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A CLASS OF COMMUTATIVE SEMIGROUPS IN WHICH THE IDEMPOTENTS ARE LINEARLY ORDERED

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The study of primary semigroups is initiated by the author in [3], and it appears to be basic in view of the fact that Archimedian semigroups are primary and hence, according to a well-known theorem, every commutative semigroup is the disjoint union of Archimedian and thus primary semigroups. In primary semigroups. idempotents are linearly ordered. In [3] it is proved that among cancellative commutative semigroups the property that prime ideals being maximal is equivalent to the property that the semigroup is primary. If a commutative semigroup has identity and if its prime ideals are maximal, then the semigroup is always primary. These observations naturally prompt one to study the interconnections between the semigroups in which idempotents are linearly ordered and the semigroups in which prime ideals are maximal and to obtain the conditions when they are primary semigroups. The study of semigroups (not necessarily commutative) in which prime ideals are maximal is already initiated by Schwarz [4] and some interesting results regarding the classical radical in ring theoretic sense are obtained. In 1.1 and 1.3 the necessary and sufficient conditions are found when a commutative semigroup in which prime ideals are maximal can have its idempotents linearly ordered and conversely. In 1.4 an example is constructed to show even the commutative semigroup which has only one idempotent (hence idempotents are linearly ordered) and has only one prime and maximal ideal (hence prime ideals are maximal) need not be primary. Theorem 1.5 asserts that the only primary semigroups in which prime ideals are maximal are Archimedian. In Section 2 we shall characterize all semigroups in which every ideal is prime (a subclass of primary semigroups). Now we shall recall some facts from [3] for ready reference. An ideal A in a semigroup S is primary if $xy \in A$ and $x \notin A$, then for some positive integer n, $y^n \in A$. A commutative semigroup is said to be primary iff all of its ideals are primary. A right ideal A in an arbitrary semigroup S is called prime iff $xy \in A \Rightarrow x$ or $y \in A$. If A is an ideal in a commutative semigroup S, then $\sqrt{(A)} = \{x \in S \mid \exists\, n$ such that $x^n \in A\}$. Following the lines of proof in commutative ring theory [2; 104] it can be proved that $\sqrt{A}$ is the intersection of all prime ideals that contain A. This basic result is frequently used in this paper. The ordering of the idempotents in this paper is the same natural ordering found in [1]. A commutative semigroup S is defined to be Archimedian if, for any two elements of S, each divides some power of the other. If A is a subset of a semigroup S, we denote the complement of A in S by $S - A$.

1. SEMIGROUPS IN WHICH PRIME IDEALS ARE MAXIMAL

1.1. Theorem *Let S be a commutative semigroup with identity. Then the following are equivalent.*

i) *prime ideals are maximal*

ii) *S is either a group and so Archimedian or S has a unique prime ideal P such that $S = G \cup P$, where G is the group of units of S and P is an Archimedian subsemigroup of S.*

In either case S is a primary semigroup and S has at most one idempotent different from identity.

Proof. The proof for (i) $\Leftrightarrow$ (ii) is obvious by noting that S has a unique maximal ideal which is prime since S has identity. The proof that P is Archimedian is worked out in the proof of 1.3 below. By theorem 2.5 of [3], S is a primary semigroup. Let S be not a group. If possible let e and f be idempotents different from the identity. By virtue of the property of the radical mentioned in the introduction, $\sqrt{(eS)} = \sqrt{(fS)} = P$, which is the unique prime ideal as well as the unique maximal ideal. This implies $e = ef = f$.

1.2. Note. If a commutative semigroup has no identity, the above type of result need not be true. Let $S = \{a, b, ab : a^2 = a;\ b^2 = b;\ ab = ba\}$. S is a semigroup with the prime ideals $\{a, ab\}$ and $\{b, ab\}$ which are also maximal. The failure of the conclusion of theorem in this example may be attributed to the fact that the idempotents in S do not form a chain. In fact we have

1.3. Theorem. *Let S be a commutative semigroup without identity. Then the following are equivalent.*

i) *prime ideals are maximal and idempotents in S form a chain*

ii) *S is Archimedian or there exists only one prime ideal P in S and $S = P \cup S - P$, where P is an Archimedian semigroup and $S - P$ is a group.*

iii) *prime ideals are maximal in S and S has at most two idempotents.*

Proof. (i) $\Rightarrow$ (ii): If S has no proper prime ideals, then for any $a, b \in S$, we have $\sqrt{(a \cup aS)} = \sqrt{(b \cup bS)} = S$. Thus there exist positive integers m and n such that $b^n \in a \cup aS$ and $a^m \in b \cup bS$. Hence S is Archimedian. Suppose S has a proper prime ideal M. Firstly we shall prove that $S - M$ is a group. Let $a \in S - M$. Then $a^2 \in S - M$ since M is a prime ideal. Also M is a maximal ideal by hypothesis and hence $M \cup a \cup aS = M \cup a^2 \cup a^2S$. Thus $a = a^2$ or $a = a^2x$, $x \in S - M$ and this implies $S - M$ is a regular semigroup. From this follows that $S - M$ is a group since $S - M$ has only one idempotent. For, if e and f are two idempotents in $S - M$, then $M \cup e \cup eS = M \cup f \cup fS = S$. Hence $e = f$ or $e = fe$ and also $f = fe$ or $f = e$ and so $e = f$. Now we claim that S has a unique prime ideal. Suppose that M and N be two proper prime ideals in S. Then $S - M$ and $S - N$ are groups with identities e and f respectively. Since idempotents form a chain by hypothesis, assume for definiteness, that $e < f$, that is, $e = ef$. This implies that $f \notin M$ since $e \notin M$. But $S - M$ has only idempotent since $S - M$ is a group. So $e = f$. Now $x \in S - M \Rightarrow \exists\ y$ such that $xy = e \in S - N$. If $x \notin S - N$, then $x \in N$ and so $xy = e \in N$, a contradiction. Thus we have $S - M \subseteq S - N$ and by symmetry we assert $S - M = S - N$. Hence $M = N$. Now we have $S = M \cup S - M$, where $S - M$ is a group. Since M is the unique prime ideal in S, for any $a, b \in M$, $\sqrt{(a \cup aS)} = \sqrt{(b \cup bS)} = M$. Then there exist positive integers m and n such that $a^m = bx$ and $b^n = ay$. Clearly $a^{m+1} = b(ax)$ and $b^{n+1} = a(by)$ where ax and $by \in M$. Hence we conclude that M is an Archimedian subsemigroup. (ii) $\Rightarrow$ (iii): Suppose S is Archimedian. We assert that S has no proper prime ideals and there is at most one idempotent, which proves (iii) in this case. If possible let there exist a prime ideal $P \neq S$. Let $x \in P$. If $y \in S$, since S is Archimedian, y divides some power of x and so $y \in \sqrt{(x \cup xS)}$. Thus $\sqrt{(x \cup xS)} = S$. But $\sqrt{x \cup xS} \subseteq \sqrt{P} = P$. This implies $P = S$, a contradiction. This non-existence of proper prime ideals in S implies that $\sqrt{(eS)} = \sqrt{(fS)} = S$, where e and f are any two arbitrary idempotents in S. Hence $e = ef = f$. Assume the second condition in (ii). If P is not maximal, then P is contained properly in a proper ideal A. Let e be the identity of $S - P$. Since $A \neq P$, there exists an $x \in A - P$ such that $xy = e$ for some $y \in S$. This implies $e \in A$. Since $A \neq S$, there exists $a \in S - A$. But $a = ae$ and hence $a \in A$, a contradiction. Thus prime ideals are maximal in S. In order to prove that S has at most two idempotents, it suffices to show that P has at most one idempotent since the group $S - P$ has only one idempotent. If P has two idempotents e and f, then $\sqrt{(eS)} = \sqrt{(fS)} = P$, since P is the unique proper prime ideal in S. This clearly implies that $e = ef = f$. (iii) $\Rightarrow$ (i): Let e and f be the only two idempotents in S. Since ef is an idempotent, we must have $e = ef$ or $f = ef$. Thus idempotents are linearly ordered.

1.4. Remark. If a semigroup in which prime ideals are maximal has identity, then it is primary by Theorem 2.5 of [3]. But if the semigroup has no identity, even with the additional hypothesis that the idempotents form a chain, the semigroup need not be primary. Consider the semigroup $S = \{a, a^2, \ldots\} \cup e$ where $e^2 = e$ and $ae = ea = a^2$. $P = \{a, a^2, \ldots\}$ is the unique prime ideal in S and also is maximal in S. Also $S - P$ is a group. But the ideal $A = \{a^2, \ldots\}$ is not primary since $ae \in A$, $a \notin A$ and no power of e is in A. The failure of this semigroup to be primary is due to the fact that it is not Archimedian, which can be seen in the following. Of course an easy verification asserts that the only commutative Archimedian semigroups with identity are groups.

1.5. Theorem. *Let S be a commutative semigroup without identity. Then S is a primary semigroup in which prime ideals are maximal iff S is Archimedian.*

Proof. Let S be Archimedian. In the course of the proof of Theorem 1.3 we observed that S has no proper prime idelas. Hence it is trivially true that prime ideals are maximal. Let A be an ideal in S such that $xy \in A$ and $x \notin A$. By Archimedian property, there exists a positive integer n such that $y^n = (xy)\, k \in A$. Hence every ideal A in S is primary. Assume now that S is primary and prime ideals in S are maximal. Since S is primary, the idempotents of S are linearly ordered [3; Prop. 2.1]. Then by 1.3, S is either Archimedian or $S = P \cup S - P$, where P is the unique proper prime ideal in S such that $S - P$ is a group. We claim that the latter case is not possible. Let e be the identity of $S - P$. Since S has no identity, there exists $x \in P$, such that $x \neq ex$. Now $ex \in exS$. $x \notin exS$, otherwise $x = exy = e(exy) = ex$, a contradiction. No power of e is in exS, since otherwise, $e \in P$. Hence exS is not a primary ideal, a contradiction.

1.6. Corollary. *Let S be a commutative semigroup. Then every prime ideal in S is a union of Archimedian subsemigroups of S.*

Proof. By Theorem 4.13 [1; 132], $S = \cup S_\alpha$, where S_α is an Archimedian subsemigroup of S. Let P be a prime ideal in S. If $\{S_\beta\}$ is a collection of subsemigroups of $\{S_\alpha\}$ containing elements of P, then $P \subseteq \cup S_\beta$. But $P \cap S_\beta$ is a prime ideal of S_β. As observed in the proof of Theorem 1.5, S_β has no proper prime ideals of its own and hence $P \cap S_\beta = S_\beta$, which implies $S_\beta \subset P$. Thus $P = \cup S_\beta$.

2. SEMIGROUPS IN WHICH EVERY IDEAL IS PRIME

2.1. Proposition. *Let S be a commutative semigroup in which every ideal is prime. Then the following are true.*

1) *The ideals of S form a chain under set inclusion.*
2) *S has at most one maximal ideal.*
3) *Idempotents in S are linearly ordered.*
4) *If A is an ideal, then $A^2 = A$.*
5) *S is a regular semigroup.*

Proof. Let A and B be any two arbitrary ideals in S. If $A \not\subset B$ and $B \not\subset A$, then there exist $a \in A$ and $a \notin B$ and $b \in B$ and $\notin A$. Then $ab \in A \cap B$ and $a, b \notin A \cap B$. This contradicts that $A \cap B$ is prime. (2) and (3) can easily be verified. If A is an ideal, then for $a \in A$, $a^2 \in A^2$. Since A^2 is prime, $a \in A^2$. Thus $A = A^2$. To prove (5), observe $a \cup aS = a^2 \cup a^2 S$ by (4). Hence S is regular.

Combining 2.1 and 2.4 of [3], we have

2.2. Theorem. *Let S be a commutative semigroup. Then every ideal of S is prime iff S is regular and the idempotents in S form a chain.*

2.3. Note. In the non-commutative case the condition that every two-sided ideal is prime or even the stronger condition that every right as well as left ideal is prime need not imply that the idempotents are linearly ordered, though the semigroup is regular. This is illustrated by the example of a semigroup $S = \{a, b, ba : a^2 = a, b^2 = b. ab = a\}$. However if right ideals are linearly ordered and left ideals are linearly ordered then it can easily be verified that idempotents are linearly ordered. In general we have

2.4. Theorem. *Let S be an arbitrary semigroup in which every right ideal and left ideal is prime. Then S is a union of groups and hence is a regular semigroup.*

Proof. Let $a \in S$. Since $a \, . \, a \in a^2 \cup a^2 S$, which is prime, we have $a \in a^2 \cup a^2 S$. Hence S is right regular. Similarly S can be proved left regular. This implies that S is a union of groups by Theorem 4.3 of [1; 122].

2.5. Note. If in a commutative semigroup every ideal is prime, then it is a primary semigroup. But the converse need not be true. Consider the semigroup $S = \{a, b, z\}$ with the multiplication table

	z	a	b
z	z	z	z
a	z	a	a
b	z	a	a

$b^2 \in \{z, a\}$ and $b \notin \{z, a\}$ and so $\{z, a\}$ is not prime. But every ideal is primary.

By virtue of Theorem 2.4 of [3], we have

2.6. Theorem. *Let S be a commutative primary semigroup. Then every ideal in S is prime iff S is regular.*

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