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GENERALIZED ROBIN PROBLEM IN POTENTIAL THEORY

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Introduction. Let G be an arbitrary open set in R^m, the Euclidean m-space of dimension $m > 2$, and suppose that its boundary B is compact and non-void. Let $\mathfrak{B}$ denote the space of all finite signed Borel measures with support in B; $\mathfrak{B}$ will be treated as a Banach space with the usual norm $\|\mu\|$ equal to the total variation of μ on B. Let us fix a non-negative element $\lambda \in \mathfrak{B}$. With each $\mu \in \mathfrak{B}$ we associate its potential

$$U\mu(x) = \int_B p(x - y) \, d\mu(y)$$

corresponding to the Newtonian kernel $p(z) = |z|^{2-m}/(m - 2)$ as well as the class $\mathscr{D}_\mu$ of those infinitely differentiable functions φ with compact support in R^m for which the integral

$$I_\mu(\varphi) = \int_{B \times B} \varphi(x) \, p(x - y) \, d\lambda(x) \, d\mu(y) \tag{1}$$

converges. The symbol $\mathscr{T}\mu$ stands for the functional over $\mathscr{D}_\mu$ defined by

$$\langle \varphi, \mathscr{T}\mu \rangle = I_\mu(\varphi) + \int_G \operatorname{grad} \varphi(x) \, . \operatorname{grad} U\mu(x) \, dx \, . \tag{2}$$

If B is a smooth hypersurface with the exterior normal n and λ is absolutely continuous with respect to the area measure H on B, then, under suitable conditions concerning $U\mu$, $\langle \varphi, \mathscr{T}\mu \rangle$ transforms into

$$\int_B \varphi \left(\frac{\partial U\mu}{\partial n} + \frac{d\lambda}{dH} U\mu \right) dH \, ,$$

which shows that $\mathscr{T}\mu$ is a natural weak characterization of

$$\frac{\partial U\mu}{\partial n} + \frac{d\lambda}{dH} U\mu \, .$$

If $\lambda = 0$, $\mathscr{T}\mu$ reduces to the generalized normal derivative of $U\mu$ as investigated in [12]. In the case of G being a complementary domain of a simple closed surface in R^3 submitted to some further restrictions, the third boundary value problem (sometimes called the Robin problem) with a weak characterization of boundary values was treated in [21]. Making no a priori restrictions on G we shall give a necessary and sufficient geometrical condition (connected with the shape of G and the distribution of λ over B) in order that, for each $\mu \in \mathfrak{B}$, $\mathscr{T}\mu$ be representable by means of a unique element of $\mathfrak{B}$. Let us briefly recall the necessary notation. As in [12] we call x a hit of a half-line $S \subset R^m$ on G provided $x \in S$ and each open ball $\Omega_r(x)$ with center at x and radius $r > 0$ meets both $S \cap G$ and $S - G$ in a set of positive linear measure. Given $y \in R^m$, $\theta \in \Gamma = \{z \in R^m; |z| = 1\}$, consider the total number $n_\infty(\theta, y)$ $(0 \leqq n_\infty(\theta, y) \leqq \infty)$ of all the hits of $\{y + \varrho\theta; \varrho > 0\}$ on G. For fixed y, $n_\infty(\theta, y)$ is a Baire function of the variable $\theta \in \Gamma$ and we may put

$$v_\infty(y) = \int_\Gamma n_\infty(\theta, y)\, \mathrm{d}H_{m-1}(\theta)\,,$$

where H_{m-1} stands for the Hausdorff $(m-1)$-measure.

Using some results of [12] we prove in Section 5 the following theorem (which was announced without proof in [16]):

Theorem. *The following conditions* (i) *and* (ii) *are equivalent to each other*:

(i) $\sup\limits_{y\in B} [v_\infty(y) + \int_0^\infty \varrho^{1-m}\, \lambda[\Omega_\varrho(y)]\, \mathrm{d}\varrho] < \infty$;

(ii) *For each $\mu \in \mathfrak{B}$, there is a unique $v \in \mathfrak{B}$ such that $\langle \varphi, v\rangle = \langle \varphi, \mathscr{T}\mu\rangle$ for all $\varphi \in \mathscr{D}_\mu$.*

(Note that the integral occurring in (i) is just the value of $U\lambda$ at y.)

Suppose now that (i) holds. In view of the above theorem, $\mathscr{T}\mu$ can be identified with a unique element of $\mathfrak{B}$. The operator $\mathscr{T} : \mu \mapsto \mathscr{T}\mu$ is bounded on $\mathfrak{B}$.

Let B' denote the set of those $y \in B$ at which the m-dimensional density $d(y)$ of G equals $\frac{1}{2}$. Then B' is a Borel set with $H_{m-1}(B') < \infty$ and one may consider the set $\mathfrak{B}_H$ of all the elements of $\mathfrak{B}$ that are absolutely continuous with respect to the restriction H of H_{m-1} to B'.

In connection with solving the operator equation $\mathscr{T}\mu = v$ (connected in a natural way with the third boundary value problem) over $\mathfrak{B}$ it is of interest to clear up the relationship between $\mathfrak{B}_H$ and $\mathscr{T}(\mathfrak{B}_H)$. The following proposition is proved in Section 12 ($\mathscr{T}_{-1}(\mathfrak{B}_H)$ denotes the inverse image of $\mathfrak{B}_H$):

Proposition. *If $\lambda \in \mathfrak{B}_H$, then*

$$\mathscr{T}(\mathfrak{B}_H) \subset \mathfrak{B}_H$$

and the following conditions are equivalent:

(i) $$\mathcal{T}_{-1}(\mathfrak{B}_H) \subset \mathfrak{B}_H\,;$$

(ii) *For each* $y \in B$ *there is* $d(y) \neq 0$.

The results of this paper will be used in [17], [18].

1. Notation. Throughout this paper, $m > 2$ will be a fixed integer. For $M \subset R^m$ we shall denote by fr M, cl M and diam M the boundary of M, the closure of M and the diameter of M, respectively. As in the introduction, G will be a fixed open subset of R^m such that $B = \operatorname{fr} G$ is compact and non-void and $\mathcal{D}$ will stand for the class of all infinitely differentiable functions with compact support in R^m. The symbols p, $\mathfrak{B}$ and $U\mu$ (with $\mu \in \mathfrak{B}$) will have the meaning described in the introduction. In what follows, λ is a fixed non-negative element of $\mathfrak{B}$ and for $\mu \in \mathfrak{B}$, the functionals I_μ, $\mathcal{T}\mu$ over $\mathcal{D}_\mu$ are defined by (1), (2). The functional $\mathcal{T}\mu$ is said to be representable by an element $\nu \in \mathfrak{B}$ over $\mathcal{D}_\mu$ if

(3) $$\langle \varphi, \mathcal{T}\mu \rangle = \int_{R^m} \varphi \, d\nu$$

holds for each $\varphi \in \mathcal{D}_\mu$; every such ν will be called a representing element corresponding to $\mathcal{T}\mu$. For each $\varphi \in \mathcal{D}$ whose support does not meet B is, of course, $\varphi \in \mathcal{D}_\mu$, $I_\mu(\varphi) = 0$ and one easily verifies that also

$$\int_G \operatorname{grad} \varphi(x) \,.\, \operatorname{grad} U\mu(x) \, dx = 0$$

(compare [12], 1.1). We see that the support of any ν satisfying (3) is contained in B or, which is the same, the representing element corresponding to $\mathcal{T}\mu$ (with $\mu \in \mathfrak{B}$) belongs to $\mathfrak{B}$.

For each positive integer k and $M \subset R^m$, $H_k M$ will denote the outer Hausdorff k-dimensional measure of M defined by

(4) $$H_k M = 2^{-k} \alpha_k \liminf_{\varepsilon \to 0+} \sum_n (\operatorname{diam} M_n)^k\,,$$

where α_k is the volume of the unit k-ball and the infimum in (4) is taken over all sequences $\{M_n\}_{n=1}^\infty$ of sets M_n with $\bigcup_n M_n = M$ such that $\operatorname{diam} M_n \leqq \varepsilon$ for all n. H_m thus coincides with the Lebesgue measure in R^m.

For $x \in R^m$, $r > 0$, we shall write

$$\Omega_r(x) = \{z \in R^m;\ |z - x| < r\}\,, \quad \Gamma = \operatorname{fr} \Omega_1(0)\,, \quad A = H_{m-1}(\Gamma)\,.$$

The unit point mass (= Dirac measure) concentrated at y will be denoted by δ_y. The symbol

$$M \dot{-} N$$

will stand for the symmetric difference $(M - N) \cup (N - M)$ of the sets $M, N \subset R^m$.

2. Some background material. Let $M \subset R^m$ be a Borel set with a compact boundary and let $S \subset R^m$ be a half-line. As in [12] (see definition 1.5), $z \in S$ is termed a hit of S on M provided

$$H_1(\Omega_r(z) \cap S \cap M) > 0 \quad \text{and} \quad H_1(\Omega_r(z) \cap (S - M)) > 0$$

for each $r > 0$. Let $\theta \in \Gamma$, $y \in R^m$. The total number (possibly zero or infinite) of all hits of

$$\{z \in R^m;\ z = y + \varrho\theta,\ \varrho > 0\}$$

on M will be denoted by

$$n_\infty^M(\theta, y)\,. \tag{5}$$

For fixed y and M, (5) appears to be a Baire function of the variable $\theta \in \Gamma$ (see [12], proposition 1.6) and hence the definition

$$v_\infty^M(y) = \int_\Gamma n_\infty^M(\theta, y)\, \mathrm{d}H_{m-1}(\theta) \tag{6}$$

is justified.

Let us recall that a unit vector $\theta \in \Gamma$ is called the exterior normal of M at $y \in R^m$ in the sense of H. Federer provided the set

$$M \dot{-} \{x \in R^m;\ (x - y)\,.\,\theta < 0\}$$

has m-dimensional density 0 at y. In what follows we shall put $n^M(y) = \theta$ if $\theta \in \Gamma$ is the exterior normal of M at y and we denote by $n^M(y)$ the zero vector if there is no exterior normal $\theta \in \Gamma$ at y in the above mentioned sense.

If (6) happens to be finite, then M has a well defined m-density

$$d_M(y) = \lim_{\varrho \to 0+} (H_m[\Omega_\varrho(y) \cap M]/H_m[\Omega_\varrho(y)])$$

at y (see the proof of lemma 2.7 in [12]). In view of the inequality

$$|H_m[\Omega_r(y) \cap M] - H_m[\Omega_r(z) \cap M]| \leqq H_m[\Omega_r(y) \dot{-} \Omega_r(z)]\,,$$

$H_m[\Omega_r(y) \cap M]$ is a continuous function of y. In particular, d_M is a bounded Baire function provided (6) is finite for each $y \in B$. If there is no danger of ambiguity, we shall write simply $d(y)$, $v_\infty(y)$, $n_\infty(\theta, y)$, $n(y)$ instead of $d_G(y)$, $v_\infty^G(y)$, $n_\infty^G(\theta, y)$, $n^G(y)$.

Fix now $y \in R^m$. As shown in [12] (see propositions 1.6, 2.6 and lemma 2.2 where $C = R^m - G$; compare also lemma 3.2),

$$v_\infty(y) < \infty \tag{7}$$

is a necessary and sufficient condition for the existence of a $\nu_y \in \mathfrak{B}$ such that, for each $\varphi \in \mathscr{D}$,

$$A\, d(y)\, \varphi(y) + \langle \varphi, \nu_y \rangle = \int_G \operatorname{grad} \varphi(x) \,.\, \operatorname{grad} U\delta_y(x)\, \mathrm{d}x\,. \tag{8}$$

If (7) holds, then (8) determines ν_y uniquely and $v_\infty(y)$ equals the total variation of ν_y.

By Fubini's theorem,

$$\int_B (A\, d(y)\, \varphi(y) + \langle \varphi, \nu_y \rangle)\, \mathrm{d}\mu(y) = \int_G \operatorname{grad} \varphi(x) \,.\, \operatorname{grad} U\mu(x)\, \mathrm{d}x \tag{9}$$

for $\varphi \in \mathscr{D}$ and $\mu \in \mathfrak{B}$ (see (1.2) in [12]).

In order to prove the theorem stated in the introduction it is useful to establish some estimates. The first one expresses actually nothing else than the fact that the total variation of the spatial angle under which the oriented sphere Γ is visible from η does not exceed A.

3. Lemma. *The inequality*

$$\int_\Gamma \frac{|\theta \,.\, (\theta - \eta)|}{|\theta - \eta|^m}\, \mathrm{d}H_{m-1}(\theta) \leqq A \tag{10}$$

holds for every $\eta \in R^m$.

Proof. By lemma 2.12 in [12] it is easily seen that the integral in (10) equals $v_\infty^\Omega(\eta)$ where we put $\Omega = \Omega_1(0)$.

Consider at first $\eta \in \Omega$. Since $n_\infty^\Omega(\theta, \eta) = 1$ for every $\theta \in \Gamma$, (6) yields in this case $v_\infty^\Omega(\eta) = A$.

It remains only to consider $\eta \notin \Omega$. Denoting $\Gamma' = \{\theta \in \Gamma;\ \eta \,.\, \theta < 0\}$ we see that $n_\infty^\Omega(\theta, y) = 0$ for $\theta \notin \Gamma'$ while for $\theta \in \Gamma'$ obviously $n_\infty^\Omega(\theta, \eta) \leqq 2$. Observing that $H_{m-1}(\Gamma') = \frac{1}{2}A$ we obtain again from (6) the desired inequality $v_\infty^\Omega(\eta) \leqq A$; this completes the proof.

4. Lemma. *Let h be a continuously differentiable function of the variable $t \geqq 0$ such that h vanishes near 0, $h' \geqq 0$ and*

$$\lim_{t \to \infty} h(t) = 1\,.$$

Fix $z \in R^m$ and put

$$\varphi_h(x) = h(|x - z|)\,.$$

Then, for each $y \in R^m$,

$$\int_G |\operatorname{grad} \varphi_h(x) \,.\, \operatorname{grad} U\delta_y(x)|\, \mathrm{d}x \leqq A\,.$$

Proof. Put

$$\mathscr{J} = \int_G |\operatorname{grad} \varphi_h(x) \, . \, \operatorname{grad} U\delta_y(x)| \, \mathrm{d}x \, .$$

Simple calculation shows that

$$\mathscr{J} = \int_G h'(|x - z|) \, . \, \left| \frac{x - z}{|x - z|} \, . \, \frac{x - y}{|x - y|^m} \right| \mathrm{d}x \, .$$

Letting, for each $r > 0$,

$$\eta_r = (y - z)/r \, , \quad \mathscr{I}(r) = \int_\Gamma \frac{|\theta \, . \, (\theta - \eta_r)|}{|\theta - \eta_r|^m} \, \mathrm{d}H_{m-1}(\theta) \, ,$$

one obtains easily that

$$\mathscr{J} \leqq \int_0^\infty h'(r) \, \mathscr{I}(r) \, \mathrm{d}r \, .$$

By (10), $\mathscr{I}(r) \leqq A$ foı each $r > 0$ so that $\mathscr{J} \leqq A$.

The proof of the following theorem makes essential use of a result of [12].

5. Theorem. *The condition*

(11) $$\sup_{y \in B} \left[v_\infty(y) + U\lambda(y) \right] < \infty$$

is necessary and sufficient for each $\mathscr{T}\mu$ with $\mu \in \mathfrak{B}$ to be representable by a unique element of $\mathfrak{B}$ over $\mathscr{D}_\mu$.

Proof. Suppose first that for each $\mu \in \mathfrak{B}$, $\mathscr{T}\mu$ coincides with a uniquely determined element of $\mathfrak{B}$ over $\mathscr{D}_\mu$. Then $U\lambda < \infty$ everywhere. Indeed, $U\lambda(z) = \infty$ for some $z \in R^m$ would imply $z \in B$ and $\varphi(z) = 0$ for all $\varphi \in \mathscr{D}_{\delta_z}$, which, in view of the equality

$$\langle \varphi, \mathscr{T}\delta_z \rangle = \langle \varphi, \mathscr{T}\delta_z + k\delta_z \rangle \, , \quad k \in R^1 \, ,$$

would contradict the uniqueness of the representing element corresponding to $\mathscr{T}\delta_z$. We are going to show that

(12) $$\sup_{y \in B} U\lambda(y) < \infty \, .$$

In the opposite case there would be points $y_j \in B$ with $\lim_{j \to \infty} y_j = z \in B - \{y_1, y_2, \ldots\}$ such that

$$U\lambda(y_j) > 2^j$$

for all j. Put

$$\mu = \sum_{j=1}^{\infty} 2^{-j} \delta_{y_j}$$

and fix an arbitrary positive integer n. Further denote by λ_r the restriction of λ to $R^m - \Omega_r(z)$. Choosing $r > 0$ small enough one may suppose that

$$U\lambda_r(y_j) > 2^j, \quad 1 \leqq j \leqq n.$$

For each $\varrho > 0$ denote by h_ϱ an infinitely differentiable non-decreasing function on $\langle 0, \infty)$ such that $h_\varrho = 0$ on $\langle 0, \frac{1}{2}\varrho\rangle$ and $h_\varrho = 1$ on $\langle \varrho, \infty)$. Fix $R > r$ with

$$B \subset \Omega_R(z)$$

and define

$$\varphi(x) = h_r(|x - z|) - h_{2R}(|x - z|).$$

Since $U\lambda$ is finite and φ vanishes in some neighborhood of z, we have $\varphi \in \mathscr{D}_\mu$. Noting that $\varphi = 1$ on $B - \Omega_r(z)$ we obtain

$$I_\mu(\varphi) \geqq \langle U\lambda_r, \mu\rangle = \sum_{j=1}^{\infty} 2^{-j} U\lambda_r(y_j) > n.$$

On the other hand, lemma 4 shows that, for all j,

$$a_j = \int_G |\text{grad}\ \varphi(x) \,.\, \text{grad}\ U\delta_{y_j}(x)| \,\mathrm{d}x \leqq 2A,$$

hence

$$\left|\int_G \text{grad}\ \varphi(x) \,.\, \text{grad}\ U\mu(x)\,\mathrm{d}x\right| \leqq \sum_{j=1}^{\infty} 2^{-j} a_j \leqq 2A,$$

so that

$$\langle \varphi, \mathscr{T}\mu\rangle = I_\mu(\varphi) + \int_G \text{grad}\ \varphi(x) \,.\, \text{grad}\ U\mu(x)\,\mathrm{d}x > n - 2A.$$

We conclude that there is no element of $\mathfrak{B}$ representing $\mathscr{T}\mu$, because n was arbitrary and $|\varphi| \leqq 1$. Thus (12) is proved.

Under the condition (12), $\mathscr{D}_\mu = \mathscr{D}$ for each $\mu \in \mathfrak{B}$ and the Riesz representation theorem may by applied to assert that I_μ is representable by an element of $\mathfrak{B}$. Consequently, the distribution $NU\mu$ defined by

$$\langle \varphi, NU\mu\rangle = \langle \varphi, \mathscr{T}\mu\rangle - I_\mu(\varphi), \quad \varphi \in \mathscr{D},$$

is representable by an element of $\mathfrak{B}$ if and only if the same is true of $\mathscr{T}\mu$. In order to make the proof of our theorem complete it remains to refer to theorem 1.13 in [12] stating that $NU\mu$ is representable by an element of $\mathfrak{B}$ for each $\mu \in \mathfrak{B}$ if and only if

$$\sup_{y \in B} v_\infty(y) < \infty . \tag{13}$$

The proof is complete.

6. Remark. In [12] the distribution $NU\mu$ was called the generalized normal derivative of the potential $U\mu$. In the case of G being a complementary domain of a surface in R^3 such functionals – termed boundary flows – were treated in [2]. In [1] these boundary flows have been applied to the boundary value problem for simply connected domains $G \subset R^m$ satisfying the Carathéodory condition $\operatorname{fr} G = \operatorname{fr} (R^m - \operatorname{cl} G)$. The analoguous concept of "Randströmung" may be found in the classical treatment of J. Plemelj (see [19]). By means of that concept, boundary value problems for plane domains bounded by curves of bounded rotation were studied by J. Radon in [20].

The related weak characterization of boundary values, the so called flows of heat, has recently been considered in [13] in connection with the Fourier problem of the heat equation. Diverse weak characterizations of boundary values and further references may also be found in [22], [14], [3], [6].

The classical methods of potential theory (compare e.g. [11], [10], [7]) usually used for solving the third boundary value problem in the case when appropriate smoothness conditions on the boundary and the boundary condition are required are no longer applicable under the general condition (11). Using the concept of the boundary flow, V. D. Sapožnikova has investigated the third boundary value problem in [21] for a special kind of domains in R^3 by means of potentials having the same interior and exterior limits. An unusual formulation of the third boundary value problem on a Green space is given in [14], [6].

7. Notation. If $\mu \in \mathfrak{B}$, we shall often write $\langle f, \mu \rangle$ instead of

$$\int_B f \, \mathrm{d}\mu$$

provided the last integral is meaningful. $\mathscr{B}$ will stand for the Banach space of all bounded Baire functions on B equipped with the supremum norm $\|\ldots\|$. $\mathscr{C}$ is the subspace of all continuous functions in $\mathscr{B}$. The dual space $\mathscr{B}^*$ of $\mathscr{B}$ is formed by all additive set functions of bounded variation defined on the system of all Borel subsets of B (see [4]). Of course, $\mathfrak{B}$ is a closed subspace of $\mathscr{B}^*$.

If $\mu \in \mathfrak{B}$ and g is a function which is integrable (μ), then $g\mu \in \mathfrak{B}$ is defined by

$$\langle f, g\mu \rangle = \langle fg, \mu \rangle , \quad f \in \mathscr{B} .$$

In what follows we agree to impose (13) *and* (12) *on G and λ, respectively.*

In particular, the m-dimensional density $d(y)$ of G at y is well defined and $d \in \mathscr{B}$. According to theorem 5, for each $\mu \in \mathfrak{B}$, there is only one $\nu \in \mathfrak{B}$ representing $\mathscr{T}\mu$ on $\mathscr{D}$. For each $\mu \in \mathfrak{B}$, $\mathscr{T}\mu$ will be identified with the corresponding element of $\mathfrak{B}$, so that $\mathscr{T}$ becomes a linear operator on $\mathfrak{B}$.

Given $f \in \mathscr{B}$, we define

$$\tilde{W}f(y) = A\, d(y) f(y) + \langle f, \nu_y \rangle\,, \quad y \in B\,. \tag{14}$$

Clearly,

$$|\tilde{W}f(y)| \leqq (A + \|\nu_y\|)\, \|f\| \leqq [A + \sup_{y \in B} v_\infty(y)]\, \|f\|\,.$$

If f is continuous on B, then so is $\tilde{W}f$ (see lemma 3.4 in [12]). Hence it follows easily that $\tilde{W}f \in \mathscr{B}$ whenever $f \in \mathscr{B}$. Consequently,

$$\tilde{W} : f \mapsto \tilde{W}f$$

is a bounded operator acting on $\mathscr{B}$ with

$$\|\tilde{W}\| \leqq A + \sup_{y \in B} v_\infty(y)\,, \tag{15}$$

$$\tilde{W}\mathscr{C} \subset \mathscr{C}\,. \tag{16}$$

It should be noted here that the validity of (13) does not guarantee that B is a smooth or piecewise smooth hypersurface. On the other hand, there are open sets with a piecewise smooth boundary for which the validity of (13) is violated (see [15]). In general, of course, (12) does not imply continuity of the potential $U\lambda$.

The following proposition will be also useful for further investigations of properties of the operator $\mathscr{T}$.

8. Proposition. *Let*

$$Vf(y) = Uf\lambda(y)\,, \quad y \in B\,, \quad f \in \mathscr{B}\,. \tag{17}$$

Then $Vf \in \mathscr{B}$ whenever $f \in \mathscr{B}$ and

$$V : f \mapsto Vf$$

is a bounded operator on $\mathscr{B}$ with

$$\|V\| = \sup_{y \in B} U\lambda(y)\,.$$

If T^* is the dual operator (acting on $\mathscr{B}^*$) of $T = \tilde{W} + V$, then $\mathscr{T}$ coincides with the restriction of T^* to $\mathfrak{B} \subset \mathscr{B}^*$.

Proof. If $\varphi \in \mathscr{D}$ and $\mu \in \mathfrak{B}$, then by Fubini's theorem

$$\langle V\varphi, \mu \rangle = I_\mu(\varphi)\,.$$

In view of (9) and (14) we have

$$\langle \tilde{W}\varphi, \mu \rangle = \int_G \operatorname{grad} \varphi(x) \,.\, \operatorname{grad} U\mu(x) \, \mathrm{d}x \,,$$

hence

$$\langle T\varphi, \mu \rangle = \langle \varphi, \mathscr{T}\mu \rangle \,. \tag{18}$$

The rest is easy. It is sufficient to notice that the class of all $\varphi \in \mathscr{B}$ satisfying (18) contains the limit of each uniformly bounded pointwise convergent sequence of its elements. Consequently, (18) holds for each $\varphi \in \mathscr{B}$.

9. Remark. In particular, the above proposition shows that the operator $\mathscr{T}$ is bounded on $\mathfrak{B}$. The operators defined in this paper are more closely investigated in [17] and [18].

The formula (20) of the following simple lemma enables us to give more geometrical form to the condition (11).

10. Lemma. *Let $\chi_{r,y}$ stand for the characteristic function of $\Omega_r(y)$. If $y \in R^m$, $r > 0$ and λ_0 is a non-negative element of $\mathfrak{B}$, then*

$$U\chi_{r,y}\, \lambda_0(y) = \frac{\lambda_0[\Omega_r(y)]}{(m-2)\, r^{m-2}} + \int_0^r \varrho^{1-m}\, \lambda_0[\Omega_\varrho(y)] \, \mathrm{d}\varrho \,. \tag{19}$$

In particular, the equality

$$U\lambda(y) = \int_0^\infty \varrho^{1-m}\, \lambda[\Omega_\varrho(y)] \, \mathrm{d}\varrho \tag{20}$$

holds for each $y \in R^m$.

Proof. Since

$$U\chi_{r,y}\, \lambda_0(y) = (m-2)^{-1} \int_{\Omega_r(y)} |x-y|^{2-m} \, \mathrm{d}\lambda_0(x) =$$

$$= (m-2)^{-1} \int_0^\infty \lambda_0\{x \in \Omega_r(y);\ |x-y|^{2-m} > t\} \, \mathrm{d}t \,,$$

we get by a simple calculation

$$U\chi_{r,y}\, \lambda_0(y) = \int_0^\infty \varrho^{1-m} \lambda_0\{x \in \Omega_r(y);\ |x-y| < \varrho\} \, \mathrm{d}\varrho \,.$$

Now (19) follows immediately.

Putting $\lambda_0 = \lambda$ in (19) and assuming $r \to \infty$ we obtain (20), which completes the proof.

11. Notation. Let us recall that the set $\hat{B}$ of all y with $n(y) \neq 0$ is called the reduced boundary of G. From proposition 2.10 in [12] and from the results of E. De Giorgi and H. Federer (see [5], [9] and 2.11 in [12]) we know that

$$H_{m-1}(\hat{B}) < \infty .$$

As shown in theorem 4.5 in [8], $\hat{B}$ is a Borel subset of B. In connection with the definition of the set B' given in the introduction it should be mentioned here that

$$H_{m-1}(B' - \hat{B}) = 0$$

(see lemma 3.7 in [12]).

We denote by H the restriction of H_{m-1} to $\hat{B}$. It follows from the results of [12] (see lemma 2.12) that, for each $y \in B$, ν_y is absolutely continuous (H).

The symbol $\mathfrak{B}_H$ will stand for the set of all elements of $\mathfrak{B}$ which are absolutely continuous (H). In particular, as mentioned above, $\nu_y \in \mathfrak{B}_H$ for each $y \in B$. If $\mu \in \mathfrak{B}_H$, the Radon-Nikodym theorem may be applied to assert the existence of a function h integrable (H) over B such that

$$\mathrm{d}\mu = h \, \mathrm{d}H .$$

Then we have

$$\|\mu\| = \int_B |h| \, \mathrm{d}\mu$$

and we conclude that $\mathfrak{B}_H$ is isometrically isomorphic to the Banach space $L_1(B, H)$. In particular, $\mathfrak{B}_H$ is a closed subspace of $\mathfrak{B}$.

There is a question of the relationship between $\mathfrak{B}_H$ and $\mathscr{T}(\mathfrak{B}_H)$. The following proposition (stated in the introduction) answers this question under the natural assumption $\lambda \in \mathfrak{B}_H$. It should be noted here that to give a $\nu \in \mathfrak{B}_H$ is actually the same as to prescribe a density with respect to an "area measure" on B.

12. Proposition. *If $\lambda \in \mathfrak{B}_H$, then*

$$\mathscr{T}(\mathfrak{B}_H) \subset \mathfrak{B}_H \tag{21}$$

and the following conditions are equivalent to each other:

(i) $$\mathscr{T}_{-1}(\mathfrak{B}_H) \subset \mathfrak{B}_H ;$$

(ii) *For each $y \in B$ there is $d(y) \neq 0$.*

Proof. For each $y \in B$ we put

$$\mathrm{d}\pi_y(x) = p(x - y) \, \mathrm{d}\lambda(x) .$$

Since $\lambda \in \mathfrak{B}_H$, it is $\pi_y \in \mathfrak{B}_H$ for each $y \in B$. For any Borel set $M \subset B$ the characteristic function of M is denoted by χ_M.

Suppose that M is an arbitrary Borel subset of B with $H(M) = 0$. Recalling that $\nu_y \in \mathfrak{B}_H$ for each $y \in B$ we see that

$$T\chi_M(y) = A\, d(y)\, \chi_M(y)\,. \tag{22}$$

Consequently,

$$0 = \langle T\chi_M, \mu \rangle = \langle \chi_M, \mathcal{T}\mu \rangle$$

for each $\mu \in \mathfrak{B}_H$ and we conclude that $\mathcal{T}\mu \in \mathfrak{B}_H$; (21) is verified.

Suppose now (ii) and let $\mathcal{T}\mu \in \mathfrak{B}_H$ for a $\mu \in \mathfrak{B}$. We are going to show that $\mu \in \mathfrak{B}_H$. Let Q stand for a Borel subset of B with $H(Q) = 0$ and put $f = \chi_Q\,.\,d$. Choosing a Borel set $P \subset B$ we may apply (22) with $M = P \cap Q$ to obtain

$$0 = \langle \chi_M, \mathcal{T}\mu \rangle = \langle T\chi_M, \mu \rangle = A\langle d\,.\,\chi_M, \mu \rangle = A\langle d\chi_Q\chi_P, \mu \rangle = A\langle f\,.\,\chi_P, \mu \rangle\,.$$

Consequently, $f = 0$ almost everywhere (μ) and by the hypothesis (ii), $\chi_Q(y) = 0$ holds for μ-almost all $y \in B$. Hence $\mu(Q) = 0$ and we have $\mu \in \mathfrak{B}_H$.

As to the converse, suppose $d(y_0) = 0$ for a $y_0 \in B$. Going back to (8), we see that for each $\varphi \in \mathcal{D}$,

$$\mathcal{T}\delta_{y_0}(\varphi) = \langle \varphi, \nu_{y_0} + \pi_{y_0} \rangle\,,$$

hence

$$\mathcal{T}\delta_{y_0} = \nu_{y_0} + \pi_{y_0} \in \mathfrak{B}_H\,.$$

On the other hand, $\delta_{y_0} \notin \mathfrak{B}_H$.

The proof is complete.

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