Juraj Bosák
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CHROMATIC INDEX OF FINITE AND INFINITE GRAPHS

JURAJ BOSÁK, Bratislava

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In this paper we consider only nonoriented graphs without loops, finite or infinite; multiple edges are admissible. As a rule, we do not distinguish between isomorphic graphs.

If a positive integer k and cardinal numbers $p_1, p_2, \ldots, p_k$ are given, denote by $C(p_1, p_2, \ldots, p_k)$ the graph whose vertex set consists of k vertices (denote them by $v_1, v_2, \ldots, v_k$) and whose edge set can be expressed in a form $\bigcup_{i=1}^{k} E_i$, where the sets $E_1, E_2, \ldots, E_k$ are mutually disjoint, $|E_i| = p_i$ for $i = 1, 2, \ldots, k$ [1]) and each edge of E_i $(i = 1, 2, \ldots, k)$ joins v_i with v_{i+1} (we put $v_{k+1} = v_1$). If $p_i = 1$ for all i, we get a circuit of length k; if $p_k = 0$, but $p_i = 1$ for all $i < k$, we obtain a path of length $k - 1$. If h is a finite cardinal number, we put $G_h = C([\frac{1}{2}h], [\frac{1}{2}h]^*, [\frac{1}{2}h])$, $H_h = C([\frac{1}{2}h], [\frac{1}{2}h]^*, [\frac{1}{2}h], [\frac{1}{2}h]^*, [\frac{1}{2}h])$. [2]) If h is an infinite cardinal number, we put $G_h = C(h, h, h)$ and $H_h = C(h, h, h, h, h)$. The diagrams of G_h for $0 \leqq h \leqq 5$ are given in Fig. 1, the diagrams of H_h for $2 \leqq h \leqq 5$, in Fig. 2.

Let a graph G and a cardinal number Q be given. By an *edge-colouring of G by Q colours*, or by a *Q-edge-colouring of G* we mean a mapping of the edge set of G into a set of cardinality Q such that any two adjacent edges are assigned two different elements, so-called *colours* of the edges.

For any graph G, three characteristics of G that are cardinal numbers are defined as follows.

1. The *degree* of G is the supremum $d = d(G)$ of the degrees of all vertices of G. (The degree of a vertex v of G is the cardinality $D = D(v)$ of the set of all edges incident to v in G.)

2. The *multiplicity* of G is the supremum $p = p(G)$ of the multiplicities of all edges of G. (By the multiplicity of an edge e of G we understand the cardinality $P = P(e)$ of the set of all edges incident in G with both end vertices of e.)

[1]) The symbol $|M|$ denotes the cardinality of the set M.

[2]) If a is a real number, the symbol $[a]$ denotes the greatest integer $\leqq a$, and the symbol $[a]^*$ denotes the smallest integer $\geqq a$, i.e. $[a]^* = -[-a]$.

3. The *chromatic index* $q = q(G)$ of G is the least cardinal number Q such that there exists a Q-edge-colouring of G. [3]

If G has no edges, we put $d = p = q = 0$.

Evidently we have $d(G_h) = d(H_h) = h$ for any cardinal number h.

Our aim is to study estimations of q by d and p. Obviously, we always have:

$$p \leqq d \leqq q . \tag{1}$$

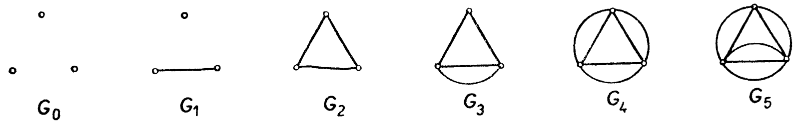

Fig. 1. Graphs G_h $(0 \leqq h \leqq 5)$.

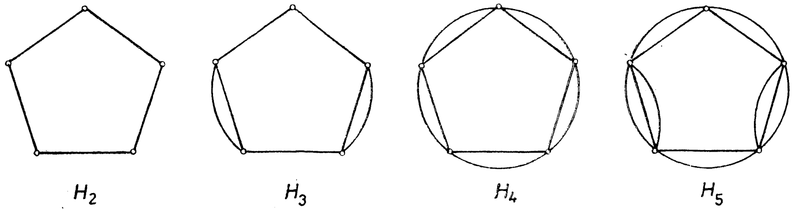

Fig. 2. Graphs H_h $(2 \leqq h \leqq 5)$.

Theorem 1. *If a graph G has an infinite degree d, then its chromatic index $q = d$.*

Proof. Let $d \geqq \aleph_0$. According to (1) we have $q \geqq d$. To prove that $q \leqq d$, we consider any component C of G. Choose a vertex v of G. Denote by C_i $(i = 0, 1, 2, \ldots)$ the set of all vertices of G whose distance from v is i. By induction it is easy to prove that every C_i has at most d vertices. It follows that C has also at most $\aleph_0 \,.\, d = d$ vertices. As every vertex of C is incident to at most d edges, the component C has at most $d \,.\, d = d$ edges. Thus for any component C of G we have $q(C) \leqq d$. Consequently, $q \leqq d$.

Theorem 2. *Let k be a finite cardinal number. If for every finite subgraph H of a graph G there is a k-edge-colouring of H (i.e. $q(H) \leqq k$), then there exists a k-edge-colouring of G (i.e. $q(G) \leqq k$).*

First proof. If G has no edges, the assertion evidently holds. Therefore suppose that G has at least one edge. Form a graph $L(G)$ whose vertices are the edges of G; we join vertices u and v of $L(G)$ by just one edge provided that the edges u and v in G

[3]) The chromatic index is also called the line-chromatic number [6], the edge chromatic number [9], the edge coloration number [9], and the chromatic class [12], [13], [14].

have at least one end vertex in common; otherwise we do not join u and v in $L(G)$ by any edge. Evidently all finite subgraphs of $L(G)$ can be vertex-coloured by k colours. According to [3] (see also [8], Theorem 14.1.3) the graph $L(G)$ can be also vertex-coloured by k colours. It follows that G can be edge-coloured by k colours. The theorem follows.

The theorem of DE BRUIJN and ERDÖS used in the proof is based on a deep theorem by RADO. Therefore we give another proof which is formally longer, but in fact it is simpler because it is based only on a KÖNIG's theorem.

Second proof. (Cf. [7], XIII, §§ 1 and 4.) It suffices to prove our assertion for any component C of G with an infinite number of edges. From the assumptions it follows that G, and consequently also C, has a finite degree. Therefore C has a countable number of vertices and edges (cf. [7], VI, Theorem 1). Arrange all the edges of C into a sequence $\{e_1, e_2, \ldots, e_i, \ldots\}$. Let D_i be the subgraph of C generated by the edges $e_1, e_2, \ldots, e_i$ $(i = 1, 2, 3, \ldots)$. Further denote by Π_i the set of all edge-colourings of D_i by colours $1, 2, \ldots, k$. Obviously every Π_i is a finite nonempty set.

Form a graph P whose vertex set is $\bigcup_{i=1}^{\infty} \Pi_i$; join a vertex $u \in \Pi_i$ with a vertex $v \in \Pi_{i+1}$ if and only if colourings u and v assign the same colour to all edges of D_i; suppose P contains no other edges. Evidently the graph P fulfils the suppositions of KÖNIG's Theorem 6 from [7], VI (see also [1], III, Corollary 2 of Theorem 2); therefore there exists in G an infinite path $v_1 v_2 v_3 \ldots$ such that $v_i \in \Pi_i$ for all $i = 1, 2, 3, \ldots$

Define an edge-colouring of C in the following way: If e_i is an edge of C, assign to e_i the same colour that is assigned to e_i by the edge-colouring v_i of D_i (and thus by every v_j where $j > i$). Obviously we obtain a k-edge-colouring of C.

Remark. From Theorem 1 it follows that we can restrict our considerations to graphs with finite degrees.

Theorem 3. *Let G be a graph of finite degree d, multiplicity p and chromatic index q. Then we have:*

(2) $q \leqq d + p$;
(3) $q \leqq [\frac{3}{2}d]$;
(4) *if $d \geqq 4$ and G does not contain the subgraph G_d, then* $q \leqq [\frac{3}{2}d] - 1$.

Proof. In the case of finite graphs the estimations hold. (2) has been proved by VIZING [12], (3) by SHANNON [11], (4) again by VIZING [13]. (For proofs of (2) and (3) see also [2], [9], [13] and [14].) The validity of these results can be easily extended to infinite graphs by means of Theorem 2. It is sufficient in each case to define the number k and to check the assumptions of Theorem 2.

(2) As d is finite, by (1) p is also finite. Put $k = d + p$. Evidently for every finite subgraph H of G we have

$$q(H) \leqq d(H) + p(H) \leqq d + p = k\,.$$

Theorem 2 yields

$$q(G) \leqq k = d + p .$$

(3) Put $k = [\frac{3}{2}d]$. For any finite subgraph H of G we have

$$q(H) \leqq [\tfrac{3}{2}d(H)] \leqq [\tfrac{3}{2}d] = k ,$$

so Theorem 2 implies

$$q(G) \leqq k = [\tfrac{3}{2}d] .$$

(4) Put $k = [\frac{3}{2}d] - 1$. Let H be a finite subgraph of G. Distinguish two cases:

(i) $d(H) = d$. Since G does not contain the subgraph G_d, H also does not contain the subgraph $G_d = G_{d(H)}$. According to (4) already proved for finite graphs, we have:

$$q(H) \leqq [\tfrac{3}{2}d(H)] - 1 \leqq [\tfrac{3}{2}d] - 1 = k .$$

(ii) $d(H) \leqq d - 1$. Then by (3) we get

$$q(H) \leqq [\tfrac{3}{2}d(H)] \leqq [\tfrac{3}{2}(d - 1)] \leqq [\tfrac{3}{2}d] - 1 = k .$$

In both cases we have obtained $q(H) \leqq k$. According to Theorem 2 we get

$$q(G) \leqq k = [\tfrac{3}{2}d] - 1 .$$

Corollary. (For finite graphs see [2], [12], [13], [14].) *For a graph of finite degree d, of chromatic index q and without multiple edges we always have $q = d$ or $q = d + 1$.*

Proof follows from (1) and (2) for $p = 1$.

Remark. Relations (2) and (3) can be generalized as follows. Let G be a graph of finite degree d with a chromatic index q. Denote by V the vertex set of G and for $u \in V$ put

$$D^*(u) = D(u) + \max_{v \in V} P(u, v) ,$$

where $D(u)$ is the degree of u and $P(u, v)$ is the number of edges joining u and v. Then we have:

$$(2') \qquad q \leqq \max_{u \in V} D^*(u) ,$$

$$(3') \qquad q \leqq \max \{d, \max_{(x,y,z)} [\tfrac{1}{2}(D(x) + D(y) + D(z))]\} ,$$

where the second maximum is related to all paths (x, y, z) of length two in G. Relation (2′) for finite graphs has been proved in [9], Theorem 14.4.1 and [2], XII, Corollary 1 of Theorem 6, relation (3′) in [9], Theorem 14.3.1 and [2], XII, Theorem 7.

The validity of (2′) and (3′) can be extended into infinite graphs of finite degrees using Theorem 2 analogously as in the proof of Theorem 3.

A generalization of (4) will be studied in Theorem 5.

Theorem 4. (Cf. [2], XII, § 2.) *Let G be a graph of degree $d \leqq 5$. Then for the chromatic index q of G we have*:

(5) *If G contains the subgraph G_5 (Fig. 1), then $q = 7$.*

(6) *If G does not contain the subgraph G_5, but G contains the subgraph G_4 (Fig. 1), then $q = 6$.*

(7) *If G does not contain G_4 as a subgraph, then $q = d$ or $q = d + 1$.*

Proof. (5) If G contains G_5, then evidently $q \geqq 7$. From (3) it follows that $q \leqq 7$.

(6) If G contains G_4, then evidently $q \geqq 6$ and (3) implies $d \geqq 4$. If $d = 4$, then (3) implies $q \leqq 6$. If $d = 5$, then by (4) we again get $q \leqq 6$. Therefore $q = 6$.

(7) The inequality $d \leqq q$ follows from (1). The inequality $q \leqq d + 1$ for $d \leqq 3$ follows from (3), for $d = 4$ and $d = 5$ it follows from (4).

Remark. (7) does not hold in general for graphs of a finite degree $d \geqq 6$. From the proof of (10) given below it follows that to every positive integer d there exists a graph G of degree d with chromatic index $q = [\frac{3}{2}d] - [\frac{1}{4}d]$ not containing G_4 (it is sufficient to take $G = H_d$); obviously, for any integer $d \geqq 6$ we have $q(H_d) \geqq$ $\geqq d + 2$.

On the other hand, for a graph of degree $d \leqq 2$ it is very easy to determine its chromatic index q. Evidently if $d = 0$ or $d = 1$, then $q = d$. Further, if $d = 2$, then $q = d$ if the graph is bipartite and $q = d + 1$ otherwise.

Lemma 1. *Let $s \geqq 4$ and d be cardinal numbers and let G be a graph of degree $\leqq d$ not containing the graph G_s as a subgraph. Then there exists a regular graph H of degree d not containing G_s as a subgraph such that G is a subgraph of H.*

Proof. (Cf. the proof of Theorem in [4].) Let V be the vertex set of G. For $v \in V$ denote by A_v the set of all vertices of G adjacent to v. Obviously there is a set system $\{B_v\}_{v \in V}$ such that

1° $|A_v \cup B_v| = d$ for all $v \in V$;
2° $V \cap B_v = \emptyset$ for all $v \in V$;
3° $B_u \cap B_v = \emptyset$ for all $u, v \in V$.

Put $B = \bigcup_{v \in V} B_v$. First suppose that B has (finite and) odd number of elements. Add to the set V one vertex $x \notin V \cup B$ and to the system $\{B_v\}_{v \in V}$ one set B_x of cardinality $|B_x| = d$ in such a way that $B_x \cap B = \emptyset$ and $B_x \cap V' = \emptyset$, where $V' = V \cup \{x\}$. Put $B' = B \cup B_x$. If B has an even or an infinite number of elements, put $V' = V$, $B' = B$. From the well-known fact (see e.g. [7], II, Theorem 3) that a finite graph has always an even number of vertices of odd degree it follows that in any case the set B' has an even or an infinite number of elements.

Construct a graph H as follows. The vertex set of H is $V' \cup B'$. All the edges of G are also edges of H. Moreover, for any $v \in V'$ join each vertex of B'_v (provided that $B'_v \neq \emptyset$) by one new edge with vertex v. Arrange all the elements of B' in an arbitrary way into pairs. Join the vertices belonging to the same pair by $d - 1$ edges if d is finite, and by d edges if d is infinite. It is easy to show that H thus constructed fulfils all the conditions of our lemma.

Remarks. 1. For $s = 1$ Lemma 1 does not hold. For $s = 2$ and $s = 3$ it takes place, but must be proved in a different way.

2. The construction given in the proof has the property that if G and d are finite, then H is finite as well.

Theorem 5. *Let $s \geqq 4$ and $d \geqq 0$ be integers such that $d \leqq 2s$ or $d \equiv 0 \pmod 2$. Then for the chromatic index q of any graph of degree d not containing G_s as a subgraph we have:*

$$d \leqq q \leqq \left[\frac{3d}{2}\right] - \left[\frac{d}{s}\right]. \tag{8}$$

Remark. For d and s even (and finite graphs) (8) was proved by Berge [2], XII, Theorem 8. He conjectured that (8) holds for any integers $s \geqq 4$ and $d \geqq 0$.

Proof. The lower estimation follows from (1). The upper estimation for $0 \leqq d \leqq s - 1$ follows from (3), for $s \leqq d \leqq 2s - 1$ from (4). We shall prove (8) for $d = 2s$. Let G be a graph of degree $2s$ not containing G_s. We must prove that there exists an edge-colouring of G by $3s - 2$ colours. According to Lemma 1 there is a regular graph H of degree $2s$ not containing G_s such that G is a subgraph of H.

If s is even, then by [7], XIII, Theorem 2 H is decomposable into two regular factors of degree s. For each of these two factors with respect to (8) already proved for $d = s$ there exists an edge-colouring by $\frac{3}{2}s - 1$ colours. It follows that for H and consequently also for G there exists an edge-colouring by $3s - 2$ colours.

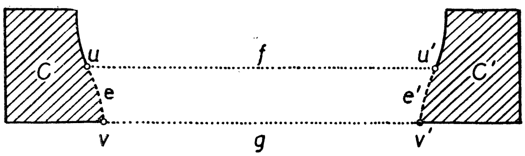

Fig. 3. Graph D from the proof of Theorem 5.

Suppose now that s is odd. Let C be any component of H. If C has an even or an infinite number of vertices, then by [5] (§§ 7–8) C is decomposable into two regular factors of degree s. Each of them can be edge-coloured by $[\frac{3}{2}s] - 1 = \frac{1}{2}(3s - 3)$ colours so that C is edge-colourable by $3s - 3 \leqq 3s - 2$ colours.

If C has an odd number of vertices, take another specimen (an isomorphic copy) C' of C. Suppose that an isomorphism of C onto C' assigns to an edge e of C with end vertices u and v an edge e' of C' with end vertices u' and v'. (Fig. 3.) Form a graph D consisting of all elements (vertices and edges) of C and C' except edges e and e' and, moreover, let D contain an edge f with end vertices u, u' and an edge g with end vertices v, v'. C and C' are finite connected regular graphs of an even degree $2s$ so that they have no bridges (after the removal of a bridge two components with just one vertex of an odd degree would arise, which is impossible). Therefore D is connected. Further, D is a regular graph of even degree $2s$ with an even number of vertices. According to [7], p. 160 or [5], p. 148, D can be decomposed into two regular factors of degree s. Each of the two factors can be edge-coloured by $[\frac{3}{2}s] - 1 = \frac{1}{2}(3s - 3)$ colours. Therefore all edges of C with the exception of e can be coloured by $3s - 3$ colours. When we colour the remaining edge e by another colour, we obtain an edge-colouring of C by $3s - 2$ colours.

If we repeat this argument for every component of H, we prove the existence of an edge-colouring of H and thus also of G by $3s - 2$ colours. So (8) is proved for $d = 2s$ and, consequently, for all $d \leqq 2s$.

Suppose now that for a fixed s (8) holds for all even $d < k$, where k is even, and $k > 2s$. We shall prove that (8) holds for $d = k$ as well. Let G be a graph of degree k not containing G_s. By Lemma 1 there is a regular graph F of degree k not containing G_s. According to [7], XIII, § 1 or [5], §§ 7–8 F is decomposable into a regular factor F_1 of degree $2s$ and a regular factor F_2 of degree $k - 2s$. From (8) already proved for $d = 2s$ it follows that F_1 can be edge-coloured by $3s - 2$ colours. According to induction hypothesis, F_2 can be edge-coloured by

$$\frac{3(k - 2s)}{2} - \left[\frac{k - 2s}{s}\right] = \frac{3k}{2} - \left[\frac{k}{s}\right] - 3s + 2$$

colours. It follows that F and, consequently, also G can be edge-coloured by

$$\frac{3k}{2} - \left[\frac{k}{s}\right]$$

colours, q.e.d.

Remark. We do not know whether the upper estimation in (8) is valid for odd $d \geqq 2s + 1$.

Corollary 1. *Let an integer $s \geqq 4$ be given. Then for the chromatic index of any graph G of odd degree d not containing G_s, we have*:

$$d \leqq q \leqq 3\left[\frac{d + 1}{2}\right] - \left[\frac{d + 1}{s}\right]. \tag{9}$$

Proof. The lower estimation follows from (1). To prove the upper estimation it suffices to take into consideration that according to Lemma 1 there exists a graph H

of even degree $d + 1$ not containing G_s such that G is a subgraph of H. From (8) it follows:

$$q = q(G) \leqq q(H) \leqq \left[\frac{3(d+1)}{2}\right] - \left[\frac{d+1}{s}\right] = 3\left[\frac{d+1}{2}\right] - \left[\frac{d+1}{s}\right].$$

Corollary 2. *Let an even integer $s \geqq 4$ be given. Then for the chromatic index q of any graph of finite degree d not containing G_s the inequalities* (9) *hold.*

Proof. For even d (9) coincides with (8). For odd d the assertion has been proved in Corollary 1.

Remark. In some cases when s is odd and d is even (9) need not hold. For $s = 5$ and $d = 4$, G_4 is such a counterexample.

Lemma 2. *In a bipartite graph of any degree d, the chromatic index $q = d$.*

Proof. For graphs of a finite degree d the assertion is proved in [7], XI, Theorem 15 and XIII, § 4.

For graphs of an infinite degree d the assertion follows from Theorem 1.

Lemma 3. *Let an integer $k \geqq 2$ and cardinal numbers $p_1, p_2, \ldots, p_k$ be given. Denote by d the degree and by q the chromatic index of the graph $C(p_1, p_2, \ldots, p_k)$. Then we have:*

(i) *If some of the cardinal numbers p_i $(i = 1, 2, \ldots, k)$ is infinite, then $q = d$.*
(ii) *If some $p_i = 0$, then $q = d$.*
(iii) *If k is even, then $q = d$.*
(iv) *If k is odd and all p_i are positive integers, then*

$$q = \max\left\{d, \left[\frac{2(p_1 + p_2 + \ldots + p_k)}{k-1}\right]^*\right\}.$$

Proof. (i) follows from Theorem 1, (ii) and (iii) from Lemma 2, (iv) is proved in [9], Theorem 14.1.4 and [2], XII, Theorem 5.

Remark. It is evident that the lower estimation of (8) is sharp for every $s \geqq 2$ and d, because it is attained for any bipartite graph of degree d. For studying the upper estimation it will be useful to denote by $S(d, s)$ the maximal chromatic index of a graph of degree d not containing G_s as a subgraph. Obviously $S(d, s)$ is defined for any integers $d \geqq 0$ and $s \geqq 2$.

Lemma 4. *Let integers $d \geqq 0$ and $s \geqq 2$ be given. Then we have*:

(10) $$S(d, s) \geqq [\tfrac{3}{2}d] - [\tfrac{1}{4}d];$$

(11) $$S(d, s) \geqq [\tfrac{1}{2}s] + d - 1, \quad \text{if} \quad s \leqq d.$$

Proof. (10) follows from the fact that the graph H_d has degree d, it contains no subgraph G_s where $s \geqq 2$ and it has chromatic index $[\frac{3}{2}d] - [\frac{1}{4}d]$, which may be checked directly or by using Lemma 3.

(11) follows from the fact that the graph $C([\frac{1}{2}d], [\frac{1}{2}d]^*, [\frac{1}{2}s] - 1)$ has degree d, it does not contain G_s and it has chromatic index $[\frac{1}{2}s] + d - 1$.

Lemma 5. *Let integers $d \geqq 0$ and $s \geqq 4$ be given. Then we have:*

$$S(d, s) \leqq \left[\frac{3d}{2}\right] - \left[\frac{d}{s}\right] \quad \textit{if } d \leqq 2s \textit{ or if } d \textit{ is even}; \tag{12}$$

$$S(d, s) \leqq 3\left[\frac{d+1}{2}\right] - \left[\frac{d+1}{s}\right] \quad \textit{if } d \textit{ is odd or if } s \textit{ is even}. \tag{13}$$

Proof. (12) follows from Theorem 5.

(13) follows from Corollaries 1 and 2 of Theorem 5.

Theorem 6. *Let integers $s \geqq 2$ and $d \geqq 0$ be given. Then for the maximal chromatic index $S(d, s)$ of a graph of degree d not containing G_s as a subgraph we have:*

(14) $[\frac{3}{2}d] - [\frac{1}{4}d] \leqq S(d, 2) \leqq S(d, 3) \leqq S(d, 4) \leqq 3[\frac{1}{2}(d + 1)] - [\frac{1}{4}(d + 1)]$.

(15) *If $4 < s < \frac{1}{3}d + 3$, then*

$$\left[\frac{3d}{2}\right] - \left[\frac{d}{4}\right] \leqq S(d, s) \leqq \begin{cases} \left[\frac{3d}{2}\right] - \left[\frac{d}{s}\right] & \textit{if } d \textit{ is even}; \\ 3\left[\frac{d+1}{2}\right] - \left[\frac{d+1}{s}\right] & \textit{if } d \textit{ is odd.} \end{cases}$$

(16) *If $\frac{1}{3}d + 3 \leqq s < \frac{1}{2}d$, then*

$$\left[\frac{s}{2}\right] + d - 1 \leqq S(d, s) \leqq \begin{cases} \left[\frac{3d}{2}\right] - \left[\frac{d}{s}\right] & \textit{if } d \textit{ is even}; \\ 3\left[\frac{d+1}{2}\right] - \left[\frac{d+1}{s}\right] & \textit{if } d \textit{ is odd.} \end{cases}$$

(17) *If $\frac{1}{2}d \leqq s < d$ and $s \geqq 4$, then*

$$\left[\frac{s}{2}\right] + d - 1 \leqq S(d, s) \leqq \left[\frac{3d}{2}\right] - \left[\frac{d}{s}\right].$$

(18) *If $s = d \geqq 4$, then*

$$S(d, s) = [\tfrac{3}{2}d] - 1.$$

(19) *If $s = d < 4$ or if $s > d$, then*

$$S(d, s) = [\tfrac{3}{2}d].$$

Proof. (14) follows from (10) and (13).

(15) follows from (10), (12) and (13).
(16) follows from (11), (12) and (13).
(17) follows from (11) and (12) as $d \leqq 2s$.
(18) follows from (11) and (12).

It remains to prove (19). The inequality $S(d, s) \leqq [3d/2]$ follows from (3). The converse inequality for $s = d < 4$ follows from (10), for $s > d$ from the fact that the graph G_d has degree d, chromatic index $[3d/2]$ and it does not contain G_s as a subgraph.

Remark. The number $\frac{1}{3}d + 3$ in (15) and (16) was chosen in such a way that always the better one of the estimations (10) and (11) is used.

Corollary. *If $d \leqq 8$ or if d is even, then*

$$S(d, 2) = S(d, 3) = S(d, 4) = [\tfrac{3}{2}d] - [\tfrac{1}{4}d] .$$

Proof. For d even the corollary follows from (14), for $d \leqq 8$ from (14) and Theorem 5.

Table 1.

$S(d, s)$		s						
		2	3	4	5	6	7	8
	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	1
	2	3	3	3	3	3	3	3
	3	4	4	4	4	4	4	4
	4	5	5	5	6	6	6	6
	5	6	6	6	6	7	7	7
	6	8	8	8	8	8	9	9
	7	9	9	9	9	9	9	10
	8	10	10	10	10—11	10—11	10—11	11
d	9	11—12	11—12	11—12	11—12	11—12	11—12	12
	10	13	13	13	13	13—14	13—14	13—14
	11	14—15	14—15	14—15	14—15	14—15	14—15	14—15
	12	15	15	15	15—16	15—16	15—17	15—17
	13	16—18	16—18	16—18	16—18	16—18	16—18	16—18
	14	18	18	18	18—19	18—19	18—19	18—20
	15	19—20	19—20	19—20	19—21	19—21	19—21	19—21
	16	20	20	20	20—21	20—22	20—22	20—22
	17	21—23	21—23	21—23	21—24	21—24	21—24	21—24
	18	23	23	23	23—24	23—24	23—25	23—25

Problem 1. Improve estimations (15), (16) and (17).

Remarks. 1. Using the above results, the values $S(d, s)$ given in Table 1 were calculated.

2. From Corollary 2 of Theorem 5 for $s = 4$ we obtain that for the chromatic index q of any graph of a finite degree d not containing G_4 (Fig. 1) as a subgraph, we have:

$$d \leqq q \leqq 3\left[\frac{d+1}{2}\right] - \left[\frac{d+1}{4}\right]. \tag{20}$$

This result (for finite graphs) was established by FIAMČÍK and JUCOVIČ [4] (see also [2], XII, Corollary of Theorem 8). As mentioned in Remark before Lemma 4, the lower estimation of (20) is sharp for every d. From Corollary of Theorem 6 or from (14) it follows that the upper estimation of (20) cannot be improved for even d. Further, from (14) it follows that the upper estimation of (20) can be improved at most by two if $d \equiv 1 \pmod 4$ and at most by one if $d \equiv 3 \pmod 4$. We believe that these improvements really take place, i.e.

$$S(d, 4) = \left[\frac{3d}{2}\right] - \left[\frac{d}{4}\right] = \left[\frac{5d+2}{4}\right]$$

for all nonnegative integers d. This assertion may be expressed also in the following form:

Conjecture 1. For the chromatic index q of any graph of a finite degree d not containing subgraph G_4 (Fig. 1), we have:

$$d \leqq q \leqq \left[\frac{5d+2}{4}\right]. \tag{21}$$

Remark. Corollary of Theorem 6 implies the validity of (21) for $d \leqq 8$ and for even d. From the above considerations it follows that if the upper estimation of (21) holds for some d, then it is sharp; in fact, it is attained for the graph H_d.

The following conjecture is concerned also with graphs containing subgraphs G_4.

Conjecture 2. Let G be a graph of finite degree d. Denote by $t = t(G)$ the maximal number of edges of a subgraph of G with at most three vertices.[4]) Then for the chromatic index q of G we have:

$$\text{If} \quad t \geqq \left[\frac{5d+2}{4}\right], \quad \text{then} \quad q = t. \tag{22}$$

$$\text{If} \quad t < \left[\frac{5d+2}{4}\right], \quad \text{then} \quad \max\{d, t\} \leqq q \leqq \left[\frac{5d+2}{4}\right]. \tag{23}$$

[4]) Evidently, we have $p \leqq t \leqq [3d/2]$, where p is the multiplicity of G.

Remark. From (1), (3) and (4) it can be deduced that Conjecture 2 is true for $d \leqq 7$. Moreover, it can be proved that the validity of Conjecture 2 for some positive integer d implies the validity of Conjecture 1 for the same d.

Lemma 6. *Let three integers p, k and c be given such that $p \geqq 1$, $k \geqq 1$, and $0 \leqq c \leqq p(k-1)$. Then there exist graphs $G(p, k, c)$ and $H(p, k, c)$ such that*

(i) *$G(p, k, c)$ has $2k+1$ vertices, $(2k+1)kp - 2kc - c$ edges, degree $2kp - 2c$ and multiplicity p;*

(ii) *$H(p, k, c)$ has $2k+1$ vertices, $(2k+1)kp - k(2c+1) - c - 1$ edges, degree $2kp - 2c - 1$ and multiplicity p.*

Proof. Let G be a graph on $2k+1$ vertices in which any two different vertices are joined just by p edges. G contains a factor F that is a complete graph on $2k+1$ vertices. It is well-known ([6], Theorem 9.6) that F is decomposable into k Hamiltonian lines $h_1, h_2, \ldots, h_k$. Decompose G into k factors $F_1, F_2, \ldots, F_k$ in such a way that an edge e of G belongs to F_i $(i = 1, 2, \ldots, k)$ if and only if in h_i there is an edge with the same end vertices as e. Obviously every F_i can be decomposed into p Hamiltonian lines $h_{i1}, h_{i2}, \ldots, h_{ip}$. Denote by H the set of all Hamiltonian lines h_{ij}, where $i \in \{1, 2, \ldots, k-1\}$, $j \in \{1, 2, \ldots, p\}$. Evidently $|H| = p(k-1)$. Let $H^{(c)}$ be a c-element subset of H. Remove from G all edges belonging to some Hamiltonian line from $H^{(c)}$. We obtain a graph $G(p, k, c)$. Further, remove from $G(p, k, c)$ $k+1$ edges of F_k in such a way that every vertex of G is incident with at least one removed edge. We get a graph $H(p, k, c)$. It is easy to show that the graphs $G(p, k, c)$ and $H(p, k, c)$ have the required properties.

Remark. It is well-known [2], [11], [12], [13], [14] that the estimation (3) is sharp in the sense that to every finite cardinal number d there exists a graph G such that $q(G) = [\frac{3}{2}d(G)]$; it suffices to put $G = G_d$. From (18) it follows that (4) is also sharp for every integer $d \geqq 4$. The investigation of (2) from this point of view was suggested by VIZING [12]. For studying (2) or the estimation

$$q \leqq \min\{[\tfrac{3}{2}d], d + p\}, \tag{24}$$

which follows from (2) and (3), it will be useful to define a function P as follows. If d and p are positive integers such that $d \geqq p$, denote by $P(d, p)$ the greatest positive integer q for which there is a graph G with $d(G) = d$, $p(G) = p$, and $q(G) = q$. Evidently, under the given conditions $P(d, p)$ is always defined. From (1), (2) and (3) we immediately obtain

$$d \leqq P(d, p) \leqq d + p, \tag{25}$$

$$d \leqq P(d, p) \leqq [\tfrac{3}{2}d]. \tag{26}$$

Theorem 7. *Let d and p be positive integers. Then for the maximal chromatic index $P(d, p)$ of a graph of degree d and multiplicity p we have:*

(27) *If* $d \geqq 2p$, *then* $P(d, p) \geqq 3p$.

(28) *If* $d \geqq 2p - 1$, *then* $P(d, p) \geqq d + [[d/2]/[d/2p]^*]^*$.

Proof. (27) The inequality $3p \leqq P(d, p)$ for $3p \leqq d$ follows from (25); for $3p \geqq d$ follows from the existence of a graph which consists of the graph G_{2p} with vertices u, v and w and of $d - 2p$ other vertices of degree 1 each of which is adjacent to u. (Fig. 4.) Evidently, the multiplicity of this graph is p, the degree is d, and the chromatic index is $3p$.

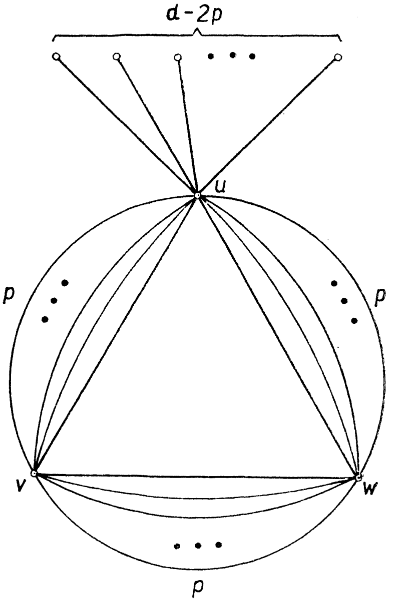

Fig. 4. The graph from the proof of (27).

(28) Obviously, this is all that is needed: for every pair of positive integers p and d, where $d \geqq 2p - 1$, to construct a graph of multiplicity p, degree d and chromatic index

$$q \geqq d + \left[\frac{\left[\frac{d}{2}\right]}{\left[\frac{d}{2p}\right]^*}\right]^* .$$

Put

$$k = \left[\frac{d}{2p}\right]^*,$$

$$c = p\left[\frac{d}{2p}\right]^* - \left[\frac{d+1}{2}\right].$$

It is easy to ascertain that the assumptions of Lemma 6 are fulfilled. Thus the corresponding graphs $G(p, k, c)$ and $H(p, k, c)$ of multiplicity p and with $2k + 1$ vertices exist.

Let any edge-colouring of $G(p, k, c)$ and any edge-colouring of $H(p, k, c)$ be given. It is evident that in each of these two graphs at most k edges may be coloured by the same colour. Distinguish two cases:

I. d is even. Then $G(p, k, c)$ has degree $d = 2kp - 2c$. It has $(2k + 1)\,kp - 2kc - c$ edges and $2k + 1$ vertices, therefore according to Theorem 4 of [2], XII its chromatic index

$$q \geqq \frac{(2k + 1)\,kp - 2kc - c}{k} = (2k + 1)\,p - 2c - \frac{c}{k} = d + p - \frac{c}{k}.$$

It follows that

$$q \geqq d + p - \left[\frac{c}{k}\right] = d + p - \left[p - \frac{\left[\frac{d+1}{2}\right]}{\left[\frac{d}{2p}\right]^*}\right] = d + \left[\frac{\left[\frac{d}{2}\right]}{\left[\frac{d}{2p}\right]^*}\right]^* .$$

II. d is odd. Then $H(p, k, c)$ has degree $d = 2kp - 2c - 1$. As it has $(2k + 1) kp - k(2c + 1) - c - 1$ edges and $2k + 1$ vertices, its chromatic index

$$q \geqq \frac{(2k + 1) kp - k(2c + 1) - c - 1}{k} =$$

$$= (2k + 1) p - 2c - 1 - \frac{c+1}{k} = d + p - \frac{c+1}{k} .$$

It follows that

$$q \geqq d + p - \left[\frac{c+1}{k}\right] = d + p - \left[p - \frac{\left[\frac{d-1}{2}\right]}{\left[\frac{d}{2p}\right]^*}\right] = d + \left[\frac{\left[\frac{d}{2}\right]}{\left[\frac{d}{2p}\right]^*}\right]^* .$$

Corollary 1. (For finite graphs see [12].) *If d and p are positive integers such that d is divisible by $2p$, then $P(d, p) = d + p$.*

Proof follows immediately from (25) and (28), because in this case we have

$$\left[\frac{\left[\frac{d}{2}\right]}{\left[\frac{d}{2p}\right]^*}\right]^* = p .$$

Remark. Now we shall be concerned with the case $d > 2p$. In this case d may be written in the form $d = 2px + y$, where x and y are integers, $x \geqq 1$, $0 < y \leqq 2p$. In fact, it suffices to put

$$x = \left[\frac{d-1}{2p}\right] = \left[\frac{d}{2p}\right]^* - 1 ,$$

$$y = d - 2px .$$

Corollary 2. *Let d, p, x and y be positive integers such that $d = 2px + y$ and $y \leqq 2p$. Let l be such a nonnegative integer that $(l + 1) x + [\frac{1}{2}y] \geqq p - l$. Then we have: $P(d, p) \geqq d + p - l$.*

Proof.

$$\left[\frac{[d/2]}{[d/2p]^*}\right]^* = \left[\frac{[px + y/2]}{[x + y/2p]^*}\right]^* = \left[\frac{p(x+1) - p + [y/2]}{x+1}\right]^* =$$

$$= p + \left[\frac{-p + [y/2]}{x+1}\right]^* \geqq p + \left[\frac{-l - (l+1)\,x}{x+1}\right]^* = p + \left[-l - \frac{x}{x+1}\right]^* = p - l\,.$$

Therefore the assertion follows from (28).

Corollary 3. *Let d and p be positive integers such that $d \geqq 2p$. Then we have:*

$$d + \frac{x}{x+1}\,p \leqq P(d, p) \leqq d + p\,,$$

where $x = [(d-1)/2p]$.

Proof. The upper estimation follows from (25). The lower estimation follows from (28), because $d = 2px + y$, $0 < y \leqq 2p$ so that

$$\left[\frac{[d/2]}{[d/2p]^*}\right]^* \geqq \left[\frac{px}{x+1}\right]^* \geqq \frac{x}{x+1}\,p\,.$$

Corollary 4. *Let d and p be positive integers such that $d \geqq 2p$. Then we have:*

$$d + [\tfrac{1}{2}p]^* \leqq P(d, p) \leqq d + p\,.$$

Proof. For $d = 2p$ the assertion follows from Corollary 1. For $d > 2p$ we have

$$x = \left[\frac{d-1}{2p}\right] \geqq 1\,,$$

so the assertion follows from Corollary 3.

Corollary 5. *Let d and p be positive integers. Suppose that there exists an integer x such that*

$$0 \leqq x < p$$

and

$$2px + 2p - 2x \leqq d \leqq 2px + 2p\,.$$

Then $P(d, p) = d + p$.

Proof. (25) implies $P(d, p) \leqq d + p$. The inequality $P(d, p) \geqq d + p$ for $x = 0$ follows from Corollary 1, for $x \geqq 1$ from Corollary 2 if $l = 0$. In fact, we have

$$x + [\tfrac{1}{2}y] \geqq x + [\tfrac{1}{2}(2p - 2x)] = p\,.$$

Theorem 8. *Let p and d be positive integers such that $2p < d < 3p$. Then any graph of degree d and of multiplicity p has chromatic index $q \leqq d + p - 1$.*

Proof. First we prove the assertion for finite graphs. We use the method of VIZING ([13], p. 32; see also [2], XII, § 2) that was applied by him to the case $d = 2p + 1$.

Assume that there exists a finite graph of degree d and multiplicity p $(2p < d < 3p)$ whose chromatic index is $d + p$. Let G be a graph of these properties with the least number of edges. Then every vertex x of G is incident just with two "sheaves" of p mutually parallel (i.e. with the same end vertices) edges. In fact, the number of these sheaves cannot be $\geqq 3$, since the degree of G is less than $3p$. On the other hand, if x is not incident with two such sheaves, delete from G one edge incident with x of maximal possible multiplicity. Taking into account the minimality of G the edges of the graph obtained in such a way (denote it by G') can be coloured by $d + p - 1$ colours provided that $d(G') = d$ and $p(G') = p$; however, if $d(G') < d$ or $p(G') < p$, then $d(G') + p(G') \leqq d + p - 1$, and we get the same result from (2). According to [2], XII, Theorem 6, G then can be edge-coloured by $d + p - 1$ colours and we arrived at a contradiction.

It follows that in G there exists a regular factor F_1 of degree $2p$. The complementary factor of F_1 with respect to G will be denoted by F_2 (it need not be regular).

Suppose that F_1 contains a triangle. Then F_1 contains three vertices u, v and w such that any two of them are joined by p edges. If u, v and w are contracted into single new vertex V, a new graph G^* arises. Evidently, the degree of V is at most $3(d - 2p)$. Since $d < 3p$, we have $3(d - 2p) < d$. Therefore the degree of G^* is at most d. It is easy to see that the multiplicity of G^* is at most p and that it has less edges than G has. From the minimality of G it follows that G^* can be edge-coloured by $d + p - 1$ colours. But then we can construct an edge-colouring of G as follows. Edges of G not joining two of vertices u, v, w are coloured in the same way as in G^*. Denote by u_1, v_1, w_1 the degrees of u, v, w in F_2, respectively. Choose $u_1[v_1, w_1]$ edges from p edges joining in G v and w [u and w, u and v] and colour them by the colours of the edges incident with $u[v, w$, respectively$]$ in F_2. As $d \geqq 2p + 1$, the remaining $3p - u_1 - v_1 - w_1$ edges can be coloured by the remaining $p + d - 1 - u_1 - v_1 - w_1$ colours. Thus there is an edge-colouring of G by $d + p - 1$ colours, which is a contradiction to our assumption.

We have proved that F_1 does not contain a triangle. It follows that F_1 does not contain G_4 as a subgraph. As $d(F_1) = 2p$, according to (8) we have:

$$q(F_1) \leqq 3p - [\tfrac{1}{2}p] .$$

F_2 is a graph of degree $d - 2p$. So (3) yields

$$q(F_2) \leqq [\tfrac{3}{2}(d - 2p)] = [\tfrac{3}{2}d] - 3p .$$

Therefore

$$q(G) \leqq q(F_1) + q(F_2) \leqq 3p - [\tfrac{1}{2}p] + [\tfrac{3}{2}d] - 3p =$$
$$= [\tfrac{3}{2}d] - [\tfrac{1}{2}p] \leqq d + p - 1$$

if $d \leqq 3p - 2$, or if $d = 3p - 1$ with p even. In the case that $d = 3p - 1$ but p is odd, we proceed as follows. We decompose the regular factor F_1 of degree $2p$ into a factor F_3 of degree $2p - 2$ and a factor F_4 of degree 2.([2], [5], [7], [10].) Evidently F_3 does not contain G_4, therefore (8) yields

$$q(F_3) \leqq 3(p - 1) - [\tfrac{1}{2}(p - 1)] = \tfrac{1}{2}(5p - 5).$$

If we unify the factors F_2 and F_4, we obtain a factor F_5 of degree $d - 2p + 2 = p + 1$. Obviously F_5 does not contain G_{p+1}. From the assumptions of the theorem it follows that $p > 1$. As p is now odd, we have $p \geqq 3$, i.e. $p + 1 \geqq 4$. Using (4) we get

$$q(F_5) \leqq [\tfrac{3}{2}(p + 1)] - 1 = \tfrac{1}{2}(3p + 1).$$

It follows that

$$q(G) \leqq q(F_3) + q(F_5) = \tfrac{1}{2}(5p - 5) + \tfrac{1}{2}(3p + 1) =$$
$$= 4p - 2 = d + p - 1,$$

so the theorem for finite graphs is proved.

Now, if G is an infinite graph of finite degree d and multiplicity p, where $2p < d < 3p$, put $k = d + p - 1$. According to what has been proved above, for all finite subgraphs H of G we have $q(H) \leqq d(H) + p(H) - 1 \leqq d + p - 1 = k$. From Theorem 2 it follows that $q(G) \leqq k = d + p - 1$. The theorem follows.

Theorem 9. *Let d and p be positive integers. Then for the maximal chromatic index $P(d, p)$ of a graph of degree d and multiplicity p we have:*

(29) *If $p \leqq d \leqq 2p$, then $P(d, p) = [\tfrac{3}{2}d]$.*

(30) *If $d = 2p + 1$, then $P(d, p) = d + p - 1$.*

(31) *If $2p + 2 \leqq d \leqq \tfrac{12}{5}p$, then $3p \leqq P(d, p) \leqq d + p - 1$.*

(32) *If $\tfrac{12}{5}p < d \leqq 3p - 1$, then $d + [[d/2]/[d/2p]^*]^* \leqq P(d, p) \leqq d + p - 1$.*

(33) *If $3p \leqq d \leqq 2p^2 - 2p + 1$, then $d + [[d/2]/[d/2p]^*]^* \leqq P(d, p) \leqq d + p$.*

(34) *If $d \geqq 2p^2 - 2p + 2$, then $P(d, p) = d + p$.*

Proof. (29) Let $p \leqq d \leqq 2p$. According to (26) we have $P(d, p) \leqq [\tfrac{3}{2}d]$. To prove the reverse inequality, we consider a graph with two components, the first one having the form $C(p, 0)$ (two vertices joined by p edges), the second one being G_d. Evidently the multiplicity of this graph is equal to p, the degree equals d and the chromatic index is $[\frac{3}{2}d]$.

(30) and (31) follows from Theorem 8 and (27).

(32) follows from Theorem 8 and (28).

(33) follows from (25) and (28).

(34) follows from (25) and Corollary 2 of Theorem 7 for $l = 0$, since if $d \geqq 2p^2 - 2p + 2$, $d = 2px + y$, $0 < y \leqq 2p$, then either $x = p - 1$ and $y \geqq 2$, or $x \geqq p$. Thus we always have $x + [\frac{1}{2}y] \geqq p$.

Remarks. 1. The number $\frac{12}{5}p$ in (31) and (32) is chosen in such a way that always the best one of the estimations (27) and (28) is used.

2. On the basis of Theorem 9 and the preceding results, values $P(d, p)$ given in Table 2 were calculated.

Table 2.

$P(d, p)$		p					
		1	2	3	4	5	6
d	1	1	—	—	—	—	—
	2	3	3	—	—	—	—
	3	4	4	4	—	—	—
	4	5	6	6	6	—	—
	5	6	6	7	7	7	—
	6	7	8	9	9	9	9
	7	8	9	9	10	10	10
	8	9	10	10	12	12	12
	9	10	11	11—12	12	13	13
	10	11	12	13	13	15	15
	11	12	13	14	14	15	16
	12	13	14	15	15—16	15—16	18
	13	14	15	15—16	16—17	16—17	18
	14	15	16	17	18	18	18—19
	15	16	17	18	19	19—20	19—20
	16	17	18	19	20	20—21	20—21

Problem 2. Improve the estimations (31), (32) and (33).

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Author's address: Bratislava, Štefánikova 41, ČSSR (Matematický ústav SAV).