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Czechoslovak Mathematical Journal, Vol. 22 (1972), No. 2, 195–209

Persistent URL: http://dml.cz/dmlcz/101090

Czechoslovak Mathematical Journal, 22 (97) 1972, Praha

BOUNDED FUNCTIONS WITH POSITIVE REAL PART

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(Received May 16, 1970)

1. Introductory Remarks. Let $\mathcal{P}$ denote the class of functions $P(z)$ which are regular in the open unit disk centered at the origin, symbolized by Δ, and satisfying the conditions

$$(1.1)\qquad P(0) = 1 \quad \text{and} \quad \operatorname{Re}\{P(z)\} > 0 \quad \text{for} \quad z \quad \text{in} \quad \Delta\,.$$

The class $\mathcal{P}$ has interesting properties and many useful applications, particularly in the study of special classes of univalent functions; it has, as a consequence, experienced a long and detailed history.

Recently, several authors have examined some properties of functions $P(z)$ in $\mathcal{P}$ satisfying the additional requirement that

$$(1.2)\qquad |P(z) - M| < M\,, \quad z \in \Delta\,,$$

for a fixed M, $M > \frac{1}{2}$; the resulting subclass of $\mathcal{P}$ has been written $\mathcal{P}_M$.

KACZMARSKI [6] obtained sharp coefficient estimates for meromorphic and univalent functions

$$(1.3)\qquad F(z) = \frac{1}{z} + \alpha_1 z + \alpha_2 z^2 + \ldots\,, \quad z \in \Delta$$

which for a fixed α, $0 \leqq \alpha < 1$, satisfy the condition

$$(1.4)\qquad \frac{1}{(\alpha - 1)}\left[\frac{z\,F'(z)}{F(z)} + \alpha\right] \in \mathcal{P}_M\,.$$

All $F(z)$ meeting these conditions form a subclass of the starlike functions of order α introduced by POMMERENKE [10]. Kaczmarski also gave the coefficient bounds for $F(z)$ which are meromorphically spirallike in Δ, [13], and meet a condition similar to (1.4).

[1]) Supported in part by N.S.F. Grant No. 11726, Renewal of GP 7439.
[2]) Supported in part by the University of Delaware Research Foundation.

GOEL [3] gave the coefficient estimates and some distortion theorems for an arbitrary $P(z)$ in $\mathscr{P}_M$, $M \geqq 1$, and applied his results to subclasses of close-to-convex functions introduced by LIBERA [7].

SINGH [12] obtained coefficient and distortion bounds for regular starlike functions

$$f(z) = z + a_2 z^2 + \dots, \quad z \in \Delta, \tag{1.5}$$

such that $z f'(z)/f(z) \in \mathscr{P}_1$. JANOWSKI [4] established radii of starlikeness for functions of the form (1.5) which are defined by operations on regular starlike functions and members of $\mathscr{P}_M$, $M \geqq 1$; Janowski's functions may not be univalent throughout the disk Δ. More recently he has announced additional results [6].

If $P(z)$ is in $\mathscr{P}_M$, then $P[\Delta]$, the image of Δ under $P(z)$, is contained entirely in the disk of radius M centered at M; and the converse holds too. In this paper we study functions $P(z)$ in $\mathscr{P}$ for which $P[\Delta]$ is contained in an arbitrary (but fixed) disk contained in the right half-plane and which contains the point 1; these results are then used to generalize and extend some of the work mentioned above.

2. The Class $\mathscr{P}[\alpha, t, \varrho]$. A disk of radius ϱ, $\varrho > 0$, lying in the right half-plane is tangent to a line $w = \alpha + iv$, $\alpha \geqq 0$, v real, at a point $\alpha + it$ and its center is at $(\varrho + \alpha) + it$. If it is required that 1 be in this disk, then $|1 - (\varrho + \alpha + it)| < \varrho$, or

$$D = D(\alpha, t, \varrho) = 2\varrho(1 - \alpha) + \alpha(2 - \alpha) - (1 + t^2) > 0\,. \tag{2.1}$$

Except for a rotation of Δ the linear transformation mapping Δ onto the above disk with 0 corresponding to 1 is

$$L(z) = \frac{Az + \varrho}{Bz + \varrho}, \tag{2.2}$$

with

$$A = A(\alpha, t, \varrho) = \varrho(1 - 2\alpha) + \alpha(1 - \alpha) - t^2 + it \quad \text{and} \tag{2.3}$$
$$B = B(\alpha, t, \varrho) = 1 - \varrho - \alpha + it\,.$$

It is useful to observe that the discriminant of $L(z)$ is $\varrho(A - B) = \varrho D$.

Definition 1. $P(z)$ is in $\mathscr{P}[\alpha, t, \varrho]$, $0 \leqq \alpha < 1$, t real, $\varrho > \frac{1}{2}$ and $D(\alpha, t, \varrho) > 0$ if and only if there is a function $\omega(z)$ regular in Δ such that

$$P(z) = \frac{\varrho + A\,\omega(z)}{\varrho + B\,\omega(z)} \quad \text{and} \quad |\omega(z)| \leqq |z| \quad \text{for} \quad z \quad \text{in} \quad \Delta\,, \tag{2.4}$$

and A and B are defined by (2.3).

As a consequence of the principle of subordination [9, p. 226] we conclude that every $P(z)$ in $\mathscr{P}[\alpha, t, \varrho]$ is subordinate to $L(z)$, therefore $P[\Delta]$ is contained in the open disk of radius ϱ and centered at $(\varrho + \alpha + it)$. The class $\mathscr{P}_M$, mentioned above, is the same as $\mathscr{P}[0, 0, M]$. For every admissible ϱ and t, $\mathscr{P}[\alpha, t, \varrho]$ is a subclass of the functions of "positive real part of order α" which are defined implicitly in [10] and [11] and explicitly in [8].

The subsequent parts of this section deal with the coefficient and distortion bounds on functions in $\mathscr{P}[\alpha, t, \varrho]$; the coefficient bounds which follow (and those given in later theorems) are derived by using the Method of CLUNIE [2].

Theorem 1. *If* $P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ *is in* $\mathscr{P}[\alpha, t, \varrho]$, *then*

$$|p_n| \leqq 2(1 - \alpha) + \frac{\alpha(2 - \alpha) - (t^2 + 1)}{\varrho}, \quad n = 1, 2, \dots ; \tag{2.5}$$

these results are sharp for all admissible α, t *and* ϱ.

Proof. The representation for $P(z)$ in (2.4) is equivalent to

$$[A - B\,P(z)]\,\omega(z) = \varrho[P(z) - 1], \tag{2.6}$$

or

$$[A - B \sum_{k=0}^{\infty} p_k z^k]\,\omega(z) = \varrho \sum_{k=1}^{\infty} p_k z^k, \quad p_0 = 1. \tag{2.7}$$

This can be rewritten

$$[(A - B) - B \sum_{k=1}^{n-1} p_k z^k]\,\omega(z) = \varrho \sum_{k=1}^{n} p_k z^k + \sum_{k=n+1}^{\infty} q_k z^k, \tag{2.8}$$

the last term also being absolutely and uniformly convergent *in compacta* on Δ. Writing $z = re^{i\theta}$, performing the indicated integration and making use of the bound $|\omega(z)| \leqq |z| < 1$ for z in Δ gives

$$\begin{aligned} |A - B|^2 + |B|^2 \sum_{k=1}^{n-1} |p_k|^2 r^{2k} &= \frac{1}{2\pi} \int_0^{2\pi} |(A - B) + B \sum_{k=1}^{n-1} p_k r^k e^{ik\theta}|^2 \, d\theta \geqq \\ &\geqq \frac{1}{2\pi} \int_0^{2\pi} |\{(A - B) + B \sum_{k=1}^{n-1} p_k r^k e^{ik\theta}\}\,\omega(re^{i\theta})|^2 \, d\theta \geqq \\ \geqq \frac{1}{2\pi} \int_0^{2\pi} |\varrho \sum_{k=1}^{n} p_k r^k e^{ik\theta} + \sum_{k=n+1}^{\infty} q_k r^k e^{ik\theta}|^2 \, d\theta &\geqq \varrho^2 \sum_{k=1}^{n} |p_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |q_k|^2 r^{2k}. \end{aligned} \tag{2.9}$$

The last term is non-negative and $r < 1$, therefore

$$|A - B|^2 + |B|^2 \sum_{k=1}^{n-1} |p_k|^2 \geqq \varrho^2 \sum_{k=1}^{n} |p_k|^2, \tag{2.10}$$

or

$$\varrho^2 |p_n|^2 \leqq |A - B|^2 + (|B|^2 - \varrho^2) \sum_{k=1}^{n-1} |p_k|^2. \tag{2.11}$$

A brief calculation shows that $|B|^2 - \varrho^2 = -D(\alpha, t, \varrho)$, which is negative, (2.1), hence

$$|p_n| \leqq \frac{|A - B|}{\varrho} = \frac{D(\alpha, t, \varrho)}{\varrho}, \tag{2.12}$$

and this is equivalent to (2.5). If $\omega(z) = z^n$, then

$$P(z) = 1 + \frac{D(\alpha, t, \varrho)}{\varrho} z^n + \dots, \tag{2.13}$$

which makes (2.5) sharp.

If $P(z) \in \mathscr{P}[\alpha, t, \varrho]$, then

$$Q(z) = \frac{P\left[\dfrac{z + \zeta}{1 + \bar{\zeta} z}\right] - i \operatorname{Im} P(\zeta)}{\operatorname{Re} P(\zeta)} = 1 + \frac{P'(\zeta)(1 - |\zeta|^2)}{\operatorname{Re} P(\zeta)} z + \dots \tag{2.14}$$

is in $\mathscr{P}$ for each ζ in Δ. Furthermore $Q[\Delta]$ is contained in the disk of radius $\varrho/\operatorname{Re} P(\zeta)$ centered at $[(\alpha + \varrho) + i(t - \operatorname{Im} P(\zeta))]/\operatorname{Re} P(\zeta)$, consequently $Q(z)$ is in

$$\mathscr{P}\left[\frac{\alpha}{\operatorname{Re} P(\zeta)}, \frac{t - \operatorname{Im} P(\zeta)}{\operatorname{Re} P(\zeta)}, \frac{\varrho}{\operatorname{Re} P(\zeta)}\right]$$

and in this case (2.12) reads

$$\frac{|p'(\zeta)|}{\operatorname{Re} P(\zeta)}(1 - |\zeta|^2) \leqq \frac{\operatorname{Re} P(\zeta)}{\varrho} D\left(\frac{\alpha}{\operatorname{Re} P(\zeta)}, \frac{t - \operatorname{Im} P(\zeta)}{\operatorname{Re} P(\zeta)}, \frac{\varrho}{\operatorname{Re} P(\zeta)}\right). \tag{2.15}$$

Summarizing and rewriting these results we have the following

Corollary 1. *If $P(z) \in \mathscr{P}[\alpha, t, \varrho]$ and $z \in \Delta$, then*

$$|P'(z)|(1 - |z|^2) \leqq 2(\operatorname{Re} P(z) - \alpha) - \frac{(\operatorname{Re} P(z) - \alpha)^2 + (t - \operatorname{Im} P(z))^2}{\varrho}. \tag{2.16}$$

Theorem 2. *If $P(z) \in \mathscr{P}[\alpha, t, \varrho]$, then*

$$\frac{|(1 - r^2)\varrho^2 + r^2(\varrho + \alpha) D + ir^2 tD| - r\varrho D}{(1 - r^2)\varrho^2 + r^2 D} \leqq \tag{2.17}$$

$$\leqq |P(z)| \leqq \frac{|(1 - r^2)\varrho^2 + r^2(\varrho + \alpha) D + ir^2 tD| + r\varrho D}{(1 - r^2)\varrho^2 + r^2 D}$$

and

$$1 - \frac{r[r(1-\alpha) + \varrho(1-r)]\,D}{(1-r^2)\,\varrho^2 + r^2 D} \leqq \operatorname{Re}\{P(z)\} \leqq \tag{2.18}$$

$$\leqq 1 - \frac{r[r(1-\alpha) - \varrho(1+r)]\,D}{(1-r^2)\,\varrho^2 + r^2 D},$$

for $|z| \leqq r$. *These bounds are sharp for each* $r, 0 < r < 1$, *and all admissible* α, t *and* ϱ.

Proof. (In the theorem and below $D = D(\alpha, t, \varrho)$.) A calculation shows that $L(z)$, (2.2), maps the disk $\{z : |z| \leqq r\}$ onto the disk with its center at

$$w_0 = \frac{A}{B} - \frac{\varrho^2(A - B)}{B[\varrho^2 - |B|^2 r^2]} \tag{2.19}$$

and radius

$$R = \frac{r\varrho|A - B|}{|\varrho^2 - |B|^2 r^2|}. \tag{2.20}$$

From (2.1) and (2.3) we conclude that

$$|\varrho^2 - |B|^2 r^2| = (1 - r^2)\,\varrho^2 + r^2 D > 0, \tag{2.21}$$

therefore

$$w_0 = \frac{\varrho^2 - A\bar{B}r^2}{\varrho^2 - |B|^2 r^2} = 1 - \frac{r^2 \bar{B} D}{(1 - r^2)\,\varrho^2 + r^2 D} \tag{2.22}$$

and

$$R = \frac{r\varrho D}{(1 - r^2)\,\varrho^2 + r^2 D}. \tag{2.23}$$

If $P(z)$ is in $\mathscr{P}[\alpha, t, \varrho]$, it follows by subordination that

$$|w_0| - R \leqq |P(z)| \leqq |w_0| + R \tag{2.24}$$

and

$$\operatorname{Re}\{w_0\} - R \leqq \operatorname{Re}\{P(z)\} \leqq \operatorname{Re}\{w_0\} + R, \tag{2.25}$$

for $|z| \leqq r$; and these are equivalent to (2.17) and (2.18) respectively.

For any choice of values of α, t and ϱ members of $\mathscr{P}[\alpha, t, \varrho]$ are necessarily bounded in Δ, consequently $\bigcup \mathscr{P}[\alpha, t, \varrho]$, the union being taken over all α, t and ϱ, is a proper subset of $\mathscr{P}$. However, for any fixed t and α, the class $\bigcup_{\varrho > 1/2} \mathscr{P}[\alpha, t, \varrho]$ is dense in the family $\mathscr{P}[\alpha]$ of functions $P(z)$ in $\mathscr{P}$ for which $\operatorname{Re} P(z) > \alpha$ for z in Δ. Therefore, the preceding results can be extended to $\mathscr{P}[\alpha]$, $0 < \alpha < 1$, and $\mathscr{P}[0] = \mathscr{P}$, by taking appropriate limits.

If, in particular, we choose $\alpha = t = 0$ and let $\varrho \to \infty$, then Theorem 1 gives the classical coefficient bound of CARATHÉODORY [1] and Theorem 2 gives known distortion bounds [9, p. 173] for every $P(z)$ in $\mathscr{P}$.

In a similar way, setting $t = 0$ and letting $\varrho \to \infty$ in (2.5), gives sharp coefficient bounds for $\mathscr{P}[\alpha]$, which appear in [8]. The same procedure in Corollary 1 and Theorem 2 gives the following distortion bounds.

Corollary 2. *If $P(z)$ is in $\mathscr{P}[\alpha]$, for α fixed, $0 < \alpha < 1$ and $|z| \leqq r < 1$, then*

$$\frac{|z\,P'(z)|}{\operatorname{Re}\{P(z) - \alpha\}} \leqq \frac{2r}{1 - r^2}, \tag{2.26}$$

$$\frac{1 - 2(1-\alpha)\,r + (1 - 2\alpha)\,r^2}{1 - r^2} \leqq |P(z)| \leqq \frac{1 + 2(1-\alpha)\,r + (1 - 2\alpha)\,r^2}{1 - r^2}, \tag{2.27}$$

and

$$\frac{1 - 2(1-\alpha)\,r + (1 - 2\alpha)\,r^2}{1 - r^2} \leqq \operatorname{Re}\{P(z)\} \leqq \frac{1 + 2(1-\alpha)\,r + (1 - 2\alpha)\,r^2}{1 - r^2}. \tag{2.28}$$

Choosing $\alpha = t = 0$ and $\varrho = M$ in Definition 1 gives the class $\mathscr{P}_M$ defined by (1.2) and studied earlier by Goel [3], Kaczmarski [6] and Janowski [4]; however they restricted their attention to the case $M \geqq 1$. In general, we have the following corollary.

Corollary 3. *If $P(z)$ is in $\mathscr{P}[0, 0, M] = \mathscr{P}_M$ for fixed M, $M \geqq \frac{1}{2}$, and $P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$, then*

$$|p_k| \leqq 2 - \frac{1}{M}, \quad k = 1, 2, \ldots; \tag{2.29}$$

$$|P'(z)|\,(1 - |z|^2) \leqq 2 \operatorname{Re} P(z) - \frac{|P(z)|^2}{M}, \quad z \in \Delta; \tag{2.30}$$

$$\frac{M(1 - |z|)}{M + (M-1)\,|z|} \leqq |P(z)| \leqq \frac{M(1 + |z|)}{M - (M-1)\,|z|}, \quad z \in \Delta; \tag{2.31}$$

and

$$\frac{M(1 - |z|)}{M + (M-1)\,|z|} \leqq \operatorname{Re} P(z) \leqq \frac{M(1 + |z|)}{M - (M-1)\,|z|}, \quad z \in \Delta. \tag{2.32}$$

3. Other Results. In this section we will indicate some of the many possible applications of the preceding results.

Definition 2. $F(z) = z^{-1} + \sum_{k=0}^{\infty} a_k z^k$, meromorphic in Δ, is in $\Sigma^*[\alpha, t, \varrho]$ if and only if

$$\frac{-z\,F'(z)}{F(z)} \quad \text{is in} \quad \mathscr{P}[\alpha, t, \varrho]\,. \tag{3.1}$$

The restrictions on α, t and ϱ are those in Definition 1. For each α, $\Sigma^*[\alpha, t, \varrho]$ is a subset of the class of meromorphic starlike functions of order α, [10], hence all functions covered by Definition 2 are univalent in Δ.

Theorem 3. *If* $F(z) = z^{-1} + \sum_{k=0}^{\infty} a_k z^k$ *is in* $\Sigma^*[\alpha, t, \varrho]$, *then*

$$|a_n| \leqq \frac{D(\alpha, t, \varrho)}{\varrho(n+1)} = \frac{2(1-\alpha)}{(n+1)} + \frac{\alpha(2-\alpha) - (1+t^2)}{\varrho(n+1)}\,, \quad n = 0, 1, 2, \dots\,; \tag{3.2}$$

each of these inequalities is rendered sharp by the function $F(z)$ *in* $\Sigma^*[\alpha, t, \varrho]$ *defined by*

$$-\frac{z\,F'(z)}{F(z)} = \frac{\varrho + Az^{n+1}}{\varrho + Bz^{n+1}}\,, \tag{3.3}$$

for all α, t *and* ϱ *admitted above, with* D, A *and* B *defined by* (2.1) *and* (2.3).

Proof. Using Definitions 1 and 2 we write

$$\frac{-z\,F'(z)}{F(z)} = \frac{\varrho + A\,\omega(z)}{\varrho + B\,\omega(z)}\,, \quad \text{for } z \text{ in } \Delta\,, \tag{3.4}$$

with $\omega(z)$ satisfying Schwarz's Lemma [9, p. 165]. Using the power series representation for $F(z)$ we may rewrite (3.4) as

$$-\sum_{k=0}^{\infty} \varrho(k+1)\,a_k z^k = \left[\frac{A-B}{z} + \sum_{k=0}^{\infty} (A + Bk)\,a_k z^k\right] \omega(z)\,. \tag{3.5}$$

The last equation implies that $\varrho a_0 = (A - B)\,\omega'(0)$, and since $|\omega'(0)| \leqq 1$, it follows that

$$|a_0| \leqq \frac{A-B}{\varrho} = \frac{D}{\varrho}\,. \tag{3.6}$$

On the other hand for $n > 0$ and because $\omega(0) = 0$, (3.5) can be rewritten

$$-\sum_{k=0}^{n} \varrho(k+1)\,a_k z^k + \sum_{k=n+1}^{\infty} d_k z^k = \left\{\frac{D}{z} + \sum_{k=0}^{n-1} (A + Bk)\,a_k z^k\right\} \omega(z)\,, \tag{3.7}$$

$\sum_{k=n+1}^{\infty} d_k z^k$ being absolutely and uniformly convergent on compact subsets of Δ. Letting $z = re^{i\theta}$ and using the assumption that $\omega(z)$ is bounded by 1, we conclude that

$$(3.8) \qquad \frac{1}{2\pi}\int_0^{2\pi} \left| -\sum_{k=0}^{n} \varrho(k+1)\, a_k r^k e^{ik\theta} + \sum_{k=n+1}^{\infty} d_k r^k e^{ik\theta} \right|^2 d\theta \leqq$$

$$\leqq \frac{1}{2\pi}\int_0^{2\pi} \left| Dr^{-1} e^{-i\theta} + \sum_{k=0}^{n-1} (A + Bk)\, a_k r^k e^{ik\theta} \right|^2 d\theta\,,$$

which, by an application of Parseval's identity [9, p. 100] is equivalent to

$$(3.9) \quad \sum_{k=0}^{n} \varrho^2 (k+1)^2 |a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k} \leqq \frac{D^2}{r^2} + \sum_{k=0}^{n-1} |A + Bk|^2 |a_k|^2 r^{2k}\,.$$

Since the infinite series is non-negative and $0 < r < 1$, we can write

$$(3.10) \qquad \sum_{k=0}^{n} \varrho^2 (k+1)^2 |a_k|^2 \leqq D^2 + \sum_{k=0}^{n-1} |A + Bk|^2 |a_k|^2\,,$$

or

$$(3.11) \qquad \varrho^2 (n+1)^2 |a_n|^2 \leqq D^2 + \sum_{k=0}^{n-1} \left[|A + Bk|^2 - \varrho^2 (k+1)^2 \right] |a_k|^2\,.$$

The remainder of the proof consists of showing that the coefficient of $|a_k|^2$ is not positive for $k = 0, 1, 2, \ldots$

For each k,

$$(3.12) \qquad |A + Bk|^2 - \varrho^2 (k+1)^2 = \{(1 - \varrho - \alpha)^2 + t^2 - \varrho^2\}\, k^2 +$$
$$+ \{2(1 - \varrho - \alpha)\left[\varrho(1 - 2\alpha) + \alpha(1 - \alpha) - t^2\right] + 2t^2 - 2\varrho^2\}\, k +$$
$$+ \{\left[\varrho(1 - 2\alpha) + \alpha(1 - \alpha) - t^2\right]^2 + t^2 - \varrho^2\}\,.$$

The coefficient of k^2 is $-D(\alpha, t, \varrho)$, (2.1), hence it is negative. Rewriting the coefficient of k as a quadratic in t gives us the form

$$(3.13) \quad 2(\varrho + \alpha)\left[(1 - \alpha)(1 - 2\varrho - \alpha) + t^2\right] = 2(\varrho + \alpha)\left[-D(\alpha, t, \varrho)\right] < 0\,.$$

The constant term in (3.12) can be rewritten and bounded in the following way

$$(3.14) \qquad \left[\varrho(1 - 2\alpha) + \alpha(1 - \alpha) - t^2\right]^2 + t^2 - \varrho^2 =$$
$$= \alpha(1 - \alpha)(2\varrho + \alpha)(1 - 2\varrho - \alpha) + \left[2\varrho\alpha + \alpha^2 - (1 - \alpha)(2\varrho + \alpha - 1)\right] t^2 + t^4 =$$
$$= \alpha(2\varrho + \alpha)(-D - t^2) + (2\varrho\alpha + \alpha^2 - D - t^2)\, t^2 + t^4 =$$
$$= -D(2\varrho\alpha + \alpha^2 + t^2) < 0\,.$$

Consequently, (3.11) implies that

$$\varrho^2(n + 1)^2 |a_n|^2 \leqq D(\alpha, t, \varrho)^2, \quad n = 1, 2, \dots; \tag{3.15}$$

which is equivalent to (3.2).

If for any fixed n, $n = 0, 1, 2, \dots$, we let $F(z)$ be defined by (3.3) and have the series representation of the hypotheses of Theorem 3, then

$$\frac{-z\,F'(z)}{F(z)} = 1 + \frac{A - B}{\varrho} z^n + \dots \tag{3.16}$$

or

$$-z^{-1} - \sum_{k=0}^{\infty} k a_k z^k = \left\{1 + \frac{D}{\varrho} z^n + \dots\right\}\left(z^{-1} + \sum_{k=0}^{\infty} a_k z^k\right). \tag{3.17}$$

A comparison of coefficients shows that $a_k = 0$, for $k = 0, 1, \dots, n - 1$ and that

$$|a_n| = \frac{D(\alpha, t, \varrho)}{\varrho(n + 1)}. \tag{3.18}$$

This completes the proof of Theorem 3.

An examination of the preceding proof yields the following analog of the Area Theorem [9, p. 210].

Corollary 4. *If* $F(z) = z^{-1} + \sum_{k=0}^{\infty} a_k z^k$ *is in* $\Sigma^*[\alpha, t, \varrho]$, *then*

$$\sum_{k=0}^{\infty} [k^2 + 2(\varrho + \alpha)\,k + (2\varrho\alpha + \alpha^2 + t^2)]\,|a_k|^2 \leqq D(\alpha, t, \varrho). \tag{3.19}$$

Proof. Applying Parseval's identity to (3.5) and making use of the bound on $\omega(z)$ gives

$$\sum_{k=0}^{\infty} [\varrho^2(k + 1)^2 - |A + Bk|^2]\,|a_k|^2 \leqq |A - B|^2, \tag{3.20}$$

which, by making use of the calculations in (3.12), (3.13) and (3.14), may be written

$$\sum_{k=0}^{\infty} [Dk^2 + 2(\varrho + \alpha)\,Dk + D(2\varrho\alpha + \alpha^2 + t^2)]\,|a_k|^2 \leqq D^2. \tag{3.21}$$

This form is equivalent to (3.19).

By specializing the choices of the parameters α, t and ϱ in Theorem 3, we get some interesting special cases which have appeared elsewhere.

Corollary 5. *If* $F(z) = z^{-1} + \sum_{k=0}^{\infty} a_k z^k$ *is in* $\Sigma^*[0, 0, M]$, *then*

$$|a_n| \leqq \frac{2M - 1}{M(n + 1)}, \quad n = 0, 1, 2, \ldots \tag{3.22}$$

This result appears in [6], with the additional proviso that $a_0 = 0$; many other theorems appearing in [6] for meromorphic starlike, or spiral, functions can be extended by the above methods.

Let $\Sigma^*[\alpha]$ be the class of meromorphic starlike functions of order α, [10]. For any fixed t, $\bigcup_{\varrho > 1/2} \Sigma^*[\alpha, t, \varrho]$ forms a dense subclass of $\Sigma^*[\alpha]$, consequently, letting $\varrho \to \infty$ in (3.2) we obtain the following theorem of Pommerenke [10].

Corollary 6. *If* $F(z) = z^{-1} + \sum_{k=0}^{\infty} a_k z^k$ *is in* $\Sigma^*[\alpha]$, *then*

$$|a_n| \leqq \frac{2(1 - \alpha)}{n + 1}, \quad n = 0, 1, 2, \ldots \tag{3.23}$$

Theorem 4. *If* $F(z)$ *is in* $\Sigma^*[\alpha, t, \varrho]$, A *and* B *are as in* (2.3) *and* $|z| = r, 0 < r < 1$, *then*

$$\frac{1}{r}\left(\frac{\varrho + |B|\, r}{\varrho - |B|\, r}\right)^{(B-A)/2|B|} \left(\frac{\varrho^2 - |B|^2 r^2}{\varrho^2}\right)^{(B-A)\operatorname{Re} B/2|B|^2} \leqq |F(z)| \leqq \tag{3.24}$$

$$\leqq \frac{1}{r}\left(\frac{\varrho + |B|\, r}{\varrho - |B|\, r}\right)^{(A-B)/2|B|} \left(\frac{\varrho^2}{\varrho^2 - |B|^2 r^2}\right)^{(A-B)\operatorname{Re} B/2|B|^2}$$

and these inequalities are made sharp when $t = 0$, *in which case* A *and* B *are real, by*

$$F(z) = \frac{1}{z}\left(\frac{\varrho + Bz}{\varrho}\right)^{(B-A)/A}. \tag{3.25}$$

Proof. From (3.4) we have

$$-z \frac{\mathrm{d}}{\mathrm{d}z} \log [z\, f(z)] = \frac{(A - B)\, \omega(z)}{\varrho + B\, \omega(z)}, \tag{3.26}$$

or

$$F(z) = \frac{1}{z} \exp\left\{-D \int_0^z \frac{\omega(\zeta)}{\zeta(\varrho + B\, \omega(\zeta))} \,\mathrm{d}\zeta\right\}, \tag{3.27}$$

therefore

$$|F(z)| = \frac{1}{r} \exp\left\{-D \int_0^r \operatorname{Re}\left(\frac{\omega(s e^{i\theta})}{\varrho + B\, \omega(s e^{i\theta})}\right) \frac{\mathrm{d}s}{s}\right\}, \tag{3.28}$$

recalling that $A - B = D > 0$.

Excepting the constant term $(A - B)$, the right side of (3.26) is subordinate to the linear transformation

$$l(z) = \frac{z}{\varrho + Bz} \tag{3.29}$$

which for any r, $0 < r < 1$, maps the disk $\{z : |z| \leqq r\}$ onto the disk with center and radius

$$\frac{-r^2 \bar{B}}{\varrho^2 - |B|^2 r^2} \quad \text{and} \quad \frac{\varrho r}{\varrho^2 - |B|^2 r^2}, \tag{3.30}$$

having observed in (2.21) that the denominators in (3.30) are positive. Therefore, as in the proof of Theorem 2, we conclude that

$$-\frac{\varrho r + r^2 \operatorname{Re} B}{\varrho^2 - |B|^2 r^2} \leqq \operatorname{Re}\left(\frac{\omega(z)}{\varrho + B\,\omega(z)}\right) \leqq \frac{\varrho r - r^2 \operatorname{Re} B}{\varrho^2 - |B|^2 r^2}. \tag{3.31}$$

The left side of this inequality yields

$$\exp\left\{-D \int_0^r \operatorname{Re}\left(\frac{\omega(se^{i\theta})}{\varrho + B\,\omega(se^{i\theta})}\right)\frac{ds}{s}\right\} \geqq \exp\left\{D \int_0^r \frac{\varrho + s \operatorname{Re} B}{\varrho^2 - |B|^2 s^2}\, ds\right\} = \tag{3.32}$$

$$= \exp D \left\{\frac{1}{2|B|} \log\left(\frac{\varrho + |B|\, r}{\varrho - |B|\, r}\right) + \frac{\operatorname{Re} B}{2|B|^2} \log\left(\frac{\varrho^2}{\varrho^2 - |B|^2 r^2}\right)\right\}.$$

Combining this with (3.28) gives the upper bound in (3.24); the lower bound is obtained in much the same way.

If $t = 0$, then B is real and (3.31) reads

$$-\frac{r}{\varrho - Br} \leqq \operatorname{Re}\left(\frac{\omega(z)}{\varrho + B\,\omega(z)}\right) \leqq \frac{r}{\varrho + Br}. \tag{3.33}$$

These bounds are sharp when $\omega(z) = z$ with equality on the left occurring for $z = -r$ and on the right for $z = r$, $0 \leqq r < 1$; consequently for this choice of $\omega(z)$ and with $\theta = \pi$, (3.32) is sharp, whereas with $\theta = 0$ the corresponding lower bound is sharp. The function given in (3.25) comes from this choice of $\omega(z)$.

In concluding the proof of Theorem 4 we should like to remark that equality on either side of (3.31) obtains at a point z, $|z| = r$, if and only if $\omega(z) = cz$, $|c| = 1$. However, a calculation shows that equality will occur on either side of (3.31) at points $re^{i\theta_1}$ and $re^{i\theta_2}$, θ_1 and θ_2 being dependent on r. As a result, the estimates made in (3.32) are not sharp in general because the integration is performed along a ray from the origin.

The remaining portion of this section deals with applications to regular functions.

Theorem 5. *If* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ *is regular in* Δ *and, for a fixed* ϱ, $\frac{1}{2} < \varrho < 1$,

$$\left|\frac{z f'(z)}{f(z)} - \varrho\right| < \varrho\,, \quad z \in \Delta\,, \tag{3.34}$$

then

$$|a_n| \leqq \frac{2\varrho - 1}{\varrho(n-1)}\,, \quad n = 2, 3, \ldots. \tag{3.35}$$

These equalities are sharp, equality being attained for each n, by

$$f(z) = z\left[1 + \left(\frac{1-\varrho}{\varrho}\right) z^{n-1}\right]^{(2\varrho-1)/(1-\varrho)(n-1)}. \tag{3.36}$$

This theorem has been given for $\varrho = 1$ by Singh [12] and Janowski [5] recently announced partial results for $\varrho > 1$.

Proof. Condition (3.34) is equivalent to requiring that $z f'(z)/f(z)$ be in $\mathscr{P}[0, 0, \varrho]$. In this case Definition 1 gives

$$\frac{z f'(z)}{f(z)} = \frac{\varrho + \varrho\,\omega(z)}{\varrho + (1-\varrho)\,\omega(z)}\,, \quad z \in \Delta\,, \tag{3.37}$$

$\omega(z)$ satisfying Schwarz's Lemma. Substitution of the Maclaurin series for $f(z)$ enables us to write

$$\sum_{k=2}^{\infty} (k-1)\, a_k z^k = \omega(z) \sum_{k=1}^{\infty} \left(1 - \left(\frac{1-\varrho}{\varrho}\right) k\right) a_k z^k\,, \tag{3.38}$$

or because $\omega(0) = 0$,

$$\sum_{k=2}^{n} \varrho(k-1)\, a_k z^k + \sum_{k=n+1}^{\infty} c_k z^k = \omega(z) \sum_{k=1}^{n-1} (\varrho(k+1) - k)\, a_k z^k, \tag{3.39}$$

the series $\sum_{k=n+1}^{\infty} c_k z^k$ being uniformly and absolutely convergent on compact subsets of Δ. Since $|\omega(z)| \leqq 1$, $z \in \Delta$,

$$\frac{1}{2\pi}\int_0^{2\pi} \Big|\sum_{k=2}^{n} \varrho(k-1)\, a_k r^k e^{ik\theta} + \sum_{k=n+1}^{\infty} c_k r^k e^{ik\theta}\Big|^2 \, d\theta \leqq \tag{3.40}$$

$$\leqq \frac{1}{2\pi}\int_0^{2\pi} \Big|\sum_{k=1}^{n-1} (\varrho(k+1) - k)\, a_k r^k e^{ik\theta}\Big|^2 \, d\theta\,,$$

for $0 < r < 1$. Applying Parseval's identity to both sides of (3.40) gives

$$(3.41)\quad \sum_{k=2}^{n} \varrho^2(k-1)^2 |a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |c_k|^2 r^{2k} \leqq \sum_{k=1}^{n-1} (\varrho(k+1)-k)^2 |a_k|^2 r^{2k},$$

and this, upon letting $r \to 1$ and neglecting the (non-negative) infinite series, gives

$$(3.42)\qquad \sum_{k=2}^{n} \varrho^2(k-1)^2 |a_k|^2 \leqq \sum_{k=1}^{n-1} (\varrho(k+1)-k)^2 |a_k|^2 .$$

In consequence,

$$(3.43)\quad \varrho^2(n-1)^2 |a_n|^2 \leqq (2\varrho-1)^2 + \sum_{k=2}^{n-1} \{(\varrho(k+1)-k)^2 - \varrho^2(k-1)^2\} |a_k|^2 =$$
$$= (2\varrho-1)^2 + \sum_{k=2}^{n-1} k(2\varrho-k)(2\varrho-1) |a_k|^2 .$$

By assumption, $\frac{1}{2} < \varrho < 1$, therefore $0 < 2\varrho - 1 < 1$ and $2\varrho - k \leqq 0$, $k = 2, 3, \ldots, n-1$; hence from (3.43) we infer that

$$(3.44)\qquad \varrho^2(n-1)^2 |a_n|^2 \leqq (2\varrho-1)^2 ,$$

and this is equivalent to (3.35).

Theorem 6. *If* $f(z)$ *is regular in* Δ, $f(0) = f'(0) - 1 = 0$ *and it satisfies* (3.34) *for* $\varrho \geqq 1$, *then* $f(z)$ *is convex for*

$$(3.45)\qquad |z| < \frac{(4\varrho-1) - \sqrt{(1 - 8\varrho + 12\varrho^2)}}{2\varrho} ;$$

moreover there is a function satisfying the given conditions which is not convex in a larger disk.

This result was announced recently by Janowski [5] where he indicates that his proofs depend on variational methods; the proof given below makes use of a classical inequality for bounded functions [9, p. 165]. Singh gives the bound (3.45) for the case $\varrho = 1$.

Proof. From the hypotheses, in particular (3.34), we can say that there is a function $\phi(z)$ regular and bounded by 1 in Δ such that

$$(3.46)\qquad \frac{z f'(z)}{f(z)} = \varrho(1 + \phi(z)) , \quad z \in \Delta , \quad \text{and} \quad \phi(0) = \frac{1-\varrho}{\varrho} .$$

Therefore,

$$(3.47)\qquad 1 + \frac{z f''(z)}{f'(z)} = \frac{z\phi'(z) + \varrho(1+\phi(z))^2}{(1+\phi(z))} =$$
$$= \frac{z\phi'(z)(1+\overline{\phi(z)}) + \varrho|1+\phi(z)|^2 (1+\phi(z))}{|1+\phi(z)|^2} .$$

It is well-known [9, p. 224] that $f(z)$ maps the circle $|z| = r$ onto the boundary of a convex set whenever the real part of the form in (3.47) is non-negative for $|z| = r$, and conversely.

$$(3.48)\qquad \begin{aligned} &\operatorname{Re}\{z\,\phi'(z)\,(1+\overline{\phi(z)}) + \varrho|1+\phi(z)|^2\,(1+\phi(z))\} \geqq \\ &\geqq \varrho|1+\phi(z)|^2\,(1+\operatorname{Re}\phi(z)) - |z\,\phi'(z)|\,|1+\phi(z)| = \\ &= |1+\phi(z)|\,\{\varrho|1+\phi(z)|\,(1+\operatorname{Re}\phi(z)) - |z\,\phi'(z)|\} \geqq \\ &\geqq |1+\phi(z)|\,\{\varrho(1-|\phi(z)|)^2 - |z\,\phi'(z)|\}\,, \end{aligned}$$

and making use of the inequality [9, p. 168],

$$(3.49)\qquad |\phi'(z)| \leqq \frac{1-|\phi(z)|^2}{1-|z|^2}\,, \quad z \in \Delta\,,$$

we write

$$(3.50)\qquad \begin{aligned} &\varrho(1-|\phi(z)|)^2 - |z\,\phi'(z)| \geqq \varrho(1-|\phi(z)|)^2 - \frac{|z|\,(1-|\phi(z)|^2)}{1-|z|^2} = \\ &= (1-|\phi(z)|)\left\{\varrho(1-|\phi(z)|) - \frac{|z|\,(1+|\phi(z)|)}{1-|z|^2}\right\}. \end{aligned}$$

Since [9, p. 167]

$$(3.51)\qquad |\phi(z)| \leqq \frac{|\phi(0)| + |z|}{1+|\phi(0)|\,|z|} = \frac{(\varrho-1)+\varrho|z|}{\varrho+(\varrho-1)\,|z|}\,,$$

$$(3.52)\qquad \begin{aligned} &\varrho(1-|\phi(z)|) - \frac{|z|\,(1+|\phi(z)|)}{1-|z|^2} = \left(\varrho - \frac{|z|}{1-|z|^2}\right) - \\ &- \left(\varrho + \frac{|z|}{1-|z|^2}\right)|\phi(z)| \geqq \left(\varrho - \frac{|z|}{1-|z|^2}\right) - \\ &- \left(\varrho + \frac{|z|}{1-|z|^2}\right)\left(\frac{(\varrho-1)+\varrho|z|}{\varrho+(\varrho-1)\,|z|}\right) = \frac{[\varrho|z|^2 + (1-4\varrho)\,|z| + \varrho]}{(1-|z|)\,(\varrho+(\varrho-1)\,|z|)}\,. \end{aligned}$$

Consequently, $\operatorname{Re}\{1 + z\,f''(z)/f'(z)\} > 0$, $|z| = r$, whenever

$$(3.53)\qquad \varrho r^2 + (1-4\varrho)\,r + \varrho > 0\,;$$

and this quadratic is positive for the values in the disk given by (3.45).

If

$$(3.54)\qquad f(z) = z\left(1 + \left(\frac{1-\varrho}{\varrho}\right)z\right)^{(2\varrho-1)/(1-\varrho)}\,, \quad \varrho \neq 1\,,$$

then $f(z)$ satisfies (3.34) and

$$(3.55) \qquad 1 + \frac{z f''(z)}{f'(z)} = \frac{\varrho z^2 + (4\varrho - 1) z + \varrho}{(1 + z)(\varrho + (1 - \varrho) z)},$$

which is equal to zero for

$$z = - \frac{(4\varrho - 1) - \sqrt{(1 - 8\varrho + 12\varrho^2)}}{2\varrho}.$$

Therefore $f(z)$ shows the bound indicated in (3.45) is best possible for $\varrho \neq 1$. Singh [12] gives an extremal function for the case $\varrho = 1$. These methods do not seem to give sharp results for the case $\frac{1}{2} < \varrho < 1$.

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