varepsilon}^{t} \mathrm{d}\tau \int_{D^*} \ldots \int \frac{(x_i - \xi_i)^{\mu_1} (x_j - \xi_j)^{\mu_2} (x_k - \xi_k)^{\mu_3}}{(t - \tau)^{\Theta}} \times$$
$$\times \exp\left\{- \frac{\sum A^{\alpha\beta}(t, x)(x_\alpha - \xi_\alpha)(x_\beta - \xi_\beta)}{4(t - \tau)}\right\}.$$

From the assumptions about μ_i and from the symmetry of D^* it follows $h(t, 0) = 0$.

It is sufficient to know the limit of $h(t, x)$ for $x \to 0$ under the assumption that x converges to 0 so that $x_1 = x_2 = \ldots = x_{n-1} = 0$, $x_n \to 0$. In this case it can be found that $\lim_{x \to 0} h(t, x) = 0$ if all indices i, j, k differ from n, or if $k = n$, $i \neq j$, $\mu_1 = \mu_2 = 1$ or if $i \neq j = k = n$, $\mu_1 = \mu_2 = 1$ or $\Theta - \frac{1}{2}(\mu_1 + \mu_2 + \mu_3) < \frac{1}{2}(n + 1)$. In the cases $k = n$, $i = j \neq n$, $\mu_1 = \mu_2 = 1$, $\mu_3 = 1$ or $k = n$, $\mu_1 = \mu_2 = 0$, $\mu_3 = 3$ it is $\lim h(t, x) = 2(4\pi)^{n/2}$ and if $\mu_1 = \mu_2 = 0$, $k = n$, $\mu_3 = 1$, then $\lim h(t, x) = (4\pi)^{n/2}$. With respect to (8,6), (8,7) and to the definition of $V_{0\zeta}^*(t, x)$ it is (the expressions in the square brackets in (8,6) converge to 0 for $\zeta \to 0$)

$$(8,8) \qquad \lim_{x \to 0} V_0(t, x) = \int_\varepsilon^t \mathrm{d}\tau \int_{D^+} \frac{\partial^2 Z_0}{\partial x_i \, \partial x_n}(t, 0; \tau, \xi) \, \varphi(\tau, \xi) \frac{\partial u}{\partial \nu}(\tau, \xi) \, \mathrm{d}\sigma_\xi +$$
$$+ \int_\varepsilon^t \mathrm{d}\tau (\text{v.p.}) \int_{D^*} \ldots \int \frac{\partial^2 Z_0}{\partial x_i \, \partial x_n}(t, 0; \tau, \xi) \, \varphi(\tau, \xi) \frac{\partial u}{\partial \nu}(\tau, \xi) \, \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_{n-1} +$$
$$+ \frac{1}{4}\left[\frac{\partial \det A}{\partial x_i}(t, 0) + \sum_{\alpha=1}^{n-1} \frac{\partial A^{\alpha\alpha}}{\partial x_i}(t, 0) - \frac{\partial A^{nn}}{\partial x_i}(t, 0)\right] - \frac{1}{2} \frac{\partial}{\partial x_i}\left(\varphi \frac{\partial u}{\partial x_n}\right).$$

Before we pass to the study of $V_m(t, x)$, some auxiliary inequalities are needed. Equality (11,12) [4] implies

$$\left|\frac{\partial K}{\partial x_i}(t, x; \tau, y) + \frac{\partial K}{\partial y_i}(t, x; \tau, y)\right| \leqq c_1(t - \tau)^{-\frac{1}{2}(n+2-\alpha)} \exp\left\{-c_2 \frac{|x - y|^2}{t - \tau}\right\}.$$

Due to (8,4),

$$\left|\frac{\partial K_m}{\partial x_i} + \frac{\partial K_m}{\partial y_i}\right| \leqq \int_\tau^t \mathrm{d}\lambda \int_{R_n} \left|\frac{\partial K}{\partial x_i} + \frac{\partial K}{\partial y_i}\right| . \left|K_{m-1}\right| \mathrm{d}\xi +$$
$$+ \int_\tau^t \mathrm{d}\lambda \int_{R_n} |K| . \left|\frac{\partial K_{m-1}}{\partial x_i} + \frac{\partial K_{m-1}}{\partial y_i}\right| \mathrm{d}\xi \leqq$$
$$\leqq \hat{c}_m(t - \tau)^{-\frac{1}{2}(n+2-m\alpha)} \exp\left\{-c_2 \frac{|x - y|^2}{t - \tau}\right\}$$

where $\hat{c}_m$ are some positive constants such that $\sum \hat{c}_m < \infty$. This yields

$$\left|\frac{\partial^2 Z_m}{\partial x_i \, \partial x_n} + \frac{\partial^2 Z_m}{\partial \xi_i \, \partial x_n}\right| \leqq \int_\tau^t \mathrm{d}\lambda \int_{R_n} \left|\frac{\partial^2 Z_0}{\partial x_i \, \partial x_n} + \frac{\partial^2 Z_0}{\partial \xi_i \, \partial x_n}\right| . \left|K_m\right| \mathrm{d}\xi +$$
$$+ \int_\tau^t \mathrm{d}\lambda \int_{R_n} \left|\frac{\partial Z_0}{\partial x_n}\right| . \left|\frac{\partial K_m}{\partial x_i} + \frac{\partial K_m}{\partial y_i}\right| \mathrm{d}\xi \leqq$$
$$\leqq c_m^*(t - \tau)^{-\frac{1}{2}(n+1-m\alpha)} \exp\left\{-c_2 \frac{|x - y|^2}{t - \tau}\right\}.$$

The last inequality means that $V_m(t, x)$ $(m > 0)$ can be treated in the same way as $V_0(t, x)$, but since $\partial Z_m/\partial x_n$ has a weaker singularity than $\partial Z_0/\partial x_n$ no additive terms appear by the limit procedure so that

$$\lim_{x \to 0} \int_\varepsilon^t \mathrm{d}\lambda \int_{\dot{D}} \frac{\partial^2 Z_m}{\partial x_i \, \partial x_n} (t, x; \tau, \xi) \, \varphi(\tau, \xi) \frac{\partial u}{\partial v} (\tau, \xi) \, \mathrm{d}\sigma_\xi =$$

$$= \int_\varepsilon^t \mathrm{d}\tau (\text{v.p.}) \int_{\dot{D}} \frac{\partial^2 Z_m}{\partial x_i \, \partial x_n} (t, 0; \tau, \xi) \, \varphi(\tau, \xi) \frac{\partial u}{\partial v} (\tau, \xi) \, \mathrm{d}\sigma_\xi \, .$$

The last equality together with (8,6) gives relation (8,2).

Remark 9. Denote more precisely $D_\zeta^* = \dot{D} \cap S(0, \zeta)$, $D_\zeta^+ = \dot{D} - D_\zeta^*$. The principal value in (8,2) is defined by

$$(\text{v.p.}) \int_{\dot{D}} \frac{\partial^2 Z}{\partial x_i \, \partial x_n} (t, 0; \tau, \xi) \, \varphi(\tau, \xi) \frac{\partial u}{\partial v} (\tau, \xi) \, \mathrm{d}\sigma_\xi =$$

$$= \int_{D_n{}^+} \frac{\partial^2 Z}{\partial x_i \, \partial x_n} (t, 0; \tau, \xi) \, \varphi(\tau, \xi) \frac{\partial u}{\partial v} (\tau, \xi) \, \mathrm{d}\sigma_\xi \, +$$

$$+ \lim_{\xi \to 0+} \int \dots \int_{D_n{}^* - D_\xi{}^*} \frac{\partial^2 Z}{\partial x_i \, \partial x_n} (t, 0; \tau, \xi) \, \varphi(\tau, \xi) \frac{\partial u}{\partial v} (\tau, \xi) \, \mathrm{d}\xi_1 \dots \mathrm{d}\xi_{n-1} \, .$$

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