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ISOTOPY OF DIGRAPHS

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In this paper we shall define the concept of isotopy of digraphs analogously to the concept of isotopy of groupoids introduced in [3]. We shall consider digraphs without multiple (equally directed) edges, but we shall admit loops.

The isotopy of groupoids is defined as follows. Two groupoids G_1 and G_2 are called isotopic, if and only if there exist three one-to-one mappings φ, ψ, χ of G_1 onto G_2 such that for any three elements a, b, c of G_1 the equality

$$ab = c$$

in G_1 is equivalent to the equality

$$\varphi(a)\,\psi(b) = \chi(c)$$

in G_2.

Analogously the isotopy of digraphs will be defined. Let G, G' be two digraphs, V and V' respectively their vertex sets. The graphs G, G' are called isotopic, if and only if there exist two one-to-one mappings f_1, f_2 of V onto V' such that the existence of the edge $\overrightarrow{uv}$ in G (for any $u \in V$, $v \in V$) is equivalent to the existence of the edge $\overrightarrow{f_1(u) f_2(v)}$ in G'. A pair of such mappings $\mathfrak{f} = \langle f_1, f_2 \rangle$ will be called an isotopy of G onto G'. If $f_1 \equiv f_2$, the isotopy $\mathfrak{f} = \langle f_1, f_2 \rangle$ is called an isomorphism of G onto G'.

Now we shall define the composition of isotopies. Let G, G', G'' be three digraphs, let $\mathfrak{f} = \langle f_1, f_2 \rangle$ be an isotopy of G onto G' and $\mathfrak{g} = \langle g_1, g_2 \rangle$ an isotopy of G' onto G''. Then the composition (product) $\mathfrak{g}\mathfrak{f}$ of the isotopies $\mathfrak{g}$, $\mathfrak{f}$ is the isotopy $\mathfrak{h} = \langle h_1, h_2 \rangle$ such that $h_1 = g_1 f_1$, $h_2 = g_2 f_2$. The inverse isotopy $\mathfrak{f}^{-1}$ to the isotopy $\mathfrak{f}$ is the isotopy $\mathfrak{f}^{-1} = \langle f_1^{-1}, f_2^{-1} \rangle$ of G' onto G, where f_1^{-1}, f_2^{-1} are inverse mappings to the mappings f_1, f_2 respectively.

For the investigation of isotopy we shall use a certain bipartite digraph corresponding to the given digraph. Let G be a digraph, let $a_1, \ldots, a_n$ be its vertices. The digraph $\hat{G}$ corresponding to G has the vertices $b_1, \ldots, b_n, c_1, \ldots, c_n$. In $\hat{G}$ there exists an edge $\overrightarrow{b_i c_j}$ ($1 \leqq i \leqq n$, $1 \leqq j \leqq n$) if and only if the edge $\overrightarrow{a_i a_j}$ exists in G. No two of the vertices $b_1, \ldots, b_n$ and no two of the vertices $c_1, \ldots, c_n$ are joined by an edge and there does not exist any edge $\overrightarrow{c_i b_j}$. We shall denote $B = \{b_1, \ldots, b_n\}$, $C = \{c_1, \ldots, c_n\}$.

Theorem 1. *Let G, G' be two isotopic digraphs. Then the corresponding bipartite digraphs $\hat{G}$, $\hat{G}'$ are isomorphic.*

Proof. Let $a_1, \ldots, a_n$ be the vertices of G and $a'_1, \ldots, a'_n$ the vertices of G'. The vertices of $\hat{G}$ will be $b_1, \ldots, b_n, c_1, \ldots, c_n$, the vertices of $\hat{G}'$ will be $b'_1, \ldots, b'_n, c'_1, \ldots, c'_n$. Now let there exist an isotopy $\mathfrak{f} = \langle f_1, f_2 \rangle$ of G onto G'. We shall define a one-to-one mapping $\hat{f}$ of the vertex set of $\hat{G}$ onto the vertex set of $\hat{G}'$ as follows. For $i = 1, \ldots, n$ the image $\hat{f}(b_i)$ of a vertex b_i is the vertex b_j, where j is such a number that $f_1(a_i) = a_j$. Analogously the image $\hat{f}(c_i)$ of a vertex c_i is the vertex c_k, where k is such a number that $f_2(a_i) = a_k$. Now the edge $\overrightarrow{a_k a_l}$ exists in G, if and only if the edge $\overrightarrow{b_k c_l}$ exists in $\hat{G}$. Therefore the edge $\overrightarrow{f_1(a_k) f_2(a_l)}$ exists in G' if and only if the edge $\hat{f}(b_k)\,\hat{f}(c_l)$ exists in $\hat{G}'$. We see that $\hat{f}$ induces an isomorphism of $\hat{G}$ onto $\hat{G}'$.

In the following the vertex set $\{b_1, \ldots, b_n\}$ in the graph $\hat{G}$ will be denoted by B, the set $\{c_1, \ldots, c_n\}$ by C.

Now let us have a digraph G and the corresponding graph $\hat{G}$. If π is a one-to-one mapping of B onto C in G, then $G(\pi)$ will be the digraph obtained from G by identifying all pairs b_i, $\pi(b_i)$ for $i = 1, \ldots, n$. There are $n!$ possible mappings π, therefore also $n!$ digraphs $G(\pi)$; these digraphs are evidently exactly all digraphs isotopic to G. They need not be pairwise non-isomorphic. Two graphs $G(\pi_1)$, $G(\pi_2)$ are isomorphic if and only if there exists an automorphism r of $\hat{G}$ such that $r(b_i) = b_j$ if and only if $r\,\pi_1(b_i) = \pi_2(b_j)$ for any i and j. This can be expressed in the form

$$r\,\pi_1(b_i) = \pi_2\, r(b_i)$$

for $i = 1, \ldots, n$, thus we may write the equality of mappings

$$r\pi_1 = \pi_2 r\,,$$

which implies

$$\pi_2 = r\pi_1 r^{-1}$$

Therefore the number of digraphs $G(\pi_2)$ which are isomorphic to $G(\pi_1)$ is equal to the number of mappings π_2 which are conjugated with π_1 by automorphisms of $\hat{G}$. For two automorphisms r, s of $\hat{G}$ we have

$$r\pi_1 r^{-1} = s\pi_1 s^{-1}$$

if and only if $s^{-1}r$ (which is also an automorphism of $\hat{G}$) is commutative with π_1, i.e. when $s^{-1}r(b_i) = b_j$ implies $s^{-1}r\,\pi_1(b_i) = \pi_1(b_j)$. Such an automorphism $s^{-1}r$ is an automorphism of $G(\pi_1)$. We have proved

Theorem 2. *The number of digraphs* $G(\pi)$ *isomorphic to the digraph* $G(\pi_1)$ *for a given* π_1 *is equal to the index of the automorphism group of* $G(\pi_1)$ *in the automorphism group of* $\hat{G}$.

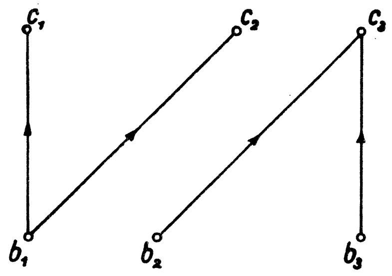

Fig. 1a.

Automorphism groups of different $G(\pi)$ (and thus of two isotopic non-isomorphic digraphs) need not be isomorphic and even need not have equal orders. In Fig. 1a we see a graph $\hat{G}$ corresponding to the digraphs in Figs. 1b and 1c. There exist six graphs $G(\pi)$ for $\hat{G}$ in Fig. 1a, four of them being isomorphic to the digraph in Fig. 1b, two being isomorphic to the digraph in Fig. 1c (whose automorphism group is of the order 2, while the automorphism group of the digraph in Fig. 1b is of the order 1).

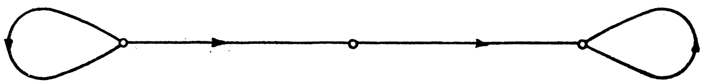

Fig. 1b.

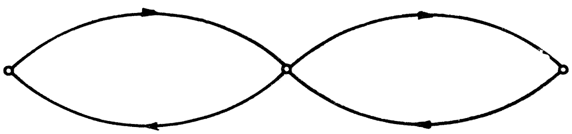

Fig. 1c.

Now we shall investigate digraphs G with the property that any digraph isotopic to G is isomorphic to G. At first we shall prove a lemma. By a bipartite graph $H(X, Y)$ we mean a graph whose vertex set is $X \cup Y$, where $X \cap Y = \emptyset$ and any edge joins a vertex of X with a vertex of Y.

Lemma. *Let* $H(X, Y)$ *be a bipartite graph, let* $|X| = |Y| = n$. *Let* k *be the maximal number of edges in a matching of* $H(X, Y)$ *and let there exist* n *pairwise disjoint matchings with* k *elements each. Then either* k *vertices of* X *are joined with all vertices of* Y, *or* k *vertices of* Y *are joined with all vertices of* X.

Proof. Let $H'(X, Y)$ be the subgraph of $H(X, Y)$ consisting of all vertices of $H(X, Y)$, and of the edges belonging to the above mentioned matchings. The graph $H'(X, Y)$ contains kn edges. Its edge chromatic number is n; if it were less than n, then there would exist at least one colour by which more than k edges would be coloured and the set of edges coloured by this colour would form a matching with more than k edges. According to [1] the edge chromatic number of a bipartite graph is equal to the maximal degree of a vertex in this graph. Therefore at least one vertex u_1 of the degree n exists in $H'(X, Y)$. At first assume that u_1 is in X. Then it is joined with all vertices of Y. Let $H_1'(X_1, Y)$ be the subgraph of $H'(X, Y)$ obtained by deleting u_1; we put $X_1 = X \dot{-} \{u_1\}$. The edges incident with u_1 in $H'(X, Y)$ belong pairwise to different matchings. Therefore in $H_1'(X_1, Y)$ there exist n pairwise disjoint matchings with $k - 1$ elements each and no matching with more than $k - 1$ elements. The edge chromatic number of $H_1'(X_1, Y)$ is again n, therefore there exists a vertex u_2 of $H_1'(X_1, Y)$ of the degree n. As $|X_1| = n - 1$, any vertex of Y has the degree at most $n - 1$ and $u_2 \in X$. By $H_2'(X_2, Y)$ we denote the graph obtained from $H_1'(X_1, Y)$ by deleting u_2 and proceed analogously as above with $H_1'(X_1, Y)$. After a finite number of steps we obtain vertices $u_1, \ldots, u_k$ which are all in X and each of which is joined with all vertices of Y. If $u_1 \in Y$, we shall find the vertices $u_1, \ldots, u_k$ of Y, each of which is joined with all vertices of X.

Theorem 3. *For a digraph G the following two properties are equivalent:*

(1) *From an arbitrary vertex of G either edges go into all vertices of G (including this vertex itself), or no edge goes out at all.*

(2) *All graphs isotopic to G are isomorphic to G.*

Proof. The proof of (1) ⇒ (2) is easy. We shall prove (2) ⇒ (1). Let a maximal matching of $\hat{G}$ consist of the edges $b_i c_i$ for $i = 1, \ldots, k$, $k \leqq n$ (without any loss of generality). Then in G there are k loops. As all graphs isotopic to G are isomorphic to G, any graph $G(\pi)$ must have also k loops, i.e. for any one-to-one mapping π of B onto C there exists a matching consisting of edges $b_{j_1} \pi(b_{j_1}), \ldots, b_{j_k} \pi(b_{j_k})$ where $j_1, \ldots, j_k$ are some k numbers from the set $\{1, \ldots, n\}$. Consider the mappings π_i for $i = 0, 1, \ldots, n - 1$ such that $\pi_i(b_j) = c_{i+j}$, where the sum $i + j$ is taken modulo n. In each of the graphs $G(\pi_i)$ there exist k loops, i.e. for each π_i we have a matching M_i with k edges, each of which joins a vertex of B with its image in π_i. Any two matchings M_{i_1}, M_{i_2} for $i_1 \neq i_2$ are disjoint, because $\pi_{i_1}(b_j) \neq \pi_{i_2}(b_j)$ for any j. Therefore the assumption of Lemma is satisfied and we have either k vertices of B joined with all vertices of C, or k vertices of C joined with all vertices of B. Without any loss of generality assume that the vertices $b_1, \ldots, b_k$ are joined with all vertices of C. Now let some b_i for $k + 1 \leqq i \leqq n$ be joined with a vertex c_j of C. Choose arbitrary k vertices $c_{i_1}, \ldots, c_{i_k}$ of c which are all different from c_j. Then the edges $b_1 c_{i_1}, \ldots, \ldots, b_k c_{i_k}, b_i c_j$ form a matching of $\hat{G}$ with $k + 1$ vertices, which is a contradiction. Thus no edge goes out from b_i for $k + 1 \leqq i \leqq n$.

In the following two theorems we shall use the concepts of $(+-)$-connectivity and of $(-+)$-connectivity. These concepts together with some others related to them (which will be also used here) are defined in [4].

Theorem 4. *Let $\mathfrak{f} = \langle f_1, f_2 \rangle$ be an isotopy of a digraph G onto G'. If two vertices x, y of G are $(+-)$-connected, then also the vertices $f_1(x), f_1(y)$ are $(+-)$-connected in G'.*

Proof. Let P be a $(+-)$-path from x to y, i.e. $P = [u_1, e_1, v_1, h_1, u_2, e_2, v_2, h_2, u_3, \ldots, u_{k-1}, e_{k-1}, v_{k-1}, h_{k-1}, u_k]$, where $u_1 = x$, $u_k = y$, $e_i = \overrightarrow{u_i v_i}$, $h_i = \overrightarrow{u_{i+1} v_i}$ for $i = 1, \ldots, k-1$. Then there exists a $(+-)$-path P' in G' such that $P' = [f_1(u_1), e'_1, f_2(v_1), h'_1, f_1(u_2), e'_2, f_2(v_2), h'_2, f_1(u_3), \ldots, f_1(u_{k-1}), e'_{k-1}, f_2(v_{k-1}), h'_{k-1}, f_1(u_k)]$, where $f_1(u_1) = f_1(x)$, $f_1(u_k) = f_1(y)$, $e'_i = \overrightarrow{f_1(u_i) f_2(v_i)}$, $h'_i = \overrightarrow{f_1(u_{i+1}) f_2(v_i)}$ for $i = 1, \ldots, k-1$.

Theorem 5. *Let $\mathfrak{f} = \langle f_1, f_2 \rangle$ be an isotopy of a digraph G onto G'. If two vertices x, y of G are $(-+)$-connected, then also the vertices $f_2(x), f_2(y)$ are $(-+)$connected in G'.*

Proof is dual to the proof of Theorem 4.

Theorem 6. *Let G be an alternatingly connected digraph. Then all digraphs isotopic to G are alternatingly connected.*

This is an immediate consequence of Theorems 4 and 5.

An analogous assertion holds neither for connectivity nor for strong connectivity. For example, let G be a cycle with six vertices, i.e. a strongly connected digraph. Let its vertices be $u_1, \ldots, u_6$ and edges $\overrightarrow{u_i u_{i+1}}$ for $i = 1, \ldots, 6$. (The sum $i + 1$ is taken modulo 6.) Let G' be the digraph consisting of two connected components which are cycles with three vertices. Let their vertices be v_1, v_2, v_3 and w_1, w_2, w_3 while their edges are $\overrightarrow{v_1 v_2}, \overrightarrow{v_2 v_3}, \overrightarrow{v_3 v_1}$, and $\overrightarrow{w_1 w_2}, \overrightarrow{w_2 w_3}, \overrightarrow{w_3 w_1}$ respectively. The graph G' is not connected. But the graphs G and G' are isotopic. Let us define f_1 and f_2 so that $f_1(u_i) = v_i$ for $i = 1, 2, 3$ and $f_1(u_i) = w_{i-3}$ for $i = 4, 5, 6$, further $f_2(u_1) = w_1$, $f_2(u_2) = v_2$, $f_2(u_3) = v_3$, $f_2(u_4) = v_1$, $f_2(u_5) = w_2$, $f_2(u_6) = w_3$. The pair of mappings $\mathfrak{f} = \langle f_1, f_2 \rangle$ is an isotopy of G onto G'.

An isotopy of a digraph G onto itself is called an autotopy of G. The autotopies of G form a group with respect to the operations of product and inverse defined above. This group is evidently isomorphic to that of all automorphisms of $\hat{G}$. Its unit element is $\mathfrak{e} = \langle e, e \rangle$, where e is the identical mapping of V onto itself.

Theorem 7. *Let H be an arbitrary finite Abelian group. Then there exists a digraph G whose group of autotopies is isomorphic to H.*

Proof. Let A be a finite cyclic group of the order k. Now G_A will be the digraph constructed as follows. Take $6k$ vertices $u_1, \ldots, u_{3k}, v_1, \ldots, v_{3k}$ and put the edges $\overrightarrow{u_i v_i}$

and $\overrightarrow{u_{i+1}v_i}$ for $i = 1, \ldots, 3k$ (the sum $i + 1$ is taken modulo $3k$). We have obtained an alternating circuit (see [3]). Now adjoin the vertices $x_j, x'_j, y_j, y'_j, y''_j, y'''_j$ for $j = 1, \ldots, k$ and the edges $\overrightarrow{u_{3j+1}x_j}, \overrightarrow{x'_jx_j}, \overrightarrow{u_{3j+2}y_j}, \overrightarrow{y'_jy_j}, \overrightarrow{y'_jy''_j}, \overrightarrow{y'''_jy''_j}$. We have obtained a bipartite digraph, whose set of sources is $\{u_1, \ldots, u_{3k}, x'_1, \ldots, x'_k, y'_1, \ldots, y'_k, y'''_1, \ldots \ldots, y'''_k\}$ and whose set of sinks is $\{v_1, \ldots, v_{3k}, x_1, \ldots, x_k, y_1, \ldots, y_k, y''_1, \ldots, y''_k\}$. We see that the set of sources and the set of sinks have the same cardinality $6k$. Let a be an automorphism of G_A such that $a(u_i) = u_{i+3}$, $a(v_i) = v_{i+3}$, $a(x_j) = x_{j+1}$, $a(x'_j) = x'_{j+1}$, $a(y_j) = y_{j+1}$, $a(y'_j) = y'_{j+1}$, $a(y''_j) = y''_{j+1}$, $a(y'''_j) = y'''_{j+1}$ for $i = 1, \ldots, 3k$, $j = 1, \ldots, k$ (the sum $i + 3$ is taken modulo $3k$, the sum $j + 1$ is taken modulo k). The cyclic group generated by a has the order k. When we take into account that any automorphism of G_A must map the alternating circuit onto itself, x_j onto some x_l etc., we see that besides the powers of a there exist no automorphisms of G_A. Thus the group of automorphisms of G_A is isomorphic to A. Now let H be an arbitrary finite Abelian group. It can be expressed as a direct product of primary cyclic groups. To each of these groups we construct the corresponding graph. The graph $\hat{G}$ whose connected components are these graphs has the group of automorphisms isomorphic to H. Any vertex of $\hat{G}$ is either a source or a sink. The number of sources is equal to the number of sinks. Now we shall choose a one-to-one correspondence between sources and sinks of $\hat{G}$ and identify the corresponding vertices. The resulting digraph will be denoted by G. (The correspondence can evidently be chosen so that G is connected.) For G the corresponding bipartite digraph is $\hat{G}$ and the group of autotopies of G is isomorphic to H.

As we have mentioned, the group of autotopies of G is isomorphic to the group of automorphisms of $\hat{G}$. To any automorphism $\mathfrak{f}$ of G (which is also an autotopy) such an automorphism $\hat{f}$ of $\hat{G}$ corresponds that $\hat{f}(b_i) = b_j$ is equivalent with $\hat{f}(c_i) = c_j$ for $1 \leqq i \leqq n$, $1 \leqq j \leqq n$. Let us denote by φ the mapping of the vertex set of $\hat{G}$ onto itself such that $\varphi(b_i) = c_i$, $\varphi(c_i) = b_i$ for $i = 1, \ldots, n$. Thus if $\hat{f}(b_i) = b_j$, then

$$\hat{f}\,\varphi(b_i) = \hat{f}(c_i) = c_j = \varphi(b_j) = \varphi\,\hat{f}(b_i)\,,$$
$$\hat{f}\,\varphi(c_i) = \hat{f}(b_i) = b_j = \varphi(c_j) = \varphi\,\hat{f}(c_i)\,.$$

We see that $\hat{f}$ and φ are commutative to each other. And also if $\hat{f}\varphi \equiv \varphi\hat{f}$ we have $\hat{f}(b_i) = b_j$ equivalent to $\hat{f}(c_i) = c_j$ and the automorphism $\hat{f}$ of $\hat{G}$ corresponds to an automorphism $\mathfrak{f}$ of G. We shall express this as

Theorem 8. *Let $S_{\hat{G}}$ be the group of all permutations of the vertex set of $\hat{G}$ which map A onto itself, let $A_{\hat{G}}$ be the group of automorphisms of $\hat{G}$ (considered only as mappings of the vertex set of G onto itself), let A_G be the group of automorphisms of $\hat{G}$ corresponding to automorphisms of G, let φ be the mapping of the vertex set of G onto itself such that $\varphi(b_i) = c_i$, $\varphi(c_i) = b_i$. Then A_G is the intersection of the subgroup of $S_{\hat{G}}$ consisting of all elements commutative with φ with the subgroup $A_{\hat{G}}$.*

Remark. The group $S_{\hat{G}}$ is isomorphic to the direct product of two groups which are both isomorphic to the symmetric group S_n.

Now we shall consider the digraphs G with the property that any autotopy of that graph is an automorphism. This means that any automorphism of $\hat{G}$ is commutative with φ. We shall give an example of such digraphs. In [2] A. Kotzig defines the concept of centrally symmetric (undirected) graph. It is a graph with the property that to any of its vertices x a unique vertex $\bar{x}$ exists so that the distance of the vertices x and $\bar{x}$ is equal to the diameter of the graph. All automorphisms of a graph preserve distance, thus if $\hat{f}$ is an automorphism of a centrally symmetric graph and $\hat{f}(x) = y$ for two vertices x, y, then $\hat{f}(\bar{x}) = \bar{y}$. Thus $\hat{f}$ is commutative with the mapping which maps each x onto $\bar{x}$. The vertices x and $\bar{x}$ are called opposite to each other. We have proved

Theorem 9. *Let $\hat{G}_0$ be a bipartite centrally symmetric graph with $2n$ vertices, whose domination number is n and whose diameter is odd. If we direct its edges so that one of its dominating sets is the set of sources and the other is the set of sinks and then identify any pair of opposite vertices, we obtain a digraph G, each of whose autotopies is an automorphism.*

Remark. We must assume that the diameter of $\hat{G}_0$ is odd, because two opposite vertices must lie in different dominating sets of $\hat{G}_0$.

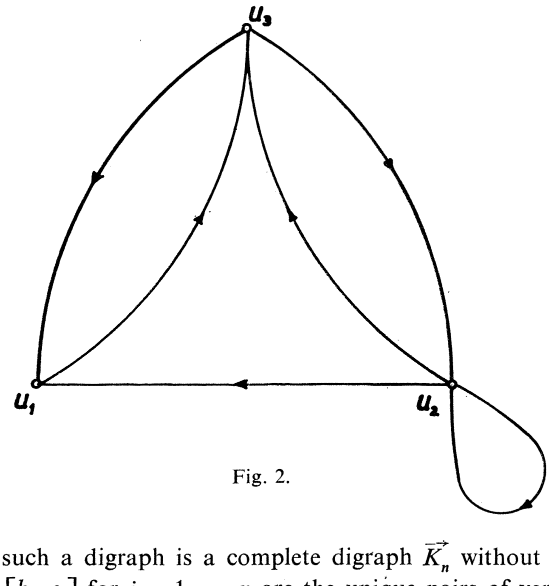

Fig. 2.

An example of such a digraph is a complete digraph $\vec{K}_n$ without loops. In the graph $\hat{G}$ the pairs $[b_i, c_i]$ for $i = 1, \ldots, n$ are the unique pairs of vertices with the distance 3, other pairs have distances 1 or 0. Other examples of bipartite centrally symmetric graphs which satisfy the assumption of Theorem 9 are Cartesian products of cycles of even lengths $C_1 \times C_2 \times \ldots \times C_k$ such that the half of the sum of their lengths is odd (this number is the diameter of such a graph).

There exist also digraphs which have no automorphisms besides the identical one, but have non-identical isotopies. An example is in Fig. 2. There exists an autotopy $\mathfrak{f} = \langle f_1, f_2 \rangle$ such that f_1 is the identical mapping of the vertex set of G onto itself and $f_2(u_1) = u_2, f_2(u_2) = u_1, f_2(u_3) = u_3$.

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