Czechoslovak Mathematical Journal

Jan Kučera

On representation of temperate distributions

Czechoslovak Mathematical Journal, Vol. 22 (1972), No. 2, 191–194

Persistent URL: http://dml.cz/dmlcz/101089

Czechoslovak Mathematical Journal, 22 (97) 1972, Praha

ON REPRESENTATION OF TEMPERATE DISTRIBUTIONS

JAN KUČERA, Pullman

(Received December 8, 1969)

Every temperate distribution can be represented as a linear combination of derivatives of square integrable on R^n functions with polynomials as coefficients (see e.g. [3]). In this paper we substitute differentiation and multiplication by a family of bounded linear operators and show that then the problem of representation is equivalent to the problem of existence of solutions of some operational equations in the dual space.

Let H, K, be Hilbert spaces with inner products $(.,.)_H$, $(.,.)_K$, respectively. For a bounded linear operator $T: H \to K$ we denote by T^* its adjoint, i.e. $(Tx, y)_K = (x, T^*y)_H$ for all $x \in H$, $y \in K$, and by T' its dual, i.e. $y(Tx) = (T'y)(x)$ for all $x \in H$, $y \in K'$, where K' is the strong dual space of K.

The space of all infinitely differentiable complex functions on R^n is denoted by $C^\infty(R^n)$ and $C_0^\infty(R^n)$ is its subspace of all functions with compact support. $\mathcal{S} = \{f \in C^\infty(R^n);\ \sup_{x \in R^n} |x^\beta D^\alpha f(x)| < +\infty$ for all multiindices $\alpha, \beta\}$, $\mathcal{S}'$ denotes its dual. For $A \subset H$, where H is a Hilbert space, l.h. A means the linear hull of A.

Due to Fréchet-Riesz Theorem for each Hilbert space L it exists a semi-linear bijective map $I_L : L \to L'$ such that $I_L(y)\, x = (x, y)_L$ for all $x, y \in L$. Evidently $T' = I_H T^* I_K^{-1}$. As an immediate consequence of this identity we get

Theorem 1. *Let T_j, $j \in J$, be a family of bounded linear operators $T_j : H \to K$, where H, K, are Hilbert spaces. Then*

$$\text{l.h.} \bigcup_{j \in J} T_j' K' = H' \quad \textit{iff} \quad \text{l.h.} \bigcup_{j \in J} T_j^* K = H\,.$$

Let us have a sequence

$$H^0 \supset H^1 \supset H^2 \supset \ldots \tag{1}$$

of Hilbert spaces with an inner product $(.,.)_k$ and a norm $\|.\|_k$ respectively. Let they satisfy the following hypothesis:

(H) For each integer $k \geqq 0$ the space H^{k+1} is dense in H^k and the identity operator $I : H^{k+1} \to H^k$ is continuous.

The intersection $H^\infty = \bigcap_{k=0}^{\infty} H^k$ is a locally convex space with a topology defined by the norms $\|.\|_k$, $k = 0, 1, 2, \ldots$ It was shown in [4] that if (H) holds then the dual space $(H^\infty)'$ equals to the space of restrictions on H^∞ of all elements of $\bigcup_{k=0}^{\infty} (H^k)'$. If we moreover assume that $H^\infty = \mathscr{S}$ (including the topology) then $\mathscr{S}'$ is a union of Hilbert spaces $(H^k)'$, $k = 0, 1, 2, \ldots$ In this particular case Theorem 1 implies

Theorem 2. *Be given a sequence* (1) *satisfying* (H) *and a family of linear bounded operators* $T_j : H^1 \to H^0, j \in J$, *such that for each integer* $k \geqq 0$ *the restriction of* T_j *on* H^{k+1} *(which we denote again by* T_j*) is a bounded operator from* H^{k+1} *into* H^k *for each* $j \in J$.

Then, for each integer $m \geqq 0$, *elements of* $(H^m)'$ *can be represented as linear combinations of* $T'_{j_m} T'_{j_{m-1}} \ldots T'_{j_1} g$, *where* $g \in (H^0)'$ *iff* l.h. $\bigcup_{j \in J} T_j^* H^k = H^{k+1}$ *for* $k = 0, 1, 2, \ldots, m - 1$.

Example 1. For any integer $k \geqq 0$ let

$$H^k = \left\{ f : R^n \to C;\ \sum_{|\alpha| + |\beta| \leqq k} \int_{R^n} x^{2\beta} |D^\alpha f(x)|^2 \, dx < +\infty \right\}, \tag{2}$$

where derivatives $D^\alpha f$ are generalized. Define operators $T_0 = 1$, $T_j = x_j$, $T_{j+n} = i(\partial/\partial x_j)$, $j = 1, 2, \ldots, n$, $T = \sum_{j=0}^{2n} T_j$. Then H^k is a Hilbert space with an inner product $(f, g)_k = \int_{R^n} (T^k f) \overline{(T^k g)} \, dx$.

The spaces (2) fulfil (1) and (H). Moreover $\bigcap_{k=0}^{\infty} H^k = \mathscr{S}$. Hence $\bigcup_{k=0}^{\infty} (H^k)' = \mathscr{S}'$. To simplify some formulae we will further write $(H^k)' = H^{-k}$. It was shown in [3] that for each $f \in H^{-k}$ and for each multiindex α, $|\alpha| \leq k$, there exists $f_\alpha \in H^0 = L^2(R^n)$ and a polynomial P_α of degree $\leqq k - |\alpha|$ such that

$$f = \sum_{|\alpha| \leqq k} P_\alpha D^\alpha f_\alpha, \tag{3}$$

where the derivative $D^\alpha f_\alpha$ is in the ditributional sense.

For each integer $k \geqq 0$ and each $j = 0, 1, \ldots, 2n$, the operator T_j is a linear bounded operator from H^{k+1} into H^k. Further $T'_j = \overline{T}_j$, where the bar denotes the complex-conjugate operator. Hence, according to Theorem 2 we have

$$\text{l.h.} \bigcup_{j=0}^{2n} T_j^* H^k = H^{k+1} \quad \text{for} \quad k = 0, 1, 2, \ldots \tag{4}$$

Fix $j = 0, 1, \ldots, 2n$, and $k = 0, 1, \ldots$ By definition $(T_j f, g)_k = (f, T_j^* g)_{k+1}$ for $f \in H^{k+1}$, $g \in H^k$. This is equivalent to $\overline{(T^k g)}(T^k T_j f) = \overline{(T^{k+1} T_j^* g)}(T^{k+1} f)$ and $\overline{(T_j T^{2k} g)} f = \overline{(T^{2k+2} T_j^* g)} f$. Hence $T_j T^{2k} g = T^{2k+2} T_j^* g$ as elements in H^{-k-1}. By (4) we have

$$T^{2k+2} H^{k+1} = \text{l.h.} \bigcup_{j=0}^{2n} T_j T^{2k} H^k, \quad k \geqq 0. \tag{5}$$

Prove that (5) implies

$$T^{2k+2} H^{k+1} = H^{-k-1} \quad \text{for} \quad k = 0, 1, 2, \ldots \tag{6}$$

Evidently $T^{2k+2} H^{k+1} \subset H^{-k-1}$. To prove the opposite inclusion let first $k = 0$. Then by (5) $T^2 H^1 = \text{l.h.} \bigcup_{j=0}^{2n} T_j H^0 = H^{-1}$. For the induction assume that (6) holds for an integer $k - 1 \geqq 0$. Then by (5) $T^{2k+2} H^{k+1} = \text{l.h.} \bigcup_{j=0}^{2n} T_j T^{2k} H^k = \text{l.h.} \bigcup_{j=0}^{2n} T_j H^{-k} = H^{-k-1}$.

Now for any integer $k \geqq 0$ we can write $H^{-k-1} = T^{2k+2} H^{k+1} = T(T^{2k+1} H^{k+1}) \subset TH^{-k}$. As the opposite inclusion is trivial we have

Theorem 3. *For any integer $k \geqq 0$ and any $g \in H^{-k-1}$ the equation*

$$\left(1 + \sum_{j=1}^{n} \left(x_j + i \frac{\partial}{\partial x_j}\right)\right) f = g \tag{7}$$

has a solution $f \in H^{-k}$.

Evidently, the representation (3) follows immediately from Theorem 3.

Example 2. Let $\vartheta : R^n \to R$ be measurable. Assume that there are constants A, B, such that $A(1 + \sum_{j=1}^{n} x_j^2) \leqq \vartheta(x) \leqq B(1 + \sum_{j=1}^{n} x_j^2)$ for all $x \in R^n$. If we define an inner product in spaces (2) by

$$(f, g)_k = \sum_{|\alpha| \leqq k} \int_{R^n} \vartheta^{k-|\alpha|} D^\alpha f D^\alpha \bar{g} \, \mathrm{d}x, \quad k = 0, 1, 2, \ldots, \tag{8}$$

then the representation (3) remains the same. Each operator T_j, $i = 0, 1, \ldots, 2n$, from H^{k+1} into H^k, $k \geqq 0$, remains bounded on H^{k+1}. Similarly as in Example 1 we get an identity (in H^{-k-1})

$$T_j\left(\sum_{|\alpha| \leqq k} D^\alpha(\vartheta^{k-|\alpha|} D^\alpha g)\right) = \sum_{|\beta| \leqq k+1} D^\beta(\vartheta^{k+1-|\beta|} D^\beta T_j^* g), \tag{9}$$

where $g \in H^k$, $k = 0, 1, 2, \ldots$, $j = 0, 1, \ldots, 2n$. Then from Theorem 2 it follows

Theorem 4. *For any* $h \in H^{-k}$ *the equation*

$$\sum_{|\alpha| \leqq k} D^\alpha(\vartheta^{k-|\alpha|} D^\alpha f) = h \tag{10}$$

has a solution $f \in H^k$, $k = 1, 2, \ldots$

Proof by the mathematical induction.

References

[1] *L. Schwartz:* Théorie des distributions, Nouvelle édition, Hermann, Paris 1966.

[2] *L. Bers, F. John, M. Schechter:* Partial Differential Equations, Interscience 1964.

[3] *J. Kučera:* Fourier L_2-transform of Distributions, Czech. Math. J., *19* (*94*), Praha 1969, pp. 143–153.

[4] *Ю. М. Березинский:* Пространства с негативной нормой, Успехи матем. наук, т. *18*, вып. 1(109), январь—февраль 1963, стр. 63—96.

Author's address: Washington State University, Pullman, Washington 99163, U.S.A.