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DISTRIBUTIVY IN LATTICE ORDERED GROUPS

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Higher degrees of distributivity in lattice ordered groups were studied by several authors [3], [4], [8], [9], [10], [14], [15]. The results of this paper are as follows. Let G be a lattice ordered group and let α, β be any cardinals. It is proved that if G is a mixed product of lattice ordered groups A_i $(i \in I)$, then G is (α, β)-distributive if and only if each A_i is (α, β)-distributive. Let G be Archimedean and let $E(G)$ be the Dedekind completion of G; if G is (α, β)-distributive and each non-trivial interval $[a, b]$ of G contains a non-trivial interval $[a_1, b_1]$ with card $[a_1, b_1] \leqq \beta$, then $E(G)$ is (α, β)-distributive. If G is $(\alpha, 2)$-distributive and complete, then it is (α, α)-distributive (this is a partial solution of a problem by WEINBERG [14]). Assume that G is not completely distributive and denote by $\mathrm{d}G$ the least cardinal α such that G is not α-distributive. Let A be an l-ideal of G. It is proved that dG and $d(G/A)$ are mutually independent in a rather strong sense (Thm. 4.2); in particular, if $\alpha_0, \alpha_1, \ldots, \alpha_n$ are any regular cardinals, then there exist an l-group G and l-ideals $A_1 \subset A_2 \subset \ldots \subset A_n$ of G such that $\mathrm{d}G = \alpha_0$, $d(G/A_i) = \alpha_i$ $(i = 1, \ldots, n)$. The distributive radical $D(G)$ of an l-group G is defined to be the intersection of the closures of the minimal prime subgroups of G. $D(G)$ is a convex l-subgroup of G. The l-group G is completely distributive if and only if $D(G) = \{0\}$ [3]. We prove that if G is complete then G is the direct product $M(G) \times D(G)$, where $M(G)$ is the greatest convex completely distributive l-subgroup of G. Some other types of convex l-subgroups $I(G)$ of G with the property that G is completely distributive if and only if $I(G) = \{0\}$ are investigated.

Let us recall some basic definitions. For lattices and lattice ordered groups (l-groups) we shall use the standard notations, cf. [1], [7]. The group operation will be written additively, but it is not assumed to be commutative. Let α, β be cardinals and let T, S be sets, card $T \leqq \alpha$, card $S \leqq \beta$. A lattice L is said to be $(\wedge, \vee) - (\alpha, \beta)$-distributive, if the equation

$$(1) \qquad \bigwedge_{t \in T} \bigvee_{s \in S} x_{t,s} = \bigvee_{\varphi \in S^T} \bigwedge_{t \in T} x_{t,\varphi(t)}$$

holds in L identically whenever all joins and meets standing in (1) do exist in L. The $(\vee, \wedge) - (\alpha, \beta)$-distributivity is defined dually. L is (α, β)-distributive, if it is both $(\wedge, \vee)$- and $(\vee, \wedge) - (\alpha, \beta)$-distributive; L is completely distributive, if it is (α, α)-distributive for each cardinal α. An l-group G is (α, β)-distributive if its is $(\wedge, \vee) - - (\alpha, \beta)$-distributive or $(\vee, \wedge) - (\alpha, \beta)$-distributive.

An l-group G is representable if it is a subdirect product of linearly ordered groups. Any Abelian l-group is representable [1]. A set $A \subset G$ is a polar of G if it has the form $A = \{g \in G : |g| \wedge |b| = 0 \text{ for each } b \in B\}$ where B is a non-empty subset of G; we denote $A = B^{\delta}$. The system $P(G)$ of all polars of G partially ordered by inclusion is a Boolean algebra [12]. An interval $[a, b] = \{x \in L : a \leqq x \leqq b]$ of a lattice L is non-trivial, if $a < b$.

1. (α, β)-DISTRIBUTIVITY IN MIXED PRODUCTS OF l-GROUPS

Let G be a lattice ordered group and assume that G is a mixed product of l-groups G_i $(i \in I)$. From Thm. 1.4 below it follows that G is (α, β)-distributive if and only if each G_i is (α, β)-distributive. In particular, G is completely distributive if and only if each G_i is. Weinberg [15] proved that for an Archimedean l-group G the conditions

(A) G is completely distributive,

(B) the Boolean algebra $P(G)$ is atomic,

are equivalent and that (B) $\Rightarrow$ (A) for representable l-groups. He also found an example of a non-representable l-group satisfying (A) but not (B). The theorem on the (α, β)-distributivity of the mixed product enables one to show that (B) is not implied by (A) even in the case of representable l-groups.

Let I be a partially ordered set and for each $i \in I$ let G_i be a partially ordered group. The mixed product

$$G = \Omega_{i \in I} G_i \tag{2}$$

is defined as follows [7]: G is the set of all vectors

$$g = (\ldots, g_i, \ldots)_{i \in I}, \quad g_i \in G_i$$

such that the set $S(g) = \{i \in I : g_i \neq 0\}$ satisfies the descending chain condition. The operation $+$ in G is defined component-wise. For $g \in G$, $g \neq 0$ we put $g > 0$ provided $g_i > 0$ for each minimal element i of the set $S(g)$. Then G is a partially ordered group; it is an l-group if and only if (i) I is a tree (i.e., no two incomparable elements of I have a common upper bound), (ii) G_i is linearly ordered whenever i is not maximal in I, and (iii) G_i is an l-group for each maximal element $i \in I$ [7]. Let us remark that if (2) holds and G is an l-group, then the lattice operations are not, in general, performed component-wise. If the partial order on I is trivial (i.e., no two distinct elements of I are comparable) then $\Omega_{i \in I} G_i$ is the direct product $\Pi_{i \in I} G_i$. In the case of $I = \{i_1, i_2\}$, $i_1 < i_2$ we shall write $G = G_{i_1} \circ G_{i_2}$.

1.1. *Let G be an l-group, $G = A \circ B$, $B \neq \{0\}$, $g = (a, b) \in G$, $g_k = (a_k, b_k) \in \in G(k \in K)$, $g = \bigvee_{k\in K} g_k$. Then there is $k \in K$ such that $a_k = a$.*

Proof. From $g_k \leqq g$ it follows $a_k \leqq a$. Contrary to the assertion of the theorem, assume that $a_k < a$ for each $k \in K$. Since $B \neq \{0\}$, there exists $b^* \in B$ such that $0 \leqq b^*$ does not hold. Then $g_k < (a, b + b^*)$ for each $k \in K$, but (a, b) non $\leqq \leqq (a, b + b^*)$ and thus g is not the join of the system $\{g_k\}_{k\in K}$.

Analogously we can prove the dual assertion.

1.2. *Let G be an l-group, $0 < g \in G$. Then the interval $[0, g]$ is $(\wedge, \vee) - (\alpha, \beta)$-distributive if and only if it is $(\vee, \wedge) - (\alpha, \beta)$-distributive.*

Proof. Assume that $[0, g]$ is not $(\wedge, \vee) - (\alpha, \beta)$-distributive. Then the lattice $[-g, 0]$ being dually isomorphic to $[0, g]$ fails to be $(\vee, \wedge) - (\alpha, \beta)$- distributive. But the translation $\varphi : x \to x + g$ is an isomorphism of $[-g, 0]$ onto $[0, g]$, hence $[0, g]$ is not $(\vee, \wedge) - (\alpha, \beta)$-distributive. The remaining part of the proof can be performed dually.

The following simple lemma on the (α, β)-distributivity is essentially known ([14], Thm. 2.6; [8], 1.3):

1.3. *Let G be an l-group, $0 < g \in G$. Then the following conditions are equivalent:*

(i) *The interval $[0, g]$ is not (α, β)-distributive.*

(ii) *There is a system $\{x^*_{t,s} : t \in T, s \in S\} \subset [0, g]$ such that* card $T \leqq \alpha$, card $S \leqq \leqq \beta$ *and*

$$0 < v_1 = \bigvee_{s\in S} x^*_{t,s} \quad \text{for each} \quad t \in T, \tag{3}$$

$$0 = \bigwedge_{t\in T} x^*_{t,\varphi(t)} \quad \text{for each} \quad \varphi \in S^T\,. \tag{4}$$

Proof. If (ii) holds, then $v_1 \in [0, g]$ and

$$\bigwedge_{t\in T} \bigvee_{s\in S} x^*_{t,s} = v_1 > 0 = \bigvee_{\varphi\in S^T} \bigwedge_{t\in T} x^*_{t,\varphi(t)}\,,$$

thus $[0, g]$ is not (α, β)-distributive. Conversely, assume that $[0, g]$ is not (α, β)-distributive. According to 1.2 $[0, g]$ is not $(\wedge, \vee) - (\alpha, \beta)$-distributive. Then there exist in $[0, g]$ a system

$$\{x_{t,s} : t \in T,\ s \in S],\quad \text{card } T \leqq \alpha\,,\quad \text{card } S \leqq \beta$$

such that

$$v = \bigwedge_{t\in T} \bigvee_{s\in S} x_{t,s} > \bigvee_{\varphi\in S^T} \bigwedge_{t\in T} x_{t,\varphi(t)} = u\,,$$

hence

$$v_1 = v - u = \bigwedge_{t\in T} \bigvee_{s\in S} (x_{t,s} - u) > \bigvee_{\varphi\in S^T} \bigwedge_{t\in T} (x_{t,\varphi(t)} - u) = 0\,.$$

Clearly $v, v_1 \in [0, g]$. By putting $x^*_{t,s} = [(x_{t,s} - u) \vee 0] \wedge v_1$ we obtain that (ii) is valid.

From the method of the proof of 1.3, it follows easily:

1.3.1. *If G is not (α, β)-distributive, then there is $0 < g \in G$ such that the interval $[0, g]$ is not (α, β)-distributive.*

If G is a mixed product of l-groups G_i $(i \in I)$, we denote for any $i \in I$

$$\bar{G}_i = \{g \in G : g_j = 0 \text{ for each } j \in I,\ j \neq i\} .$$

1.4. Theorem. *Assume that an l-group G is a mixed product of l-groups $G_i \neq \{0\}$ $(i \in I)$. Let H be an l-subgroup of G such that*

$$\bigcup_{i\in I}\bar{G}_i \subset H .$$

Let α, β be cardinals. Then the following conditions are equivalent:

(i) *H is (α, β)-distributive.*

(ii) *Each G_i $(i \in I)$ is (α, β)-distributive.*

(iii) *G_i is (α, β)-distributive for each maximal element $i \in I$.*

Proof. Suppose that (i) holds. If $i \in I$ and i is not maximal in I, then G_i is linearly ordered, thus G_i is completely distributive. If i is maximal in I, then G_i is isomorphic to $\bar{G}_i$ and $\bar{G}_i$ is a convex l-subgroup of H. Clearly any convex l-subgroup of an (α, β)-distributive l-group is (α, β)-distributive as well; hence G_i is (α, β)-distributive.

Let us assume that (ii) is valid and suppose (contrary to the assertion ((i)) that H is not (α, β)-distributive. Then according to 1.3 there is a system $\{x^*_{t,s} : t \in T, s \in S\} \subset H$, card $T \leqq \alpha$, card $S \leqq \beta$, such that (3) and (4) are fulfilled. Let $S_1(v_1)$ be the system of all minimal elements of $S(v_1)$. Now we distinguish two cases. First assume that each element $i \in S_1(v_1)$ is maximal in I. Then $v_1(j) = 0$ for each $j \in I \setminus S_1(v_1)$ (we write $v_1(j)$ instead of $(v_1)_j$ and analogously for other elements). Let $i \in S_1(v_1)$. There exists $\bar{v}_1 \in H$ such that $\bar{v}_1(i) = v_1(i)$ and $\bar{v}_1(j) = 0$ for each $j \in I$, $j \neq i$ (since $\bar{G}_i \subset H$). Clearly $0 < \bar{v}_1 \leqq v_1$. By putting $\bar{x}_{t,s} = x^*_{t,s} \wedge \bar{v}_1$ we get from (3) and (4)

$$(5) \qquad 0 < \bar{v}_1 = \bigvee_{s\in S}\bar{x}_{t,s} \quad \text{for each} \quad t \in T,$$

$$(6) \qquad 0 = \bigwedge_{t\in T}\bar{x}_{t,\varphi(t)} \quad \text{for each} \quad \varphi \in S^T .$$

Since i is maximal in I,$\bar{G}_i$ is a convex l-subgroup of H. All elements $\bar{v}_1$, $\bar{x}_{t,s}$ belong to $\bar{G}_i$. By (5), (6) and 1.3, $\bar{G}_i$ is not (α, β)-distributive, therefore G_i is not (α, β)-distributive, which is a contradiction.

Now assume that there exists $i \in S_1(v_1)$ that is not maximal in I. Then we can choose $i_1 \in I$, $i_1 > i$ and $p \in G_i$, $p > 0$. Since $\bar{G}_{i_1} \subset H$, there is $\bar{v}_1 \in H$ such that $\bar{v}_1(i_1) = p$, $\bar{v}_1(j) = 0$ for each $j \in I, j \neq i_1$. Then $0 < \bar{v}_1 \leqq v_1$ and if we again put $\bar{x}_{t,s} = x^*_{t,s} \wedge \bar{v}_1$ we get the equations (5) and (6); thus i_1 cannot be maximal in I. Denote

$$H_0 = \{g \in H : g(j) = 0 \text{ for each } j \in I,\ j \text{ non} \geqq i_1\} ,$$

$$B = \{g \in H : g(j) = 0 \text{ for each } j \in I,\ j \text{ non} > i_1\} .$$

All elements $\bar{v}_1$, $\bar{x}_{t,s}$ belong to H_0. Clearly H_0 is isomorphic to

$$G_{i_1} \circ B$$

and since i_1 is not maximal in I, we have $B \neq \{0\}$. Therefore according to 1.1 it follows from (5) that for each $t \in T$ there is $s_t \in S$ such that

$$\bar{x}_{t,s_t}(i_1) = \bar{v}_1(i_1) = p > 0 .$$

Put $\varphi_0(t) = s_t$; thus $\bar{x}_{t,\varphi_0(t)}(i_1) = p$ for each $t \in T$. Consider the equation (6) for $\varphi = \varphi_0$; from the assertion dual to 1.1 it follows that there is $t \in T$ with the property

$$\bar{x}_{t,\varphi_0(t)}(i_1) = 0(i_1) = 0 ;$$

this is a contradiction. Therefore (ii) $\Rightarrow$ (i). Obviously (ii) and (iii) are equivalent.

1.5. Corollary. *Let G be an l-group that is a mixed product of l-groups $G_i \neq \{0\}$ $(i \in I)$. Then G is (α, β)-distributive if and only if G_i is (α, β)-distributive for each maximal element $i \in I$.*

It was proved in [8] that if G is an l-group with card $G = \alpha_0$ and if it is $(2^{\alpha_0}, \alpha_0)$-distributive, then it is completely distributive. From this and from 1.4 we obtain:

1.6. Corollary. *Let G be an l-group that is a mixed product of l-groups $G_i \neq \{0\}$ $(i \in I)$ and let H be an l-subgroup of G such that $\bigcup_{i \in I} \bar{G}_i \subset H$. Let α_0 be a cardinal such that card $G_i \leqq \alpha_0$ for each maximal element $i \in I$. If for each maximal element $i \in I$ the l-group G_i is $(2^{\alpha_0}, \alpha_0)$-distributive, then H is completely distributive.*

1.7. Example. Let I be a tree such that for each $i \in I$ there exist incomparable elements $i_1, i_2 \in I$ with $i < i_1$, $i < i_2$. For each $i \in I$ let G_i be the linearly ordered additive group of all integers, $G = \Omega_{i \in I} G_i$. According to 1.5 the l-group G is completely distributive. Since G is Abelian, it is representable. Let $X \in P(G)$, $X \neq \{0\}$. There is $x \in X$, $x > 0$. Let i be a minimal element of $S(x)$. There are incomparable elements i_1, i_2 with $i < i_1$, $i < i_2$. Let $y_k \in G$, $y_k(i_k) = 1$, $y_k(j) = 0$ for each $j \in I$, $j \neq i_k$ $(k = 1, 2)$. Then $0 < y_k \leqq 2x \in X$. Put $Y_k = \{y_k\}^{\delta} \cap X$. We have $\{0\} \neq Y_k \subset X$, $Y_k \neq X$, $Y_k \in P(G)$ $(k = 1, 2)$ and Y_1, Y_2 are incomparable (since $y_1 \in Y_2 \setminus Y_1$, $y_2 \in Y_1 \setminus Y_2$). Therefore X is not an atom in $P(G)$ and so $P(G)$ has no atoms. Thus (B) is not implied by (A) for representable l-groups.

2. DEDEKIND COMPLETION

Let L be a lattice. For any subset $\emptyset \neq X \subset L$ let $L(X)$ and $U(X)$ be the system of all lower bounds and upper bounds of X, respectively. By $E(L)$ we denote the system of all sets $L(U(X))$ where X runs over the system of all non-empty sets X that are upper bounded. $E(L)$ is partially ordered by inclusion. Then $E(L)$ is a conditionally complete lattice. If $\{A_i\}_{i\in I}$ is an upper bounded (lower bounded) subset of $E(L)$, then $\bigvee_{i\in I} A_i = L(U(X))$ where $X = \bigcup_{i\in I} A_i (\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i)$.

Now let G be an Archimedean l-group. If $A_1, A_2 \in E(G)$, the operation $+$ in $E(G)$ is defined by the rule $A_1 + A_2 = L(U(X))$, where $X = \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\}$. $E(G)$ is a complete l-group [1], [7]; it is called the Dedekind completion of G. For $g \in G$ put $\bar{g} = L(U(\{g\}))$. The system of all elements $\bar{g} (g \in G)$ is an l-subgroup of $E(G)$ isomorphic to G. $\bar{0}$ is the neutral element of $E(G)$.

If G is a completely distributive l-group, then the lattice $E(G)$ need not be completely distributive [15]. We shall assume that G is an (α, β)-distributive Archimedean l-group and we investigate a sufficient condition under which $E(G)$ is (α, β)-distributive.

2.1. *Let α, β be cardinals, $\beta \geqq \aleph_0$. Let G be an (α, β)-distributive Archimedean l-group. Let $0 < b \in G$,* card $[0, b] \leqq \beta$. *Then the interval $[\bar{0}, \bar{b}]$ of $E(G)$ is (α, β)-distributive.*

Proof. Assume to the contrary that $[\bar{0}, \bar{b}]$ is not (α, β)-distributive. Then according to 1.3 there exist elements $V, X_{t,s} \in [\bar{0}, \bar{b}]$ (card $T \leqq \alpha$, card $S \leqq \beta$) such that

(7) $$\bar{0} < V = \bigvee_{s\in S} X_{t,s} \quad \text{for each} \quad t \in T,$$

(8) $$\bar{0} = \bigwedge_{t\in T} X_{t,\varphi(t)} \quad \text{for each} \quad \varphi \in S^T.$$

Without loss of generality we may suppose that $V = \bar{v}$ for some $0 < v \in G$ (for, if this is not the case, then there is $0 < v \in V$ and instead of $X_{t,s}$ we may consider the elements $X_{t,s} \cap \bar{v}$). For any $t \in T$, $s \in S$ put

$$Y_{t,s} = \{g \in X_{t,s} : g \geqq 0\}.$$

According to (7) $Y_{t,s} \subset [0, v] \subset [0, b]$, thus we can write

(9) $$Y_{t,s} = \{y_{t,s,i} : i \in I\}, \quad \text{card } I = \beta;$$

from (7) and (9) we obtain

(10) $$v = \sup \bigcup_{s\in S} Y_{t,s} = \bigvee_{s\in S, i\in I} y_{t,s,i}$$

for each $t \in T$.

Denote $S \times I = K$, $y_{t,s,i} = y_{t,k}$, $k = (s, i) \in K$. Then card $K = \beta$ and according to (10)

(11) $$0 < v = \bigvee_{k\in K} y_{t,k} \quad \text{for each} \quad t \in T.$$

Further, it follows from (8) that for any $\chi \in I^T$ and any $\varphi \in S^T$ we have

$$\bigwedge_{t\in T} y_{t,\varphi(t),\chi(t)} = 0,$$

(12) $$\bigwedge_{t\in T} y_{t,\chi_1(t)} = 0 \quad \text{for each} \quad \chi_1 \in K^T.$$

Equations (11) and (12) show that the interval $[0, v] \subset [0, b]$ is not (α, β)-distributive, which is a contradiction. Thus the interval $[\bar{0}, \bar{b}]$ is (α, β)-distributive.

2.2. Theorem. *Let G be an Archimedean (α, β)-distributive l-group, $\beta \geqq \aleph_0$. Assume that for each $0 < a \in G$ there is $b \in G$ such that $0 < b \leqq a$,* card $[0, b] \leqq \beta$. *Then the Dedekind completion $E(G)$ of G is (α, β)-distributive.*

Proof. Suppose that $E(G)$ is not (α, β)-distributive. Then there are elements $V, X_{t,s} \in E(G)$ ($t \in T$, $s \in S$, card $T \leqq \alpha$, card $S \leqq \beta$) such that the equations (7), (8) are valid. Hence there is $a \in V$, $a > 0$ and $b \in G$, $0 < b \leqq a$ such that card $[0, b] \leqq \beta$. Put $Z_{t,s} = X_{t,s} \wedge \bar{b}$. Then

$$\bar{b} = \bigvee_{s \in S} Z_{t,s} \quad \text{for each} \quad t \in T,$$

$$\bar{0} = \bigwedge_{t \in T} Z_{t,\varphi(t)} \quad \text{for each} \quad \varphi \in S^T,$$

thus the interval $[\bar{0}, \bar{b}]$ of $E(G)$ is not (α, β)-distributive; according to 2.1 this is a contradiction.

3. THE $(\alpha, 2)$-DISTRIBUTIVITY

Let L be a lattice and α, β infinite cardinals. Consider the conditions:

(a) L is $(\alpha, 2)$-distributive.

(b) L is (α, α)-distributive.

Clearly (b) implies (a). The following two theorems on Boolean algebras are known (Pierce [10]):

3.1. *For any Boolean algebra,* (a) *implies* (b).

3.2. *Let B be a Boolean algebra and let X be the Stone space of B. Then the lattice $C(X)$ of all continuous real functions on X is (α, β)-distributive if and only if B is (α, β)-distributive.*

Weinberg [14] studied the lattice ordered group $C(X)$ of all continuous functions on a topological space X under the assumption that X is Hausdorff and completely regular; he proved the following theorem:

3.3. *For $C(X)$ the implication* (a) $\Rightarrow$ (b) *is valid.*

Weinberg also remarks that this implication is apparently unsettled for the more general case of an arbitrary lattice or even for Archimedean lattice ordered groups.

In this section we shall prove that the implication (a) $\Rightarrow$ (b) is satisfied for any complete lattice ordered group. We need the following notions:

Let G be an l-group, $e \in G$, $e > 0$. The element e is said to be a strong unit in G, if

$$\bigcup_{n=1}^{\infty} [-ne, ne] = G.$$

A complete l-group G is a K-space [13] if we can define a multiplication λx of elements $x \in G$ by reals λ such that G becomes a linear space with the property $\lambda x > 0$ for any $x > 0$ and $\lambda > 0$. Let G be a K-space and let e be a strong unit in G; we denote by $B(e)$ the set of all elements $e_1 \in G$ such that there exists $e_2 \in G$ with the property $e_1 \vee e_2 = e$, $e_1 \wedge e_2 = 0$. The set $B(e)$ is a Boolean algebra and according to the well-known theorem on the representation of K-spaces, G is isomorphic to the l-group of all bounded continuous functions on the Stone space $X(B(e))$ of the Boolean algebra $B(e)$ (cf. [13], Thm. V. 3.1.).

An l-subgroup H of an l-group G is said to be dense if for each $0 < g \in G$ there is $h \in H$ such that $0 < h \leqq g$.

3.4. *Let H be a convex dense l-subgroup of an l-group G. Then G is (α, β)-distributive if and only if H is (α, β)-distributive.*

Proof. If G is (α, β)-distributive, then obviously H is (α, β)-distributive as well. Assume that H is (α, β)-distributive and suppose (contrary to our assertion) that G is not (α, β)-distributive. Then there are elements $0 < v$, $x^*_{t,s} \in G$ such that the condition (ii) of 1.3 holds. Since H is dense in G, there exists $v_1 \in H$ with $0 < v_1 \leqq v$. If we set $x^*_{t,s} \wedge v_1 = \bar{x}_{t,s}$, we get the equations (5) and (6) showing that H is not (α, β)-distributive.

An element $0 < u \in G$ is said to be a weak unit in G if $u \wedge g > 0$ for each $0 < g \in G$. If u is a weak unit in G and H is a convex l-subgroup of G such that $u \in H$, then obviously H is dense in G. The following Corollary to 3.4 slightly generalizes a result of Weinberg ([14], Proposition 2.9):

3.4.1. Corollary. *Let G be an l-group with a weak unit u and let H be a convex l-subgroup of G such that $u \in H$. Then G is (α, β)-distributive if and only if H is (α, β)-distributive.*

3.5. *Let G be a K-space that is $(\alpha, 2)$-distributive. Then G is (α, α)-distributive.*

Proof. Assume that G is not (α, α)-distributive. Then there are elements $x^*_{t,s}$, $v_1 \in G$ such that the equations (3) and (4) hold, card $T \leqq \alpha$, card $S \leqq \beta$ and $v_1 > 0$. Let

$$G_1 = \bigcup_{n=1}^{\infty} [-nv_1, nv_1] .$$

G_1 is a K-space with the strong unit v_1, hence G_1 is isomorphic to the l-group H_1 consisting of all bounded continuous functions on the Stone space $X(B(v_1))$. Moreover. G_1 is $(\alpha, 2)$-distributive, since it is a convex l-subgroup of G. Thus H_1 is $(\alpha, 2)$-distributive. Let H be the l-group of all continuous functions on $X(B(v_1))$. Since H_1 is a convex dense l-subgroup of H, H is $(\alpha, 2)$-distributive according to 3.4 and therefore by 3.2 the Boolean algebra $B(v_1)$ is $(\alpha, 2)$-distributive as well. By 3.1 $B(v_1)$ is

(α, α)-distributive, thus, by 3.2 H is (α, α)-distributive and therefore H_1 is (α, α)-distributive. Hence G_1 is (α, α)-distributive. But, since v_1 and $x^*_{t,s}$ belong to G_1, it follows from (3), (4) that G_1 is not (α, α)-distributive, which is a contradiction.

An element s of an l-group G is said to be singular, if $s > 0$ and for each $a \in G$ such that $0 \leqq a < s$ the equation $a \wedge (s - a) = 0$ is valid. Let $S_1 = S_1(G)$ be the set of all singular elements of G.

3.6. (CONRAD [6]). *Let G be a complete lattice ordered group. Then S_1^δ is a K-space and G is isomorphic to the direct product $S_1^\delta \times S_1^{\delta\delta}$.*

3.7. *Let s be a singular element of an l-group G. Then $[0, s]$ is a Boolean algebra.*

Proof. $[0, s]$ is a distributive lattice. If $a \in [0, s]$, put $a' = s - a$. Then $a' \in [0, s]$, $a \wedge a' = 0$, $a \vee a' = a + a' = s$, thus $[0, s]$ is a complemented lattice.

3.8. *Let G be a complete l-group, $S_1 = S_1(G)$. If $S_1^{\delta\delta}$ is $(\alpha, 2)$-distributive, then it is (α, α)-distributive.*

Proof. If $S_1^{\delta\delta}$ is not (α, α)-distributive, then there exist elements $v_1, x^*_{t,s} \in S_1^{\delta\delta}$ (card $T \leqq \alpha$, card $S \leqq \alpha$) such that $v_1 > 0$ and the equations (3), (4) are fulfilled. Further there is $s \in S_1$ with the property $v_1 \wedge s = \bar{v}_1 > 0$. From 3.7 it follows that $\bar{v}_1 \in S_1$ and that $[0, \bar{v}_1]$ is a Boolean algebra. Since $S_1^{\delta\delta}$ is $(\alpha, 2)$-distributive, so is $[0, \bar{v}_1]$ and thus according to 3.1, $[0, \bar{v}_1]$ is (α, α)-distributive. But the equations (5) and (6) hold for $\bar{v}_1$, therefore $[0, \bar{v}_1]$ is not (α, α)-distributive, which is a contradiction.

3.9. Theorem. *Let G be a complete $(\alpha, 2)$-distributive l-group. Then it is (α, α)-distributive.*

Proof. The convex l-subgroups S_1^δ and $S_1^{\delta\delta}$ of G are $(\alpha, 2)$-distributive. From 3.5 and 3.6 it follows that S_1^δ is (α, α)-distributive. According to 3.8 $S_1^{\delta\delta}$ is (α, α)-distributive. Using 3.6 again we get that G is (α, α)-distributive.

4. DISTRIBUTIVITY OF FACTOR l-GROUPS

Pierce [10] investigated the higher degrees of distributivity for factor lattices B/A where B is a Boolean algebra and A is an ideal of B. We shall consider the distributivity of a factor l-group G/A determined by an l-ideal A of the l-group G. If G is not completely distributive we denote by dG the least cardinal α such that G is not α-distributive.[1]) Theorem 4.2 shows that there does not exist, in general, any relationship between dG and $d(B/A)$.

[1]) α-distributivity means (α,α) - distributivity.

4.1. *Let α be a regular cardinal. There exist an l-group G_α and an l-ideal C_α of G_α such that G_α is completely distributive and $d(G_\alpha/C_\alpha) = \alpha$.*

Proof. It was proved in [11] that there exists a Boolean algebra B_α with $dB_\alpha = \alpha$. Let X_α be the Stone space of B_α, and let H_α be the l-group of all continuous real functions defined on X_α. According to 3.2 $dH_\alpha = \alpha$. Let E be the additive group of all reals with the natural order, $E_1 = E \circ E$ and let F_α be the set of all mappings $f : X_\alpha \to E_1$. For any $x \in X_\alpha$ we have $f(x) = (y_1, y_2) \in E \circ E$; we denote $y_i = f_i(x)$ $(i = 1, 2)$. Let $G_\alpha = \{f \in F_\alpha : f_1 \in H_\alpha\}$. For $f, g \in F_\alpha$ we put $f \leqq g$ if $f(x) \leqq g(x)$ for each $x \in X$. Then F_α is an additive l-group and G_α is an l-subgroup of F_α. Assume that G_α is not β-distributive for some cardinal β. According to 1.3 there exist elements $f^{t,s}, f \in G_\alpha (t \in T, s \in S$, card $T \leqq \beta$, card $S \leqq \beta)$ such that

$$0 < f = \bigvee_{s\in S} f^{t,s} \qquad \text{for each} \quad t \in T,$$
$$0 = \bigwedge_{t\in T} f^{t,\varphi(t)} \quad \text{for each} \quad \varphi \in S^T .$$

Since $f > 0$, there is $x_1 \in X_\alpha$ with $f(x_1) = (f_1(x_1), f_2(x_1)) > 0$. If $f_1(x_1) > 0$, choose any $0 < y_0 \in E$; if $f_1(x_1) = 0$, take $y_0 \in E$ such that $0 < y_0 < f_2(x_1)$. There exists $g \in G_\alpha$ such that $g(x_1) = (0, y_0)$, $g(x) = (0, 0)$ for each $x \in X_\alpha$, $x \neq x_1$. Clearly $0 < g \leqq f$. We have

$$0 < g = \bigvee_{s\in S} f^{t,s} \wedge g \qquad \text{for any} \quad t \in T,$$
$$0 = \bigvee_{t\in T} f^{t,\varphi(t)} \wedge g \quad \text{for any} \quad \varphi \in S^T ,$$

hence the interval $[0, g]$ is not β-distributive. But $[0, g]$ is a chain and therefore it is completely distributive; this is a contradiction. Thus G_α is completely distributive. Let $C_\alpha = \{f \in G_\alpha : f_1(x) = 0$ for each $x \in X_\alpha\}$. Obviously C_α is an l-ideal of G_α and G_α/C_α is isomorphic to H_α, whence $d(G_\alpha/C_\alpha) = \alpha$.

4.2. Theorem. *Let I be a partially ordered set and let α_0, α_i $(i \in I)$ be regular cardinals. There exist a lattice ordered group G and a system of l-ideals $\{A_i\}_{i\in I}$ of G such that*

(i) *$dG = \alpha_0$,* (ii) *$d(G/A_i) = \alpha_i$ for each $i \in I$, and* (iii) *$i_1 \leqq i_2 \Leftrightarrow A_{i_1} \subset A_{i_2}$.*

Proof. Let $i_0 \notin I$, $I' = I \cup \{i_0\}$. For any $i \in I$ let G_{α_i}, C_{α_i} be l-groups constructed as in 4.1 and let $G_{\alpha_0} = H_{\alpha_0}$. Denote $G = \prod_{i\in I'} G_{\alpha_i}$. We define B_{ij} $(i \in I, j \in I')$ as follows:

$B_{ii} = C_{\alpha_i}$ for each $i \in I$, $B_{ij} = G_{\alpha_j}$ if $j < i$ or $j = i_0$ and $B_{ij} = \{0\}$ otherwise.

Denote $A_i = \prod_{j\in I'} B_{ij}$. Then A_i is an l-ideal of G and for $i_1, i_2 \in I$

$$i_1 \leqq i_2 \Leftrightarrow A_{i_1} \subset A_{i_2} .$$

The factor l-group G/A_i is isomorphic to the product $\prod_{j\in I'} D_{ij}$, where $D_{ii} = G_{\alpha_i}/C_{\alpha_i}$, $D_{ij} = \{0\}$ if $j < i$ or $j = i_0$, and $D_{ij} = G_{\alpha_j}$ otherwise; thus $d(G/A_i) = \alpha_i$. Obviously $dG = \alpha_0$.

Let G be an l-group. Conrad [4] introduced the concepts of the radical $R(G)$ and the ideal radical $L(G)$ of G and studied the relationship between the properties of $R(G)$, $L(G)$ and the complete distributivity of G. The set $R(G)$ $(L(G))$ is a convex l-subgroup (an l-ideal) of G and $L(G) \subset R(G)$. If G is Abelian, then $L(G) = R(G)$. The following results were proved in [4]:

(i) If $R(G) = \{0\}$, then G is completely distributive.

(ii) If G is completely distributive, then $L(G) = \{0\}$.

Further, it does not follow from $L(G) = \{0\}$ that G is completely distributive [9] and the complete distributivity of G does not imply that $R(G) = \{0\}$ [4].

Byrd and Lloyd [3] defined and examined the distributive radical $D(G)$ of an l-group G. $D(G)$ is a convex l-subgroup of G, $L(G) \subset D(G) \subset R(G)$ and

(iii) $D(G) = \{0\}$ if and only if G is completely distributive.

In this section we investigate some other types of convex l-subgroups $I(G)$ of G with the property that $I(G) = \{0\}$ if and only if G is completely distributive. In each l-group G there is a greatest convex completely distributive l-subgroup $M(G)$ of G [8]. We shall show that $G = M(G) \times D(G)$ whenever G is a complete l-group.

Let us recall the definition of $D(G)$. A convex l-subgroup A of G is called prime provided $a \wedge b > 0$ for any pair of positive elements $a, b \in G \setminus A$. An l-group B of G is said to be closed if whenever $\{b_i\} \subset B$ and $\bigvee b_i$ exists in G then $\bigvee b_i \in B$. The closure of an l-subgroup A of G is the intersection of all closed convex l-subgroups of G containing A. The distributive radical $D(G)$ of G is defined to be the intersection of the closures of the minimal prime subgroups of G.

5.1. ([8]) *Let G be an l-group. There exists a closed l-ideal $M(G)$ of G such that* (i) *$M(G)$ is completely distributive,* (ii) *if $0 < g \in G$ and $[0, g]$ is completely distributive, then $[0, g] \subset M(G)$ (thus $A \subset M(G)$ for any completely distributive convex l-subgroup A of G), and* (iii) *if G is complete, then $M(G)$ is a direct factor of G.*

Let $g, g^* \in G$, $g > 0$, $g^* > 0$. The pair (g, g^*) is called distributive, if whenever $g = \bigvee_{i \in I} g_i$ and $g_i \geqq 0$, then $g^* \leqq g_i$ for some $i \in I$.

5.2. (Weinberg [15]). *An l-group G is completely distributive if and only if each $0 < g \in G$ is the first term of a distributive pair.*

5.3. (Byrd - Lloyd [3]). *For an l-group G, $D(G) = \{g \in G : |g|$ is not the first term of a distributive pair$\}$. G is completely distributive if and only if $D(G) = 0$.*

5.4. *Let $0 < g \in G$ such that $[0, g]$ is completely distributive, $0 < d \in D(G)$. Then $g \wedge d = 0$.*

Proof. Assume that $0 < g_1 = g \wedge d$. Then $g_1 \in D(G)$ since $D(G)$ is convex in G and hence by 5.3 g_1 is not the first term of a distributive pair. According to 5.1 $g_1 \in$ $\in M(G)$ and therefore it follows from 5.2 that g_1 is the first term of a distributive pair, which is impossible.

5.5. *Let $0 < g \in G$ such that $g \wedge d = 0$ for each $0 < d \in D(G)$. Then $[0, g]$ is completely distributive.*

Proof. Let $X = D(G)^\delta$. Then X is a closed convex l-subgroup of G and by 5.3 each strictly positive element of X is the first term of a distributive pair. According to 5.2 X is completely distributive. Obviously $g \in X$ and thus $[0, g]$ is completely distributive.

5.6. Theorem. *For any l-group G, $D(G)^\delta = M(G)$.*

Proof. From 5.4 it follows that $M(G) \subset D(G)^\delta$ and by 5.5 $D(G)^\delta \subset M(G)$.

5.7. *If G is a complete l-group, then $D(G) = M(G)^\delta$.*

Proof. According to 5.4, $D(G) \subset M(G)^\delta$. Let $g \in M(G)^\delta$, $g > 0$ and let $\{d_i\}$ be the system of all elements $d_i \in D(G)$ such that $0 \leqq d_i \leqq g$. Put $d = \bigvee d_i$. $D(G)$ is a closed l-subgroup of G, thus $d \in D(G)$. Denote $-d + g = g_1$. If $0 < d_1 \leqq g_1$, $d_1 \in D(G)$, then $d + d_1 \leqq g$, $d + d_1 \in D(G)$, whence $d + d_1 \leqq d$, which is impossible. Therefore $d_1 \wedge g_1 = 0$ for each $d_1 \in D(G)$, $d_1 > 0$. This implies $g_1 \in D(G)^\delta =$ $= M(G)$. But then $g_1 = g \wedge g_1 = 0$ and so we obtain $g \in D(G)$.

5.8. Theorem. *Let G be a complete l-group. Then $G = M(G) \times D(G)$.*

Proof. According to 5.1 there is a direct decomposition $G = M(G) \times B$. If any direct decomposition $H = C_1 \times C_2$ of an l-group H is given, then $C_2 = C_1^\delta$. Thus $B = M(G)^\delta$. Because of 5.7, $B = D(G)$.

5.9. *For any l-group G, $M(G)^\delta$ is an l-ideal of G. The l-group G is completely distributive if and only if $M(G)^\delta = \{0\}$.*

Proof. Assume that $M(G)^\delta = \{0\}$. According to 5.4, $D(G) \subset M(G)^\delta$, hence $D(G) =$ $= \{0\}$ and thus by 5.3 G is completely distributive. If G is completely distributive, then $M(G) = G$, therefore $M(G)^\delta = \{0\}$. For any l-ideal X of G the polar X^δ is an l-ideal as well.

In view of 5.3 and 5.9 it is natural to ask whether there exist intrinsically defined convex l-subgroups $A(G)$ of G such that (i) $A(G)$ is, in general, distinct from $D(G)$ and $M(G)$, and (ii) $A(G) = \{0\}$ if and only if G is completely distributive. We intend to show that there are, roughly speaking, infinitely many such convex l-subgroups $A(G)$. Let us introduce the following notions.

Let $\mathscr{K}$ be the class of all l-groups and let f be a mapping of $\mathscr{K}$ into $\mathscr{K}$ such that

(a) $f(G)$ is a convex l-subgroup of G for any $G \in \mathscr{K}$,

(b) if $G_1, G_2 \in \mathscr{K}$ and if φ is an isomorphism of G_1 onto G_2, then $\varphi(f(G_1)) = f(G_2)$.
Under these conditions f is said to be a convex injection defined in $\mathscr{K}$. A convex injection f is called monotone (strongly monotone) if $f(G_1)$ is a convex l-subgroup of $f(G_2)$ ($f(G_1) = G_1 \cap f(G_2)$) whenever G_1 is a convex l-subgroup of G_2.

If G_1, G_2 are convex l-subgroups of an l-group G we denote by $G_1 \vee G_2(G_1 \wedge G_2)$ the l-subgroup of G generated by the set $G_1 \cup G_2$ (the l-subgroup $G_1 \cap G_2$). Let $\mathscr{F}$ be the class of all convex injections defined in $\mathscr{K}$. Let $f_1, f_2 \in \mathscr{F}$. We define $f_1 \vee f_2$, $f_1 \wedge f_2$ and f_1^δ by the rules

$$(f_1 \vee f_2)(G) = f_1(G) \vee f_2(G), \quad (f_1 \wedge f_2)(G) = f_1(G) \wedge f_2(G),$$
$$f_1^\delta(G) = f_1(G)^\delta$$

for any $G \in \mathscr{K}$. Then $f_1 \vee f_2$, $f_1 \wedge f_2$ and f_1^δ belong to $\mathscr{F}$. The convex injections $f_1 \vee f_2$ and $f_1 \wedge f_2$ are monotone (strongly monotone) whenever f_1 and f_2 are monotone (strongly monotone). Further let $(f_1 f_2)(G) = f_1(f_2(G))$ for any $G \in \mathscr{K}$. If f_1 and f_2 are monotone, then $f_1 f_2$ is a monotone injection as well.

5.10. *If f is a strongly monotone injection, then the same holds for f^δ.*

Proof. Let G_1 be a convex l-subgroup of G. If $g \in f^\delta(G) \cap G_1$, then $|g| \wedge |a| = 0$ for any $a \in f(G) \supset f(G_1)$ thus $g \in f^\delta(G_1)$. Conversely, let $g \in f^\delta(G_1)$, $a \in f(G)$. Then $0 \leqq a_1 = |g| \wedge |a| \in f(G) \cap G_1 = f(G_1)$, therefore $0 = |g| \wedge a_1 = a_1$; this shows that $g \in f^\delta(G) \cap G_1$.

For any $G \in \mathscr{K}$ let $D(G)$ and $M(G)$ have the same meaning as above. Since $D(G)$ and $M(G)$ are convex l-subgroups of G, D and M belong to $\mathscr{F}$.

5.11. *The convex injections D and M are strongly monotone.*

Proof. Let G_1 be a convex l-subgroup of G, $g \in G_1$. Then g is the first term of a distributive pair in G_1 if and only if g is the first term of a distributive pair in G, therefore $D(G_1) = D(G) \cap G_1$. From 5.1 it follows that $M(G_1) = M(G) \cap G_1$.

Let $f \in \mathscr{F}$ and let us consider the condition:

(*) *For any $G \in \mathscr{K}$,*
$f(G) = \{0\} \Leftrightarrow G$ is completely distributive.

5.12. *Let $f \in \mathscr{F}$ such that $f(G)$ is a dense l-subgroup of $M^\delta(G)$ for any $G \in \mathscr{K}$. Then f satisfies (*).*

Proof. If G is completely distributive, then according to 5.9 $M^\delta(G) = \{0\}$, thus $f(G) = \{0\}$. Conversely, assume that $f(G) = \{0\}$. Since $f(G)$ is dense in $M^\delta(G)$, we get $M^\delta(G) = \{0\}$ and then it follows from 5.9 that G is completely distributive.

5.13. Theorem. *Let f be a strongly monotone convex injection. Then f satisfies $(*)$ if and only if $f(G)$ is a dense l-subgroup of $M^\delta(G)$ for any $G \in \mathscr{K}$.*

Proof. By 5.12 it suffices to prove the assertion "only if". Assume that f satisfies $(*)$. Then $f(G) \cap M(G) = f(M(G)) = \{0\}$ because $M(G)$ is completely distributive. Therefore $f(G) \subset M^\delta(G)$. Suppose that $f(G)$ is not dense in $M^\delta(G)$. Then there is $0 < x \in M^\delta(G)$ such that $|y| \wedge x = 0$ for each $y \in f(G)$. Thus the set $X = f(G)^\delta \cap \cap M(G)^\delta$ is a convex l-subgroup of G and $X \neq \{0\}$. Since $X \subset M(G)^\delta$, no non-trivial interval of X is completely distributive. We have $f(X) = f(G) \cap X \subset f(G) \cap \cap f(G)^\delta = \{0\}$ and so according to $(*)X$ is completly distributive, which is a contradiction.

5.14. Theorem. *Let G be an l-group and let $\{G_i : i \in I\}$ be the system of all convex Abelian l-subgroups of G, $A(G) = \bigcup_{i \in I} G_i$. Then $A(G)$ is an Abelian l-ideal of G.*

Proof. Let P be the set of all elements $p \geqq 0$, $p \in G$ such that $x_1, x_2 \in [0, p]$ implies $x_1 + x_2 = x_2 + x_1$. Let $p, q \in P$, $x \in [0, p]$, $y \in [0, q]$. Denote $x \wedge y = u$, $x_1 = x - u$, $y_1 = y - u$. Then $x_1 \wedge y_1 = 0$, thus $x_1 + y_1 = y_1 + x_1$ and by the assumption $u + x_1 = x_1 + u$, $u + y_1 = y_1 + u$. Hence it follows $x + y = y + x$. Now let $p + q = r$, $a_1, a_2 \in [0, r]$. There are decompositions

$$a_1 = x + z\,, \quad a_2 = t + y$$

such that $x, t \in [0, p]$, $z, y \in [0, q]$. This implies that any two elements of the set $\{x, z, t, y\}$ are permutable, therefore $a_1 + a_2 = a_2 + a_1$. Hence the set P is closed with respect to the addition. Evidently P is a convex sublattice of G and $0 \in P$. Let $p \in P$, $x \in G$, $p' = -x + p + x$, $y_1, y_2 \in [0, p']$. Then $x_i = x + y_i - x \in [0, p]$ $(i = 1, 2)$, thus x_1, x_2 are permutable and so y_1, y_2 are permutable as well. Therefore P is a normal subset of G. Denote

$$A(G) = \{p - q : p \in P,\ q \in P\}\,.$$

Then $A(G)$ is a convex l-subgroup of G. Moreover $A(G)$ is an l-ideal of G because P is normal in G. Since any two elements of P are permutable, $A(G)$ is Abelian. If G_i is a convex Abelian l-subgroup of G, $g \in G_i$, then $|g| \in P$ and therefore $G_i \subset A(G)$. $A(G)$ is the greatest convex Abelian l-subgroup of G.

From 5.14 it follows immediately:

5.15. *A belongs to $\mathscr{F}$ and is strongly monotone.*

For any convex l-subgroup X of an l-group G the set $X \vee X^\delta$ is a dense convex l-subgroup of G. Denote

$$F(G) = A(D(G)) \vee A^\delta(D(G))\,.$$

Then $F(G)$ is a convex l-subgroup of G and a dense convex l-subgroup of $D(G)$.

From 5.6 it follows that $D(G)$ is dense in $M(G)^\delta$. Therefore $F(G)$ is a dense convex l-subgroup of $M(G)^\delta$. Hence we obtain from 5.12:

5.16. *The convex injection F satisfies* $(*)$.

Example 5.20 below shows that $F(G)$ is, in general, a proper subset of $D(G)$ and from Example 5.22 it follows that $D(G)$ need not be equal to $M^\delta(G)$. All convex injections F, D, M^δ satisfy the condition $(*)$. We can construct other types of convex injections satisfying $(*)$ by using lexicographic decompositions of G.

Let G, G_1, G_2 be l-groups and let $\varphi : G \to G_1 \circ G_2$ be an isomorphism of G onto $G_1 \circ G_2$. Then (cf. §1) G_1 is linearly ordered and G_2 is an l-group. Let $\bar{G}_i$ $(i = 1, 2)$ have the same meaning as in §1, $\bar{G}_i^0 = \varphi^{-1}(\bar{G}_i)$, $\bar{G}_1^0$ being an l-subgroup of G and $\bar{G}_2^0$ a convex l-subgroup of G. We shall write

$$(\alpha) \qquad G = \bar{G}_1^0 \circ \bar{G}_2^0 .$$

(α) is said to be a lexicographic decomposition of G. Assume that another lexicographic decomposition (β) $G = \bar{H}_1^0 \circ \bar{H}_2^0$ is given. Let Z be the linearly ordered additive l-group of all integers. Let $\sim$ denote the isomorphism of l-groups.

5.17. *If G is an l-group, $G = \bar{G}_1^0 \circ \bar{G}_2^0 = \bar{H}_1^0 \circ \bar{H}_2^0$ such that $\bar{G}_1^0 \sim Z$, $\bar{H}_1^0 \sim Z$, then $\bar{G}_2^0 = \bar{H}_2^0$.*

Proof. Let $g \in G$. The components of g with respect to the decomposition (α) will be denoted by $g(\bar{G}_1^0)$, $g(\bar{G}_2^0)$, and analogously for the decomposition (β). Suppose that $\bar{G}_2^0$ is not a subset of $\bar{H}_2^0$. Then there is $g \in \bar{G}_2^0 \setminus \bar{H}_2^0$. Thus $|g| \in \bar{G}_2^0 \setminus \bar{H}_2^0$,

$$|g| = |g|\,(\bar{H}_1^0) + |g|\,(\bar{H}_2^0)\,, \quad |g|\,(\bar{H}_1^0) > 0\,.$$

Since $\bar{H}_1^0 \sim Z$, $\bar{H}_1^0$ is Archimedean. If $g' \in G$, there exists an integer n such that

$$-n|g|\,(\bar{H}_1^0) < g'(\bar{H}_1^0) < n|g|\,(\bar{H}_1^0)\,.$$

This implies $-n|g| < g' < n|g|$, therefore $g' \in \bar{G}_2^0$ and $G \subset \bar{G}_2^0$. But there exists $g' \in G$ such that $g'(\bar{G}_1^0) \neq 0$ and then $g' \notin \bar{G}_2^0$; this is a contradiction. Thus $\bar{G}_2^0 \subset \bar{H}_2^0$ and analogously $\bar{H}_2^0 \subset \bar{G}_2^0$.

5.18. *Let H be an l-ideal of an l-group G, $H = \bar{H}_1^0 \circ \bar{H}_2^0$, $\bar{H}_1^0 \sim Z$. Then $\bar{H}_2^0$ is an l-ideal of G.*

Proof. Obviously $\bar{H}_2^0$ is a convex l-subgroup of G. Let $y \in G$. The mapping $\varphi(x) = -y + x + y (x \in H)$ is an automorphism of the l-group H, thus $H = (-y + \bar{H}_1^0 + y) \circ (-y + \bar{H}_2^0 + y)$ and $-y + \bar{H}_1^0 + y \sim \bar{H}_1^0 \sim Z$. According to 5.17, $-y + \bar{H}_2^0 + y = \bar{H}_2^0$.

Now we define $F_1 \in \mathscr{F}$ as follows. Let $G \in \mathscr{K}$. If there exists a lexicographic decomposition $G = \bar{G}_1^0 \circ \bar{G}_0^2$ with $\bar{G}_1^0 \sim Z$, $\bar{G}_2^0 \neq \{0\}$, then $\bar{G}_2^0$ is uniquely determined by 5.17 and we put $F_1(G) = \bar{G}_2^0$. If there does not exist any such lexicographic decomposition for G, we set $F_1(G) = G$.

5.19. *The convex injection F_1 is monotone.*

Proof. Let H be a convex l-subgroup of G. If $F_1(G) = G$, then obviously $F_1(H) \subset \subset H \subset G$. Assume that $G = \bar{G}_1^0 \circ \bar{G}_2^0$, $\bar{G}_1^0 \neq \{0\}$, $\bar{G}_1^0 \sim Z$. If there is $h \in H$ such that $h(\bar{G}_1^0) \neq 0$, then (analogously as in the proof of 5.17) $H = G$ and so $F_1(H) = = F_1(G)$. Assume that $h(\bar{G}_1^0) = 0$ for each $h \in H$. Thus $H \subset \bar{G}_2^0$ and therefore $F_1(H) \subset H \subset F_1(G)$.

Denote $F_n = (F_1)^n$ $(n = 1, 2, \ldots)$. According to 5.18 F_n is monotone and for any l-group G, $F_n(G)$ is an l-ideal of $G(n = 1, 2, \ldots)$. Obviously $F_n(G)$ is dense in G.

For any $G \in \mathscr{K}$ let us put

$$F_n'(G) = F_n(M^\delta(G)) \quad (n = 1, 2, \ldots).$$

Then according to 5.18 and by induction $F_n'(G)$ is an l-ideal of G. The convex injection F_n' is monotone. In Example 5.21 below $F_m'(G) \neq F_n'(G)$ whenever $m \neq n$. F_n' satisfies $(*)$ for $n = 1, 2, \ldots$

5.20. Example. $(F(G) \neq D(G))$. Let α, β be irrational numbers, $\alpha < \beta$ such that $\beta - \alpha = \gamma$ is rational and let I be the set of all rational numbers belonging to the interval (α, β). Let C_1 be a linearly ordered group that is not commutative and let $C_2 \neq \{0\}$ be an Abelian linearly ordered group. For each $i \in I$ put $G_i = C_1 \circ C_2$, $H = \Pi_{i \in I} G_i$ and let G be a subset of H with the following property: for any $g \in G$ there is a positive integer $n = n(g)$ such that

$$g(i_1) = g(i_2)$$

whenever $i_1, i_2 \in I_k^n$ $(k = 1, \ldots, n)$, where

$$I_k^n = I \cap \left[\alpha + (k-1)\frac{\gamma}{n},\ \alpha + k\frac{\gamma}{n}\right].$$

Clearly G is an l-subgroup of H. For any positive integer m let $G(m)$ be the system of all $g \in G$ such that there is $k \in \{1, \ldots, m\}$ and $0 < g^0 \in C_1 \circ C_2$ with $g(i) = g^0$ for each $i \in I_k^m$ and $g(i) = 0$ otherwise. Let $g \in G$, $g > 0$. Whenever $n(g) \mid m$, there are elements $g_{k,m} \in G(m)$ $(k = 1, \ldots, m)$ such that

$$g = \bigvee_{k=1,\ldots,m} g_{k,m}.$$

If $0 < g^* \in G$, $n(g^*) \mid m$, $g_{k,m} \in G(m)$, then g^* non $\leqq g_{k,m}$. This shows that g is not the first term of a distributive pair. According to 5.3, $D(G) = G$. Further $A(D(G)) = A(G)$ is the set of all $g \in G$ such that $g(i)(C_1) \in A(C_1)$ for each $i \in I$. Since C_1 is not commutative, $A(C_1) \neq C_1$, therefore $F(G) \neq D(G)$. The set $A(G)$ is dense in G, thus $A^\delta(D(G)) = A^\delta(G) = \{0\}$. Therefore $F(G) = A(G) \neq D(G)$.

5.21. Example. ($F'_m(G_0) \neq F'_n(G_0)$ for $m \neq n$.) Let $J = \{-1, -\frac{1}{2}, -\frac{1}{3}, \ldots, 0\}$ with the natural order. For $j \in J$, $j \neq 0$ let $H_j = Z$ and $H_0 = G$, where G has the same meaning as in 5.20. Denote $G_0 = \Omega_{j \in J} H_j$. Assume that $M(G_0) \neq \{0\}$. Then there is $0 < h \in M(G_0)$. There exists $0 < g \in G$ with $g \leq h$. Then $g \in M(G_0)$ and by 5.11 $g \in M(G)$. According to 5.20 we have $g \in D(G)$, hence by 5.6 $g = 0$, a contradiction. Therefore $M(G_0) = \{0\}$, thus $M^\delta(G_0) = G_0$. Hence we obtain

$$F'_n(G) = \Omega_{j \in J} H'_j\,,$$

where $H'_j = \{0\}$ for $j = -1, \ldots, -1/n$ and $H'_j = H_j$ otherwise. This implies that $F'_m(G_0)$ is a proper subset of $F'_n(G_0)$ whenever $m > n$.

5.22. Example. ($D(G_0) \neq M^\delta(G_0)$, $D(G_0) \neq F'_n(G_0) \neq M^\delta(G_0)$.) Let G_0 be as in 5.21. Let $g, g^* \in G_0$ such that $0 < g$, $g(j) \neq 0$ for some $j \in J$, $j \neq 0$, $0 < g^*$, $g^*(j) = 0$ for each $j \in J$, $j \neq 0$. It is easy to verify that (g, g^*) is a distributive pair. On the other hand (cf. 5.20) g is not the first term of any distributive pair. Therefore we have

$$D(G_0) = \{g \in G_0 : g(j) = 0 \text{ for each } j \in J,\ j \neq 0\}\,,$$

hence $D(G_0) \neq M^\delta(G_0)$. At the same time, according to 5.21 $M^\delta(G_0) \neq F'_n(G_0) \neq D(G_0)$ for any positive integer n.

5.23. Example. (F_1 is monotone but not strongly monotone.) Let $I = (0, 1] \cup \{2, 3\}$ with the natural order. Let G be the same as in 5.20. Put $G_3 = G$ and for any $i \in I$, $i \neq 3$, $G_i = Z$, $H = \Omega_{i \in I} G_i$. Denote

$$H_1 = \{h \in H : h(i) = 0 \quad \text{for each} \quad i \in (0, 1]\}\,,$$

$$H_2 = \{h \in H : h(i) = 0 \quad \text{for each} \quad i \in (0, 1] \cup \{2\}\}\,.$$

Then $F_1(H) = H$, $F_1(H_1) = H_2 \neq F_1(H) \cap H_1$. Thus F_1 is not strongly monotone. According to 5.19 F_1 is monotone.

Let G be an l-group and let $L(G)$ be the lattice of all convex l-subgroups of G; this lattice was studied in [5]. A convex injection f is said to be L-invariant (weakly L-invariant), if for any l-groups (complete l-groups) G_1 and G_2

$$\varphi f(G_1) = f(G_2)$$

whenever φ is an isomorphism of $L(G_1)$ onto $L(G_2)$.

5.24. *The convex injection M is weakly invariant.*

Proof. Let G be an l-group and for $0 \neq g \in G$ let L_g be the subgroup of G that is generated by the set of all l-ideals of G not containing g. Then L_g is an l-ideal of G. The ideal radical $L(G)$ of G is defined to be

$$L(G) = \bigcap L_g \quad (0 \neq g \in G) .$$

$L(G)$ is L-invariant [4] and if G is complete, then $L(G) = D(G)$ [3]. Therefore $D(G)$ is weakly invariant. According to 5.8 $M(G)$ is a complement of $D(G)$ in the lattice $L(G)$ whenever G is complete. Since $L(G)$ is distributive, the complement of $D(G)$ is uniquely determined. Therefore $M(G)$ is weakly invariant.

It remains as an open question whether $M(G)$ is L-invariant.

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