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GREEN'S RELATIONS ON A COMPACT SEMIGROUP

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Let S be a semigroup. Then $°\mathbf{K}$ will denote the equivalence on S: for $a, b \in S$, $a°\mathbf{K}b$ if and only if there exist positive integers m, n such that $a^m = b^n$. In [1] J. T. Sedlock studies necessary and sufficient conditions on a periodic semigroup S in order that $°\mathbf{K}$ coincide with any one of the Green relations [2]. In our paper [3] we considered an arbitrary semigroup having similar properties.

The fact that any element x of a compact semigroup S belongs to some idempotent (see [4]) leads us to define an equivalence $°\mathbf{K}_T$ on S by: for $a, b \in S$, $a°\mathbf{K}_T b$ if and only if the elements a, b belong to the same idempotent. The purpose of this article is to investigate the structure of compact semigroups such that $°\mathbf{K}_T$ coincides with any one of the Green relations.

Let $\mathscr{C}(S)$ denote the set of all $\mathscr{C}$-closure operations for a non-empty set S, i.e.

(0) $$\mathbf{U} \in \mathscr{C}(S) \Leftrightarrow \mathbf{U} : \exp S \to \exp S$$

and

(1) $$\mathbf{U}(\emptyset) = \emptyset\,,$$

(2) $$A \subset B \subset S \Rightarrow \mathbf{U}(A) \subset \mathbf{U}(B)\,,$$

(3) $$A \subset \mathbf{U}(A) \quad \text{for each} \quad A \subset S\,,$$

(4) $$\mathbf{U}(\mathbf{U}(A)) = \mathbf{U}(A) \quad \text{for each} \quad A \subset S$$

hold.

A subset A of S will be called $\mathbf{U}$-closed if $\mathbf{U}(A) = A$. The set of all $\mathbf{U}$-closed subsets of S will be denoted by $\mathscr{F}(\mathbf{U})$.

Let $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$. Then we define

$$\mathbf{U} \leqq \mathbf{V} \Leftrightarrow \mathbf{U}(A) \subset \mathbf{V}(A) \quad \text{for each} \quad A \subset S\,.$$

We have

(5) $$\mathscr{F}(\mathbf{U} \vee \mathbf{V}) = \mathscr{F}(\mathbf{U}) \cap \mathscr{F}(\mathbf{V}),$$

(6) $$\mathbf{U} \leqq \mathbf{V} \Leftrightarrow \mathscr{F}(\mathbf{V}) \subset \mathscr{F}(\mathbf{U}).$$

We shall denote by $\mathscr{Q}(S)$ the set of all $\mathscr{Q}$-closure operations for a set S, i.e. $\mathscr{Q}(S) \subset \mathscr{C}(S)$ and for every $\mathbf{U} \in \mathscr{Q}(S)$ and for every $A \subset S$

(7) $$\mathbf{U}(A) = \bigcup_{x \in A} \mathbf{U}(x)$$

holds. If $\mathbf{U}, \mathbf{V} \in \mathscr{Q}(S)$ then

(8) $$\mathbf{U} \leqq \mathbf{V} \Leftrightarrow \mathbf{U}(x) \subset \mathbf{V}(x) \quad \text{for each} \quad x \in S.$$

Let $\mathbf{U} \in \mathscr{C}(S)$. We define $\mathbf{U}^* \in \mathscr{Q}(S)$. If $A \subset S$ then $x \in \mathbf{U}^*(A)$ if an only if $\mathbf{U}(x) \cap A \neq \emptyset$. For $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$ we have

9) $$\mathbf{U} \leqq \mathbf{V} \Rightarrow \mathbf{U}^* \leqq \mathbf{V}^*,$$

(10) $$\mathbf{U}(x) = \mathbf{U}^{**}(x) \quad \text{for every} \quad x \in S,$$

(11) $$\mathbf{U}^* = \mathbf{U}^{***} \quad \text{and} \quad \mathbf{U}^{**} \leqq \mathbf{U}.$$

See [5].

Let $\mathbf{U} \in \mathscr{C}(S)$. We shall introduce the equivalence $°\mathbf{U}$ on S by: for $x, y \in S$, $x°\mathbf{U}y$ if and only if $\mathbf{U}(x) = \mathbf{U}(y)$. For any element x of S, let $\mathbf{U}_x$ denote the $°\mathbf{U}$-class of S containing x. If $\mathbf{U}, \mathbf{V} \in \mathscr{C}(S)$ then we have

(12) $$\mathbf{U} \leqq \mathbf{V} \Rightarrow °\mathbf{U} \subset °\mathbf{V},$$

(13) $$°(\mathbf{U} \wedge \mathbf{V}) = °\mathbf{U} \cap °\mathbf{V},$$

(14) $$x°\mathbf{U}y \Leftrightarrow x \in \mathbf{U}(y) \quad \text{and} \quad y \in \mathbf{U}(x).$$

See [3].

Let S be an arbitrary semigroup. For any $A \subset S$, $A \neq \emptyset$, let us put $\mathbf{L}(A) = SA \cup A$ and $\mathbf{R}(A) = AS \cup A$. Finally, $\mathbf{L}(\emptyset) = \emptyset = \mathbf{R}(\emptyset)$. Clearly $\mathbf{L}, \mathbf{R} \in \mathscr{Q}(S)$. Put $\mathbf{M} = \mathbf{L} \vee \mathbf{R}$, $\mathbf{H} = \mathbf{L} \wedge \mathbf{R}$. $F(\mathbf{L})$ [$F(\mathbf{R})$, $F(\mathbf{M})$, $F(\mathbf{H})$] is the set of all left [right, two-sided, quasi) ideals of S (including $\emptyset$). It is known that

(15) *$\mathbf{H}_e$ is the maximal subgroup of S belonging to the idempotent e.*

Put $\mathbf{P}(\emptyset) = \emptyset$. If $A \subset S$, $A \neq \emptyset$, then by $\mathbf{P}(A)$ we denote the subsemigroup generated by all elements of A. Evidently $\mathbf{P} \in \mathscr{C}(S)$, $\mathbf{P} \leqq \mathbf{H}$ and $\mathscr{F}(\mathbf{P})$ is the set of all subsemigroups of S (including $\emptyset$). See [5].

Let $\mathbf{K} = \mathbf{P}^* \vee \mathbf{P}^{**}$. Then $\mathbf{K} = \mathbf{K}^*$ and $x°\mathbf{K}y$ if and only if there exist positive integers n, m such that $x^n = y^m$. See [3].

Let now S be a compact (Hausdorff) semigroup. If $A \subset S$, then by $\mathbf{T}(A)$ we denote the closure of A. It is known that $\mathbf{T} \in \mathscr{C}(S)$ and

$$(16) \qquad \mathbf{T}(A \cup B) = \mathbf{T}(A) \cup \mathbf{T}(B) \quad \text{for} \quad A \subset S \quad \text{and} \quad B \subset S\,,$$

$$(17) \qquad \mathbf{T}(x) = \{x\} \quad \text{for each} \quad x \in S\,.$$

We shall prove that

$$(18) \qquad \mathbf{T}(AB) = \mathbf{T}(A)\,\mathbf{T}(B) \quad \text{for} \quad \emptyset \neq A \subset S \quad \text{and for} \quad \emptyset \neq B \subset S\,.$$

Actually, it follows from 2.1.3 [6] that $\mathbf{T}(A)\,\mathbf{T}(B) \subset \mathbf{T}(AB)$. Since $\mathbf{T}(A)$, $\mathbf{T}(B)$ are compact, it follows from 2.1.5 [6] that $\mathbf{T}(A)\,\mathbf{T}(B)$ is also compact and thus we have $\mathbf{T}(A)\,\mathbf{T}(B) \in \mathscr{F}(\mathbf{T})$. By (3), we obtain that $A \subset \mathbf{T}(A)$, $B \subset \mathbf{T}(B)$. Hence $AB \subset \subset \mathbf{T}(A)\,\mathbf{T}(B)$. Using (2), we have $\mathbf{T}(AB) \subset \mathbf{T}(\mathbf{T}(A)\,\mathbf{T}(B)) = \mathbf{T}(A)\,\mathbf{T}(B)$. This means that (18) holds.

Put $\mathbf{P}_T = \mathbf{P} \vee \mathbf{T}$. It follows from (5) that $\mathscr{F}(\mathbf{P}_T)$ is the set of all closed subsemigroups of S (including $\emptyset$). It is known from [4] that for $x \in S$

(19) $\mathbf{P}_T(x) = \mathbf{T}(\mathbf{P}(x))$ *is the commutative subsemigroup having a unique idempotent.*

Lemma 1. *Let $A \subset S$. Then $A \in \mathscr{F}(\mathbf{P}_T^*)$ if and only if*

$$(20) \qquad \mathbf{P}_T(x) \cap A \neq \emptyset \Rightarrow x \in A$$

for every $x \in S$.

Proof. Let $A \in \mathscr{F}(\mathbf{P}_T^*)$. If $\mathbf{P}_T(x) \cap A \neq \emptyset$ for some $x \in S$, then there exists y such that $y \in \mathbf{P}_T(x)$ and $y \in A$. It follows from (2) that $x \in \mathbf{P}_T^*(y) \subset \mathbf{P}_T^*(A) = A$.

Let (20) hold for every $x \in S$. Evidently $\mathbf{P}_T^* \in \mathscr{Q}(S)$. If $A \neq \emptyset$, then by (7) we have $\mathbf{P}_T^*(A) = \bigcup_{x \in A} \mathbf{P}^*(x)$. If $y \in \mathbf{P}_T^*(A)$, then $y \in \mathbf{P}_T^*(x)$ for some $x \in A$. Since $x \in \mathbf{P}_T(y)$, it follows by (20) that $y \in A$. Therefore, $\mathbf{P}_T^*(A) \subset A$. It follows from (3) that $A = = \mathbf{P}_T^*(A) \in \mathscr{F}(\mathbf{P}_T^*)$.

Remark. Let $A \subset S$, $A \neq \emptyset$. Then $\mathbf{P}_T^*(A)$ is the set of all almost nilpotent elements (in *topological sense*) with respect to A. (See [7].)

Proof. If $x \in \mathbf{P}_T^*(A)$, then there exists $y \in A \cap \mathbf{P}_T(x) = A \cap \mathbf{T}(\mathbf{P}(x))$. If O is an arbitrary beighbourhood of A, then O is also a neighbourhood of y and thus $x^n \in O$ for some positive integer n. Therefore, the element x is almost nilpotent with respect to A.

If x is an almost nilpotent element with respect to A, then in every neighbourhood of A there exists at least one element of $\mathbf{P}(x)$. Suppose that $x \notin \mathbf{P}_T^*(A)$. This implies that $\mathbf{P}_T(x) \cap A = \emptyset$. Evidently $O = S - \mathbf{P}_T(x) = S - \mathbf{T}(\mathbf{P}(x))$ is a neighbourhood of A and $A \cap \mathbf{P}(x) = \emptyset$, which is a contradiction. Therefore, $x \in \mathbf{P}_T^*(A)$.

Definition. $\boldsymbol{K}_T = \boldsymbol{P}_T^* \vee \boldsymbol{P}_T^{**}$.

Lemma 2. $\boldsymbol{K}_T = \boldsymbol{K}_T^*$.

Proof. (9) implies that $\boldsymbol{P}_T^{**} \leqq \boldsymbol{K}_T^*$ and $\boldsymbol{P}_T^{***} \leqq \boldsymbol{K}_T^*$. It follows from (11) that $\boldsymbol{K}_T = \boldsymbol{P}_T^* \vee \boldsymbol{P}_T^{**} \leqq \boldsymbol{K}_T^*$. According to (9) and (11), we have $\boldsymbol{K}_T^* \leqq \boldsymbol{K}_T^{**} \leqq \boldsymbol{K}_T$. Hence $\boldsymbol{K}_T = \boldsymbol{K}_T^*$.

Lemma 3. *Let x, $e \in S$ and let $e^2 = e$. If $e \in \boldsymbol{P}_T(x)$, then $x°\boldsymbol{K}_T e$.*

Proof. If $e \in \boldsymbol{P}_T(x)$, then $x \in \boldsymbol{P}_T^*(e) \subset \boldsymbol{K}_T(e)$. It follows from (10) that $e \in \boldsymbol{P}_T(x) = \boldsymbol{P}_T^{**}(x) \subset \boldsymbol{K}_T(x)$. (14) implies that $x°\boldsymbol{K}_T e$.

Lemma 4. *Let $e, f \in S$ and let $e^2 = e, f^2 = f$. If $e°\boldsymbol{K}_T f$, then $e = f$.*

Proof. Using (14) we obtain $e \in \boldsymbol{K}_T(f)$. Let $A = \{u \in S / f \in \boldsymbol{P}_T(u)\}$. We shall show that $A \in \mathscr{F}(\boldsymbol{K}_T) = \mathscr{F}(\boldsymbol{P}_T^* \vee \boldsymbol{P}_T^{**}) = \mathscr{F}(\boldsymbol{P}_T^*) \cap \mathscr{F}(\boldsymbol{P}_T^{**})$ (see (5)). If $\boldsymbol{P}_T(x) \cap A \neq \emptyset$ for some $x \in S$, then there exists u such that $u \in A$ and $u \in \boldsymbol{P}_T(x)$. This implies that $f \in \boldsymbol{P}_T(u) \subset \boldsymbol{P}_T(x)$ and thus we have $x \in A$. By Lemma 1, $A \in \mathscr{F}(\boldsymbol{P}_T^*)$. If $x \in \boldsymbol{P}_T^{**}(A)$, then by (7) and (10) we have $x \in \boldsymbol{P}_T^{**}(u) = \boldsymbol{P}_T(u)$ for some $u \in A$. This implies that $\boldsymbol{P}_T(x) \subset \boldsymbol{P}_T(u)$. Since $f \in \boldsymbol{P}_T(u)$, hence, by (19), $f \in \boldsymbol{P}_T(x)$ and thus $x \in A$. This means that $\boldsymbol{P}_T^{**}(A) \subset A$ and according to (3) we obtain $A = \boldsymbol{P}_T^{**}(A) \in \mathscr{F}(\boldsymbol{P}_T^{**})$. Therefore, $A \in \mathscr{F}(\boldsymbol{K}_T)$. Since $f \in A$, (2) and (4) imply $e \in \boldsymbol{K}_T(f) \subset A$ and thus we have $f \in \boldsymbol{P}_T(e) = \{e\}$ (see (17)). Therefore, $e = f$.

Theorem 1. *Let x, $y \in S$. Then $x°\boldsymbol{K}_T y$ if and only if there exists an idempotent e of S such that*

$$e \in \boldsymbol{P}_T(x) \cap \boldsymbol{P}_T(y) . \tag{21}$$

Proof. Let $x°\boldsymbol{K}_T y$. By (19) there exist e, f of S such that $e = e^2 \in \boldsymbol{P}_T(x)$ and $f = f^2 \in \boldsymbol{P}_T(y)$. Lemma 3 implies that $e°\boldsymbol{K}_T f$. According to Lemma 4, we have $f = e$ and $e \in \boldsymbol{P}_T(x) \cap \boldsymbol{P}_T(y)$.

Let (21) hold. Then according to Lemma 3, we have $x°\boldsymbol{K}_T e$ and $y°\boldsymbol{K}_T e$. This implies that $x°\boldsymbol{K}_T y$.

Lemma 5. *$\boldsymbol{K} \leqq \boldsymbol{K}_T$ and $°\boldsymbol{K} \subset °\boldsymbol{K}_T$.*

Proof. Evidently $\boldsymbol{P} \leqq \boldsymbol{P} \vee \boldsymbol{T} = \boldsymbol{P}_T$ and (9) implies that $\boldsymbol{P}^* \leqq \boldsymbol{P}_T^*$ and $\boldsymbol{P}^{**} \leqq \boldsymbol{P}_T^{**}$. Therefore, $\boldsymbol{K} \leqq \boldsymbol{K}_T$. By (12), we have $°\boldsymbol{K} \subset °\boldsymbol{K}_T$.

Lemma 6. *If e is an idempotent of S, then $e\boldsymbol{K}_{Te} = \boldsymbol{K}_{Te}e = \boldsymbol{H}_e$.*

Proof. See Theorem 8 in [4].

Put $\mathbf{L}_T = \mathbf{L} \vee \mathbf{T}$, $\mathbf{R}_T = \mathbf{R} \vee \mathbf{T}$ and $\mathbf{M}_T = \mathbf{M} \vee \mathbf{T}$. Note that $\mathbf{M}_T = \mathbf{L}_T \vee \mathbf{R}_T$. It follows from (5) that $\mathscr{F}(\mathbf{L}_T)$ [$\mathscr{F}(\mathbf{R}_T)$, $\mathscr{F}(\mathbf{M}_T)$] is the set of all closed left [right, two-sided] ideals of S (including $\emptyset$).

Lemma 7. *We have*

1. $°\mathbf{L} = °\mathbf{L}_T$ *and* $\mathbf{L} = \mathbf{L}_T^{**}$,
2. $°\mathbf{R} = °\mathbf{R}_T$ *and* $\mathbf{R} = \mathbf{R}_T^{**}$,
3. $°\mathbf{M} = °\mathbf{M}_T$ *and* $\mathbf{M} = \mathbf{M}_T^{**}$.

Proof. Let $x \in S$. It follows from (16), (17), (18) and (5) that $\mathbf{L}(x) \in \mathscr{F}(\mathbf{L}_T)$. This implies that $\mathbf{L}(x) = \mathbf{L}_T(x)$. By (14), we have $°\mathbf{L} = °\mathbf{L}_T$. Further, by (10), we obtain that $\mathbf{L}(x) = \mathbf{L}_T^{**}(x)$. According to (8), we have $\mathbf{L} = \mathbf{L}_T^{**}$.

Analogously we can prove the statements 2 and 3.

Lemma 8. $\mathbf{L} \leqq \mathbf{R}$ *if and only if* $\mathbf{L}_T \leqq \mathbf{R}_T$.

Proof. If $\mathbf{L} \leqq \mathbf{R}$, then $\mathbf{L}_T = \mathbf{L} \vee \mathbf{T} \leqq \mathbf{R} \vee \mathbf{T} = \mathbf{R}_T$. Let $\mathbf{L}_T \leqq \mathbf{R}_T$. Then $\mathbf{L}_T(x) \subset \mathbf{R}_T(x)$ for every $x \in S$. According to the proof of Lemma 7, we have $\mathbf{L}(x) \subset \mathbf{R}(x)$. It follows from (8) that $\mathbf{L} \leqq \mathbf{R}$.

Lemma 9. *If* $A \subset S$, $A \neq \emptyset$, *then*

$$\bigcap_{x \in A} xS = \bigcap_{x \in \mathbf{T}(A)} xS \tag{22}$$

holds.

Proof. Let $z \in \bigcap_{x \in A} xS$. Suppose that $z \notin \bigcap_{x \in \mathbf{T}(A)} xS$. It follows that $z \notin uS$ for some $u \in \mathbf{T}(A)$. By (17) and (18), uS is a closed subset of the compact semigroup S and there exists a neighbourhood O of uS such that $z \notin O$. Evidently $ua \in O$ for every $a \in S$. It follows from the continuity of multiplication that there exist neighbourhoods $O_a(u)$ of u and $O(a)$ of a such that $O_a(u)\, O(a) \subset O$. It is clear that $S = \bigcup_{a \in S} O(a)$. Since S is a compact semigroup, there exists a finite system $O(a_1), O(a_2), \ldots, O(a_n)$ which also covers S. If we put $O_0(u) = O_{a_1}(u) \cap O_{a_2}(u) \cap \ldots \cap O_{a_n}(u)$, then $O_0(u)\, S \subset O$. Since $O_0(u)$ is a neighbourhood of u, there exists $x \in A \cap O_0(u)$. Evidently $z \in xS$. If $z = xb$ for some $b \in S$, then $z \in O_0(u)\, S \subset O$ which is a contradiction. Hence $z \in \bigcap_{x \in \mathbf{T}(A)} xS$. According to (3), we have $A \subset \mathbf{T}(A)$ so that $\bigcap_{x \in \mathbf{T}(A)} xS \subset \bigcap_{x \in A} xS$. Hence (22) holds.

Lemma 10. *If* $A \subset S$, $A \neq \emptyset$, *then*

$$\bigcap_{x \in A} SxS = \bigcap_{x \in \mathbf{T}(A)} SxS \tag{23}$$

holds.

Proof. Let $z \in \bigcap_{x \in A} SxS$. Suppose that $z \notin \bigcap_{x \in \mathbf{T}(A)} SxS$. It follows that $z \notin SuS$ for some $u \in \mathbf{T}(A)$. By (17) and (18), SuS is a closed subset of the compact semigroup S and there exists a neighbourhood O of SuS such that $z \notin O$. Evidently $aub \in O$ for every $a, b \in S$. It follows from the proof of Lemma 9 that there exist neighbourhoods $O'_a(au)$ of au such that $O'_a(au)\, S \subset O$ for every $a \in S$. The continuity of multiplication implies that there exist neighbourhoods $O(a)$ of a and $O_a(u)$ of u such that $O(a)\, O_a(u) \subset O'_a(au)$. Evidently $S = \bigcup_{a \in S} O(a)$. Since S is a compact semigroup, there exists a finite system $O(a_1), O(a_2), \ldots, O(a_n)$ which also covers S. If we put $O_0(u) = O_{a_1}(u) \cap O_{a_2}(u) \cap \ldots \ldots \bigcap O_{a_n}(u)$, then $SO_0(u)\, S \subset (\bigcap_{i=1}^{n} O'_{a_i}(a_i u))\, S \subset O$. Since $O_0(u)$ is a neighbourhood of u, there exists $x \in A \cap O_0(u)$. Evidently $z \in SxS$. If $z = axb$ for some $a, b \in S$, then $z \in SO_0(u)\, S \subset O$ which is a contradiction. Hence $z \in \bigcap_{x \in \mathbf{T}(A)} SxS$. The rest of the proof is analogous to that of Lemma 9.

Theorem 2. *The following conditions on a semigroup S are equivalent:*

1. *S is right regular;*
2. $\mathbf{P}_T^* \leqq \mathbf{R}_T$;
3. $\mathbf{K}_T \leqq \mathbf{R}_T$;
4. $^\circ\mathbf{K}_T \subset {}^\circ\mathbf{R}$.

Proof. 1 ⇒ 2. Let S be a right regular semigroup. Let A be a closed right ideal of S, i.e. $A \in \mathscr{F}(\mathbf{R}_T)$. If $u \in \mathbf{P}_T(x) \cap A$ $(x \in S)$, then by (2) we have $\mathbf{R}_T(u) \subset A$. Since S is right regular, $x \in x^n S$ for every positive integer n. It follows from Lemma 9 that $x \in \bigcap_{v \in \mathbf{P}(x)} vS = \bigcap_{v \in \mathbf{P}_T(x)} vS$. This implies that $x \in uS \subset \mathbf{R}_T(u) \subset A$. By Lemma 1 we have $A \in \mathscr{F}(\mathbf{P}_T^*)$. It follows from (6) that $\mathbf{P}_T^* \leqq \mathbf{R}_T$.

2 ⇒ 3. Suppose $\mathbf{P}_T^* \leqq \mathbf{R}_T$. Since $\mathbf{P} \leqq \mathbf{R}$, it holds $\mathbf{P}_T \leqq \mathbf{R}_T$. According to (9) and Lemma 7, we have $\mathbf{P}_T^{**} \leqq \mathbf{R}_T^{**} = \mathbf{R} \leqq \mathbf{R}_T$. Thus $\mathbf{K}_T = \mathbf{P}_T^* \vee \mathbf{P}_T^{**} \leqq \mathbf{R}_T$.

3 ⇒ 4. This follows from (12) and from Lemma 7.

4 ⇒ 1. If $^\circ\mathbf{K}_T \subset {}^\circ\mathbf{R}$, then by Lemma 5 we have $^\circ\mathbf{K} \subset {}^\circ\mathbf{K}_T \subset {}^\circ\mathbf{R}$. It follows from Theorem 6 in [3] that S is right regular.

The dual statement reads as follows:

Theorem 3. *The following conditions on a semigroup S are equivalent:*

1. *S is left regular;*
2. $\mathbf{P}_T^* \leqq \mathbf{L}_T$;
3. $\mathbf{K}_T \leqq \mathbf{L}_T$;
4. $^\circ\mathbf{K}_T \subset {}^\circ\mathbf{L}$.

Theorem 4. *The following conditions on a semigroup S are equivalent:*

1. *S is a union of groups;*
2. $\mathbf{P}_T^* \leqq \mathbf{R}_T \wedge \mathbf{L}_T$;
3. $\mathbf{K}_T \leqq \mathbf{R}_T \wedge \mathbf{L}_T$;
4. $^\circ\mathbf{K}_T \subset {}^\circ\mathbf{H}$.

Proof. $1 \Rightarrow 2 \Rightarrow 3$. This follows from Theorem 2 and Theorem 3.

$3 \Rightarrow 4$. It follows from (12), Lemma 7 and (13) that $^\circ\mathbf{K}_T \subset {}^\circ(\mathbf{R}_T \wedge \mathbf{L}_T) = {}^\circ\mathbf{R}_T \cap \cap {}^\circ\mathbf{L}_T = {}^\circ\mathbf{R} \cap {}^\circ\mathbf{L} = {}^\circ(\mathbf{R} \wedge \mathbf{L}) = {}^\circ\mathbf{H}$.

$4 \Rightarrow 1$. If $^\circ\mathbf{K}_T \subset {}^\circ\mathbf{H}$, then by Lemma 5 we have $^\circ\mathbf{K} \subset {}^\circ\mathbf{H}$. It follows from Theorem 8 in [3] that S is a union of groups.

Theorem 5. *The following conditions on a semigroup S are equivalent:*

1. *S is intraregular;*
2. $\mathbf{P}_T^* \leqq \mathbf{M}_T$;
3. $\mathbf{K}_T \leqq \mathbf{M}_T$;
4. $^\circ\mathbf{K}_T \subset {}^\circ\mathbf{M}$.

Proof. $1 \Rightarrow 2$. Let S be an intraregular semigroup. Let A be a closed two-sided ideal of S, i.e. $A \in \mathscr{F}(\mathbf{M}_T)$. If $u \in \mathbf{P}_T(x) \cap A$ $(x \in S)$, then by (2) we have $\mathbf{M}_T(u) \subset A$. For every positive integer n, we have $x^{n+2} \in Sx^nS$. It follows from Theorem 9 of [3] and (6) that $Sx^nS \in \mathscr{F}(\mathbf{M}) \subset \mathscr{F}(\mathbf{P}^*)$. Lemma 2 in [3] implies that $x \in Sx^nS$. It follows from Lemma 10 that $x \in \bigcap_{v \in \mathbf{P}(x)} SvS = \bigcap_{v \in \mathbf{P}_T(x)} SvS$. This implies that $x \in SuS \subset \subset \mathbf{M}_T(u) \subset A$. It follows from Lemma 1 that $A \in \mathscr{F}(\mathbf{P}_T^*)$. By (6) we have $\mathbf{P}_T^* \leqq \mathbf{M}_T$.

$2 \Rightarrow 3 \Rightarrow 4$. The proof is analogous to the proof of Theorem 2.

$4 \Rightarrow 1$. If $^\circ\mathbf{K}_T \subset {}^\circ\mathbf{M}$, then by Lemma 5 we have $^\circ\mathbf{K} \subset {}^\circ\mathbf{M}$. It follows from Theorem 9 of [3] that S is intraregular.

Theorem 6. *The conditions of Theorems* 2, 3, 4 *and* 5 *and the following condition on a semigroup S are equivalent:*

$$^\circ\mathbf{K}_T = {}^\circ\mathbf{H}\,.$$

Proof. 2 of Theorem 2 $\Rightarrow$ 2 of Theorem 5. If $\mathbf{P}_T^* \leqq \mathbf{R}_T$, then $\mathbf{P}_T^* \leqq \mathbf{R}_T \leqq \mathbf{M}_T$.

2 of Theorem 5 $\Rightarrow$ 1 of Theorem 4. Let $x \in S$. It follows from (19) that $e \in \mathbf{P}_T(x)$ where $e^2 = e$. By Theorem 1 and Lemma 6 we have $ex \in \mathbf{H}_e$. (15) and (6) imply that $e \in SexS \in \mathscr{F}(\mathbf{M}_T) \subset \mathscr{F}(\mathbf{P}_T^*)$. According to Lemma 1, we obtain that $x \in SexS$. Then there exist $a, b \in S$ such that $x = aexb$. If we put $c = ae$, then $x = cexb$ and $c = ce$. This implies that $x = c^nexb^n$ and $c^n = c^ne$ for any positive integer n. Let

$f \in \mathbf{P}_T(c)$ where $f^2 = f$ (see (19)). Then by Lemma 9 we have $x \in \bigcap_{v \in \mathbf{P}(c)} vS = \bigcap_{v \in \mathbf{P}_{L}(c)} vS$ so that $x \in fS$. Since $\mathbf{P}(c) = \mathbf{P}(c)\, e$, we obtain by (18) and (17) that $\mathbf{P}_T(c) = \mathbf{P}_T(c)\, e$. Since $f \in \mathbf{P}_T(c)\, e$, it holds $f = ue$ for some $u \in \mathbf{P}_T(c)$. Therefore $f = ue = ue^2 = fe$. Since $x \in fS$, $x = fz$ holds for some $z \in S$. This implies that $x = fz = f^2z = fx = fex$. According to (19), we have $ex = xe$ and thus $\mathbf{R}(xe) \subset \mathbf{R}(x) = \mathbf{R}(fex) = \mathbf{R}(fxe) = \mathbf{R}(xe)$. Therefore $\mathbf{R}(x) = \mathbf{R}(xe) = \mathbf{R}(ex) = e\mathbf{R}(x)$. Since $x \in e\mathbf{R}(x)$, it s $x = ew$ for some $w \in \mathbf{R}(x)$. This implies that $x = ew = e^2w = ex \in \mathbf{H}_e$. Hence S s a union of groups.

4 of Theorem 4 $\Rightarrow {}^\circ\mathbf{K}_T = {}^\circ\mathbf{H}$. Suppose ${}^\circ\mathbf{K}_T \subset {}^\circ\mathbf{H}$. If ${}^\circ\mathbf{K}_T \neq {}^\circ\mathbf{H}$, then there exist $x, y \in S$ such that $\mathbf{K}_{Tx} \neq \mathbf{K}_{Ty}$ and $\mathbf{K}_{Tx} \subset \mathbf{H}_x = \mathbf{H}_y \supset \mathbf{K}_{Ty}$. Let $e \in \mathbf{P}_T(x)$ $(e^2 = e)$ and let $f \in \mathbf{P}_T(y)$ $(f^2 = f)$. Lemma 3 implies that $e \in \mathbf{K}_{Tx}$ and $f \in \mathbf{K}_{Ty}$ and thus we obtain that $e, f \in \mathbf{H}_x$. According to (15), we have $e = f$ so that $\mathbf{K}_{Tx} = \mathbf{K}_{Te} = \mathbf{K}_{Ty}$ which is a contradiction. Hence ${}^\circ\mathbf{K}_T = {}^\circ\mathbf{H}$.

${}^\circ\mathbf{K}_T = {}^\circ\mathbf{H} \Rightarrow$ 4 of Theorem 3. This follows from ${}^\circ\mathbf{H} \subset {}^\circ\mathbf{L}$ (see (12)).

2 of Theorem 3 $\Rightarrow$ 1 of Theorem 2. Let $x \in S$. It follows from (19) that $e \in \mathbf{P}_T(x)$ where $e^2 = e$. Since $e \in Se \in \mathscr{F}(\mathbf{L}_T) \subset \mathscr{F}(\mathbf{P}_T^*)$ (see (6)), hence $\mathbf{P}_T(x) \cap Se \neq \emptyset$. By Lemma 1 we have that $x \in Se$. Therefore $x = ue$ for some $u \in S$ and so $x = ue = ue^2 = xe$. According to Lemma 6 and Lemma 3, we have $x \in \mathbf{H}_e$. This implies that S is a union of groups and therefore, S is right regular.

Theorem 7. *The following conditions on a semigroup S are equivalent*:

1. *S is a semilattice of right groups*;
2. *S is a union of groups and $\mathbf{L}_T \leqq \mathbf{R}_T$*;
3. $\mathbf{P}_T^* \leqq \mathbf{L}_T \leqq \mathbf{R}_T$;
4. $\mathbf{K}_T \leqq \mathbf{L}_T \leqq \mathbf{R}_T$;
5. ${}^\circ\mathbf{K}_T \subset {}^\circ\mathbf{L} \subset {}^\circ\mathbf{R}$;
6. ${}^\circ\mathbf{K}_T = {}^\circ\mathbf{L}$.

Proof. $1 \Rightarrow 2$. It follows from Theorem 10 of [3] that S is a union of groups and $\mathbf{L} \leqq \mathbf{R}$. By Lemma 8 we have $\mathbf{L}_T \leqq \mathbf{R}_T$.

$2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$. This follows from Theorem 3, Theorem 4 and from (12).

$5 \Rightarrow 6$. If ${}^\circ\mathbf{L} \subset {}^\circ\mathbf{R}$, then by Theorem 6 and (13) we have ${}^\circ\mathbf{K}_T = {}^\circ\mathbf{H} = {}^\circ\mathbf{L}$.

$6 \Rightarrow 1$. If ${}^\circ\mathbf{K}_T = {}^\circ\mathbf{L}$, then by Theorem 6 and Lemma 5 we have ${}^\circ\mathbf{K} \subset {}^\circ\mathbf{K}_T = {}^\circ\mathbf{L} = {}^\circ\mathbf{H} \subset {}^\circ\mathbf{R}$. Theorem 10 in [3] implies that S is a semilattice of right groups.

We have:

Theorem 8. *The following conditions on a semigroup S are equivalent*:

1. *S is a semilattice of left groups*;
2. *S is a union of groups and $\mathbf{R}_T \leqq \mathbf{L}_T$*;

3. $\mathbf{P}_T^* \leqq \mathbf{R}_T \leqq \mathbf{L}_T$;
4. $\mathbf{K}_T \leqq \mathbf{R}_T \leqq \mathbf{L}_T$;
5. $°\mathbf{K}_T \subset °\mathbf{R} \subset °\mathbf{L}$;
6. $°\mathbf{K}_T = °\mathbf{R}$.

Theorem 9. *The following conditions on a semigroup S are equivalent:*

1. *S is a semilattice of groups;*
2. *S is a union of groups and* $\mathbf{L}_T = \mathbf{R}_T$;
3. $\mathbf{P}_T^* \leqq \mathbf{L}_T = \mathbf{R}_T$;
4. $\mathbf{K}_T \leqq \mathbf{L}_T = \mathbf{R}_T$;
5. $°\mathbf{K}_T \subset °\mathbf{L} = °\mathbf{R}$;
6. $°\mathbf{K}_T = °\mathbf{L} = °\mathbf{R}$;
7. $°\mathbf{K}_T = °\mathbf{M}$.

Proof. $1 \Rightarrow 2$. It follows from Theorem 12 of [3] that S is a union of groups and $\mathbf{L} = \mathbf{R}$. Thus we have $\mathbf{L}_T = \mathbf{R}_T$.

$2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6$. This follows from Theorem 7 and Theorem 8.

$6 \Rightarrow 7$. It follows from Theorems 7 and 8 that $\mathbf{L}_T = \mathbf{R}_T$. According to Lemma 8 and its dual, we have $\mathbf{L} = \mathbf{R} = \mathbf{M}$ so that $°\mathbf{K}_T = °\mathbf{L} = °\mathbf{M}$.

$7 \Rightarrow 1$. Theorem 6 implies that $°\mathbf{H} = °\mathbf{K}_T = °\mathbf{M} = °\mathbf{L} = °\mathbf{R}$. According to Lemma 5, we have $°\mathbf{K} \subset °\mathbf{L} = °\mathbf{R}$. It follows from Theorem 12 in [3] that S is a semilattice of groups.

References

[1] *Sedlock J. T.:* "Green's relations on a periodic semigroup", Czech. Math. J., *19* (1969), 318—323.
[2] *Green J. A.:* "On the structure of semigroups", Annals of Math., *54* (1951), 163—172.
[3] *Pondělíček B.:* "A certain equivalence on a semigroup", Czech. Math. J., *21* (1971), 109—117.
[4] *Schwarz Št.:* „К теории хаусдорфовых бикомпактных полугрупп". Czech. Math. J., *5* (1955), 1—23.
[5] *Pondělíček B.:* "On a certain relation for closure operations on a semigroup", Czech. Math. J., *20* (1970), 220—231.
[6] *Pondělíček B.:* "On periodic and recurrent compact groupoids", Čas. pro pěst. mat. *93* (1968), 262—272.
[7] *Šulka R.:* "Note on the nilpotency in compact H-semigroups", Mat. čas. *18* (1968), 105—112.

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