Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 21 (1971), No. 4, 625–632

Persistent URL: http://dml.cz/dmlcz/101061

Czechoslovak Mathematical Journal, 21 (96) 1971, Praha

A GENERALIZATION OF K-COMPACT SPACES

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(Received April 10, 1970)

1. INTRODUCTION

The object of the present paper is to introduce a new class of spaces (named almost K-compact) which is a generalization of the concept of K-compact spaces introduced by H. HERRLICH [5] recently. These spaces also generalize, as can be seen immediately from the definition given in section 3, the concepts of almost compact spaces and almost realcompact spaces [2]. It will be shown that this very general class of spaces has a number of useful properties like productivity, feeble hereditaryness and others.

Before going on to the main object of this paper we present in section 2, some results connected with the theory of K-compact spaces. In section 3, we introduce the definition of almost K-compact spaces some of whose properties are worked out in section 4. Finally in section 5, we characterize them in terms of a Completeness property, the idea of which is due to E. ČECH and which has been stated explicitely by FROLÍK [2]. A brief mention is made in section 2 of a type of space that we have called σ_K-compact. Its properties will be studied elsewhere.

Notations and terminology. We denote a family of sets as $\mathscr{U} = \{U_a : a \in A\}$ or simply $\mathscr{U} = \{U_a\}$, where U_a is indexed by the elements a of an index set A. We will denote union and intersection of all the members of such a family $\mathscr{U}$ by $\bigcup\mathscr{U}$ or $\bigcup_a A_a$ and $\bigcap\mathscr{U}$ or $\bigcap_a U_a$ respectively. If $\mathscr{U}$ is a family of subsets of a space X and 0 a subset of space X, then $\mathscr{U} \cap 0$ will be used to denote the family of all $U \cap 0$, $U \in \mathscr{U}$. The closure of a subset O of a space X will be denoted by $\bar{O}^X$, or simply $\bar{O}$, if no confusion can result thereby. $\overline{\mathscr{U}}$ will be used to denote the family of sets $\{\bar{U} : U \in \mathscr{U}\}$. The cardinality of a set A will be denoted by card (A). Throughout this paper K will denote an infinite cardinal number.

All through the paper we assume the spaces consideted to be Hausdorff except in section 2, where complete regularity is also assumed.

A family $\mathfrak{F}$ of sets is said to have K-intersection property if the intersection of every subfamily $\mathfrak{F}' \subset \mathfrak{F}$ with card $(\mathfrak{F}') < K$ is non empty. K-intersection property is known as centredness or finite intersection property (respectively countable intersection property) when $K = \aleph_0$ (respectively $K = \aleph_1$). A subset of a space X is said to be regular closed if it is the closure of an open set or equivalently if it is the closure of its own interior.

2. *K*-COMPACT AND *K*-LINDELÖF SPACE

In this section, all spaces are assumed to be completely regular. According to H. Herrlich, a space is K-compact if every Z-ultrafilter with K-intersection property is fixed. This can also be stated as follows:

Definition 2.1. A space is *K-compact* if every maximal centred family of zero-sets with K-intersection property has non empty intersection.

Remark 2.1. This is to be distinguished from the concept of (K, f)-compactness to be defined in section 4.

Now, K-Lindelöf space, introduced by Frolík [3], can be stated as:

Definition 2.2. A topological space X is said to be *K-Lindelöf* if every family of closed sets of X with K-intersection property has non empty intersection.

Theorem 2.1. *Every K-Lindelöf space is K-compact.*

Proof. Infact, a space has K-Lindelöf property if every z-filter with K-intersection property is fixed and obviously it implies the following condition: every maximal Z-filter with K-intersection property is fixed, which is nothing but a K-compact space.

The converse of the above theorem may not be true. However, it can be shown to hold under some additional conditions.

BAGLEY and MCNIGHT [1] introduced the concept of I-Space. In a similar manner we can define what may be termed as an I_K-space.

Definition 2.3. A space is said to be *I_K-space* if each collection of closed sets with K-intersection property is contained in a collection of closed sets which is maximal with respect to this property.

Theorem 2.2. *A normal, K-compact, I_K-Space is K-Lindelöf.*

Proof. Let X be the space satisfying the given conditions and let $\{F_a\}$ be a maximal family of closed sets of X with K-intersection property. Let us denote by $\mathscr{Z} = Z_b$ the collection of all zero sets lying in the above system. Evidently $\mathscr{Z}$ has K-intersection property. We shall show that it is maximal.

If $\mathscr{Z} = \{Z_b\}$ is not maximal, then there exists a zero set Z which intersects all the elements of $\{Z_b\}$ but is not a member of $\{Z_b\}$. Thus, Z is not in $\{F_a\}$ and since this collection is maximal with K-intersection property it contains a set F such that $Z \cap F = \emptyset$. In view of the normality of X, there exists a zero set Z_0 for which $Z_0 \supset F$ and $Z_0 \cap Z = \emptyset$. But $Z_0 \in \{F_a\}$. Then this contradicts the assumption that Z intersects each member of $\mathscr{Z}$. Thus $\mathscr{Z}$ must be maximal with respect to the K-intersection property.

Since X is a K-compact space, we have

$$\bigcap_{Z_b \in \mathscr{Z}} Z_b \neq \emptyset$$

and since a closed set in a completely regular space is the intersection of zero sets containing it,

$$\bigcap_a Z_b = \bigcap_a F_a \,.$$

This shows that each maximal family of closed sets with K-intersection property has a non empty intersection and in an I_k-space this clearly implies that X is a K-Lindelöf space.

A variant of the definition of σ-compact spaces can, however, be shown to imply the K-Lindelöf property.

Definition 2.4. If $X = \bigcup\{C : C \in \mathscr{G}\}$ where each member of the family $\mathscr{G}$ is a compact subspace of a space X and card $(\mathscr{G}) < K$, then X will be said to be σ_K-compact.

Clearly every σ_K-compact space is K-Lindelöf and hence is K-compact.

σ_K-compact space will be studied elsewhere.

3. ALMOST K-COMPACT SPACE

In this section we define a new class of topological spaces, called, almost K-compact, which is a generalization of K-compact spaces of Herrlich [5].

Definition 3.1. A space X is said to be *almost K-compact* if $\mathscr{U}$ is a maximal centred family of open sets such that $\overline{\mathscr{U}}$ has the K-intersection property, then $\bigcap\overline{\mathscr{U}} \neq \emptyset$.

Equivalently, a space X is almost K-compact if $\mathscr{U}$ is a maximal centred family of open sets with $\bigcap\overline{\mathscr{U}} = \emptyset$, then $\bigcap\overline{\mathscr{V}} = \emptyset$ for some subfamily $\mathscr{V}$ of $\mathscr{U}$ with card $(\mathscr{V}) < K$.

Every completely regular K-compact space is almost K-compact and so from the definitions themselves, it follows that every completely regular K-Lindelöf space is almost K-compact.

A space is almost $\aleph_0$-compact when it is almost compact and almost $\aleph_1$-compact when it is almost realcompact. Every almost $\aleph_0$-compact space (i.e. almost compact) is almost $\aleph_1$-compact (i.e. almost realcompact). In general, we can say that if $\mathfrak{k}$ and $\mathfrak{l}$

are two cardinal numbers with $\mathfrak{k} \leq \mathfrak{l}$, then every almost $\mathfrak{k}$-compact space is almost $\mathfrak{l}$-compact.

Remark 3.1. If the space is completely regular and if 'open sets' are replaced by 'zero sets' in definition 3.1, then we reach the definition of K-compact spaces.

4. PROPERTIES OF ALMOST K-COMPACT SPACES

4.1. Subsets. It is well known that realcompact and K-compact spaces are weakly hereditary. Our almost K-compact space is not hereditary. It is not even weakly hereditary, since even almost realcompact spaces do not have this property.

Definition 4.1.1. A property P of a topological space is said to be *feebly hereditary* if every regular closed sub-space of the space has the property P.

Almost compact spaces and almost realcompact spaces are feebly hereditary, and we will show that almost K-compact spaces are also feebly hereditary.

Theorem 4.1.1. *A regular closed subset of an almost K-compact space is almost K-compact.*

Proof. Let X be an almost K-compact space and O be an open subset of X. Then a regular closed subset of the space X is a set of the form $\bar{O}$. We have to prove that $\bar{O}$ is an almost K-compact space.

Let $\mathcal{U}$ be a maximal centred family of open subsets of O such that the intersection of closures of sets from $\mathcal{U}$ in $\bar{O}$ is empty i.e. $\bigcap\{\bar{U} \cap \bar{O} : U \in \mathcal{U}\} = \emptyset$.

Let $\mathcal{V}$ be a maximal centred family of open subsets of X with $\mathcal{V} \supset \mathcal{U} \cap 0$, such that $\bigcap\bar{\mathcal{V}} = \emptyset$. Since X is an almost K-compact space, we have $\bigcap\bar{\mathcal{G}} = \emptyset$, for some subfamily $\mathcal{G}$ of $\mathcal{V}$ such that card $(\mathcal{G}) < K$. Now it follows from the maximality of $\mathcal{U}$ that $\mathcal{G} \cap \mathcal{O} \subset \mathcal{U}$ and $\bigcap\overline{\mathcal{G} \cap \mathcal{O}} = \emptyset$. This means that the intersection of closures of sets from some subfamily of $\mathcal{U}$ whose cardinality is less than K in $\bar{O}$ is empty.

Thus $\bar{O}$ is almost K-compact.

Theorem 4.1.2. Intersection of almost K-compact subspaces of a space is itself almost K-compact.

Proof. Let (X_a) be a family of almost K-compact subspaces of a space X and let $I = \bigcap_a X_a$. We have to prove that I is almost K-compact.

Let $\mathcal{U}$ be a maximal centred family of open subsets of I with $\bigcap\bar{\mathcal{U}}^I = \emptyset$. $\bigcap\bar{\mathcal{U}}^X$ is at most a singleton, and so we can choose an almost K-compact space X_{a_i} containing I such that $\bigcap\bar{\mathcal{U}}^{X_{a_i}} = \emptyset$. Let $\mathcal{V}$ be a maximal centred family of open subsets of X_{a_i} such that $\mathcal{U} \subset \mathcal{V} \cap I$. It follows that then we have actually $\mathcal{V} \cap I = \mathcal{U}$, and we get

$$\emptyset = \bigcap\bar{\mathcal{U}}^{X_{a_i}} = \bigcap\bar{\mathcal{V}}^{X_{a_i}}.$$

Since the space X_{a_i} is almost K-compact, there exists a subfamily $\mathscr{G}$ of $\mathscr{V}$ with card $(\mathscr{G}) < K$ such that $\bigcap \overline{\mathscr{G}}^{X_{a_i}} = \emptyset$. But we have $\mathscr{G} \cap I \subset \mathscr{U}$ and from above $\bigcap \overline{\mathscr{G} \cap I}^{X_{a_i}} = \emptyset$. Since $I \subset X_{a_i}$, this remains true also in I.

Hence I is an almost K-compact Space.

Corollary 4.1.1. *Every closed subspace of a regular (or semi-regular) almost K-compact space is almost K-compact.*

We know that every closed subspace in a regular (or semi-regular) space is the intersection of regular closed subspaces and hence the corollary follows from the theorems 4.1.1 and 4.1.2.

4.2. Products. Now we examine the behaviour of almost K-compactness with regard to products. Herrlich [5] proved that the product of an arbitrary family of K-compact spaces is K-compact. We find that the same is also true for almost K-compact spaces.

Theorem 4.2.1. *An arbitrary product of almost K-compact spaces is an almost K-compact space.*

Proof. Let X be the topological product of a family $\{X_a : a \in A\}$ of almost K-compact spaces, i.e., $X = \times\{X_a : a \in A\}$ where each X_a is an almost K-compact space. We have to prove that X is an almost K-compact space.

Let $\mathscr{U}$ be a maximal centred family of open subsets of X such that $\overline{\mathscr{U}}$ has K-intersection property and π_a the projections of X onto the co-ordinate space X_a such that

$$\pi_a(x) = x_a \quad \text{for} \quad x = \{x_a : a \in A\} .$$

Since π_a is open continuous, the family $\mathscr{U}_a = \{\pi_a(U) : U \in \mathscr{U}\}$ is centred family of open subsets of X_a. We have to show that $\mathscr{U}_a$ is maximal. Let us suppose O to be an open subset of X_a intersecting every member of $\mathscr{U}_a$. Then $\pi_a^{-1}(O)$ is a subset of X intersecting every member of $\mathscr{U}$. Hence by the maximality of $\mathscr{U}$, $\pi_a^{-1}(O) \in \mathscr{U}$. Therefore,

$$\pi_a(\pi_a^{-1}(O)) = O \in \mathscr{U}_a .$$

Thus $\mathscr{U}_a$ is also a maximal family of open subsets of X_a with finite intersection property. Now it remains to show that $\overline{\mathscr{U}}_a$ has K-intersection property.

Now, for any subfamily $\mathscr{U}_0$ of $\mathscr{U}$ with cardinality less than K and such that $\bigcap_{U \in \mathscr{U}} \overline{U} \neq \emptyset$, we have

$$\emptyset \neq \pi_a(\bigcap \overline{U}) \subset \bigcap_{U \in \mathscr{U}_0} \pi_a(\overline{U}) \subset \bigcap_{U \in \mathscr{U}_0} \overline{\pi_a(U)} .$$

This implies that $\overline{\mathscr{U}}_a$ has K-intersection property, since its members are of the form $\pi_a(U)$, $U \in \mathscr{U}$. The space X_a being an almost K-compact space, there exists a point $(x_a) \in \bigcap \overline{\mathscr{U}}_a^{X_a}$, where x_a is a point in X_a for every $a \in A$.

Consequently $(x) = (\{x_a : a \in A\}) \in \bigcap \overline{\mathscr{U}}^X$ where $x = \{x_a\}$ is in the space X, that is, $\bigcap \overline{\mathscr{U}}^X \neq \emptyset$.

Hence the product space X is almost K-compact.

4.3. Mappings. In his paper [5], Herrlich proved that, the inverse image of each K-compact subset of a space under a continuous mapping is K-compact. In our case, however, we are able to prove a corresponding theorem only under an additional hypothesis of regularity for both the spaces.

Theorem 4.3.1. *If f is a continuous mapping from a regular almost K-compact space X into a regular space Y, then the inverse image of each almost K-compact subset of Y is almost K-compact.*

Proof. Let S be an almost K-compact subspace of Y. It is well known that $f^{-1}(S)$ is homeomorphic to the set $\{(x, y) : x \in X \text{ and } y = f(x) \in S\}$ and this set is closed in $X \times S$. Since regularity is a hereditary property and it is also productivity, $X \times S$ is regular and so by the theorem 4.2.1, $X \times S$ is almost K-compact. The theorem, then, follows from the corollary 4.1.1.

The most interesting result of almost K-compact space is that it can be preserved under a (K, f)-perfect mapping which is a notion weaker than that of a perfect mapping.

Definition 4.3.1. A space is said to be (K, f)-*compact* if every open covering of cardinality $<K$ has a finite subcovering.

Definition 4.3.2. A mapping $f : X \to Y$ is (K, f)-*perfect* if f is closed and $f^{-1}(y)$ is (K, f)-compact for every point y in Y.

Remark 4.3.1. If $K = \aleph_1$, then we call the mapping quasi perfect.

Theorem 4.3.2. *The image under a (K, f)-perfect continuous mapping of an almost K-compact space is almost K-compact.*

Proof. Let f be a (K, f)-perfect continuous mapping of an almost K-compact space X onto a space Y. We have to prove that Y is an almost K-compact space.

Let $\mathscr{U} = \{U_a\}$ be a maximal centred family of open-subsets of Y such that $\overline{\mathscr{U}}$ has K-intersection property. Let $\mathscr{V} = f^{-1}(\mathscr{U}) = \{f^{-1}U : U \in \mathscr{U}\}$ be a centred family of open sets of X, where $\overline{\mathscr{V}}$ has K-intersection property. Let us supplement it to a maximal centred family $\mathscr{V}'$ of open sets of X. Now we will show that $\overline{\mathscr{V}}'$ has K-intersection property.

Let us suppose that there exists a subfamily $\mathscr{V}'_0 = \{V_b\}$ of the family $\mathscr{V}'$ with card $(\mathscr{V}'_0) < K$. If $\overline{\mathscr{V}}'$ does not have the K-intersection property, then we can take the subfamily such that $\bigcap \overline{\mathscr{V}}'_0 = \emptyset$. Taking complements on both sides, we have the family $\mathfrak{M} = \{X - \overline{V}_b : V_b \in \mathscr{V}'_0\}$ which is an open covering of X such that card

$(\mathfrak{M}) < K$. This family will surely cover the subset $f^{-1}(y)$ of X for any $y \in Y$. Since for each $y \in Y$, $f^{-1}(y)$ is (K, f)-compact, we can pick up a finite family $\{\overline{V}_{b_i,y}\}$ from the family $\overline{\mathscr{V}}'_0$ of cardinality $<K$ such that for an integer n_y, the family $\{X - \overline{V}_{b,y}\}$ also covers $f^{-1}(y)$, where $i = 1, 2, 3, \ldots, n_y$.

Since the mapping f is closed and continuous, we have $f(\overline{V}) = \overline{f(V)}$ for each subset of X.

So, for each $y \in Y$, $y \in \bigcup_{i=1}^{n_y} \{Y - \overline{f(V)}\}$.

Thus the family $\mathfrak{N} = \{Y - \overline{f(V_b)} : V_b \in \mathscr{V}'_0\}$ is an open cover of Y, such that card $(\mathfrak{N}) < K$.

Hence there exists a V_b in $\mathscr{V}'_0$ such that

$$Y - \overline{f(V_b)} \in \mathscr{U},$$

then $f^{-1}[Y - \overline{f(V_b)}] \in \mathscr{V}'$. From $V_b \in \mathscr{V}'$ and $V_b \cap f^{-1}[Y - \overline{f(V_b)}] = \emptyset$, we get a contradiction of the fact that $\mathscr{V}'$ is centred. Hence $\bigcap \overline{\mathscr{V}}'_0 \neq \emptyset$, i.e., $\overline{\mathscr{V}}'$ has K-intersection property.

Since the space X is an almost K-compact space, we have $\bigcap_{V_b \in \mathscr{V}'} \overline{V}_b \neq \emptyset$. This gives us $\bigcap_a \overline{f^{-1}(U_a)} \neq \emptyset$ so that

$$\bigcap_a \overline{U}_a \neq \emptyset .$$

Thus Y is an almost K-compact space. Hence the theorem.

Remark 4.3.1. This theorem leds to a slight improvement of the Frolík's theorem [2] if we put $K = \aleph_1$. His theorem remains true if we take quasi perfect mapping in place of perfect mapping.

5. CHARACTERIZATION OF ALMOST K-COMPACT SPACES BY A COMPLETENESS PROPERTY

In his paper [2] Frolík defined completeness in the following way.

Definition 5.1. Let $\alpha = \{\mathscr{U}\}$ be a collection of open coverings of a space X. An α-Cauchy family is a centred family $\mathscr{V}$ of open subsets of X such that for every $\mathscr{U}$ in α there exists an U in $\mathscr{U}$ and a V in $\mathscr{V}$ with $V \subset U$.

The collection α is called complete if $\bigcap \overline{\mathscr{V}} \neq \emptyset$ for every α-Cauchy family $\mathscr{V}$.

Note 5.1. Let α be a collection of open covering of a space X and let $\mathscr{V}$ be a maximal centred family of open sub-sets of X. $\mathscr{V}$ is an α-Cauchy family iff $\mathscr{U} \cap \mathscr{V} \neq \emptyset$ for every $\mathscr{U}$ in α [2].

Using the above definition of completeness we shall prove the following theorem.

Theorem 5.1. *A space X is almost K-compact if and only if the collection δ of all open coverings, with cardinality less than K, of X is complete.*

Proof. Necessity. Let δ be the collection of all open coverings with cardinality less than K and let $\mathscr{V}$ be a δ-Cauchy family. Without loss of generality we may assume that $\mathscr{V}$ is a maximal centred family since every centred family of open sets is contained in a maximal one. Now we have to show that $\bar{\mathscr{V}}$ has K-intersection property.

Let $\mathscr{V}_0$ be a subfamily of $\mathscr{V}$ of cardinality less than K, such that $\bigcap\bar{\mathscr{V}}_0 = \emptyset$.

Let $\mathscr{U} = \{X - \bar{V} : V \in \mathscr{V}_0\}$. By our assumption, $\mathscr{U}$ belongs to δ. Since $\mathscr{V}$ is δ-Cauchy family, by note 5.1, we can choose an $U = X - \bar{V}$ in $\mathscr{U} \cap \mathscr{V}$. We have

$U \in \mathscr{V}$, $V \in \mathscr{V}$, $U \cap V = \emptyset$ which contradicts the finite intersection property of $\mathscr{V}$.

Thus $\bigcap\bar{\mathscr{V}}_0 \neq \emptyset$, that is, $\bar{\mathscr{V}}$ has K-intersection property.

The space X being almost K-compact, $\bigcap\bar{\mathscr{V}} \neq \emptyset$, which shows that the collection δ of all open coverings of X of cardinality less then K is complete for every δ-Cauchy family $\mathscr{V}$.

Sufficiency. Let $\mathscr{V}$ be a maximal centred family of open subsets of X such that $\bar{\mathscr{V}}$ has the K-intersection property and let δ be the collection of all open coverings with cardinality less than K. First we have to prove that $\mathscr{V}$ is a δ-Cauchy family.

If $\mathscr{V}$ is not δ-Cauchy family, then by note 5.1 we have $\mathscr{U} \cap \mathscr{V} = \emptyset$ where $\mathscr{U} \in \delta$. Then evidently, for every U in $\mathscr{U}$, the sets of the form $X - \bar{U}$ belong to $\mathscr{V}$ and form a subfamily $\mathscr{V}_0$ of $\mathscr{V}$ with card $(\mathscr{V}_0) < K$.

Now $X - \bar{U} \subset X - U$, or, $\overline{X - \bar{U}} \subset \overline{X - U} = X - U$. Thus $\bigcap\{\overline{X - \bar{U}} : : U \in \mathscr{U}\} \subset \bigcap\{X - U : U \in \mathscr{U}\} \subset X - \bigcap\mathscr{U} = \emptyset$, which contradicts the K-intersection property of $\bar{\mathscr{V}}$. Hence $\mathscr{V}$ is δ-Cauchy family.

Since δ is complete, then for every δ-Cauchy family $\mathscr{V}$ we have

$$\bigcap\bar{\mathscr{V}} \neq \emptyset .$$

Hence X is almost K-compact.

Addendum. 1. In the theorem 2.1 of Section 2, inplace of K-Lindelöf space we can take, K-quasi-Lindelöf spaces, as according to Frolík [3] a completely regular space is K-Lindelöf iff it is K-quasi-Lindelöf.

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